Diffusion due to chaos in 1D quantum walks

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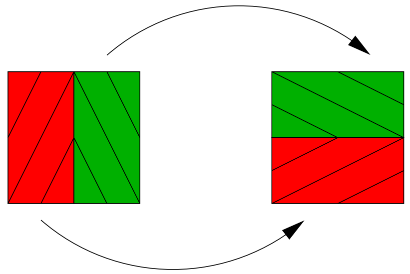
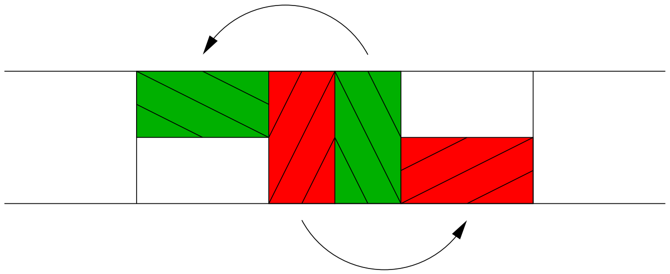
Classical baker and multibaker maps

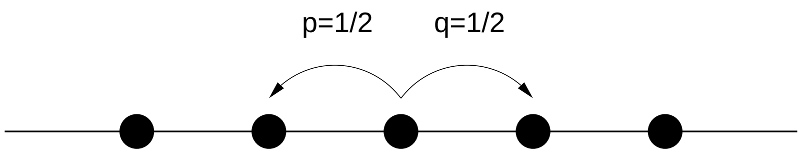
Baker map



Multibaker map

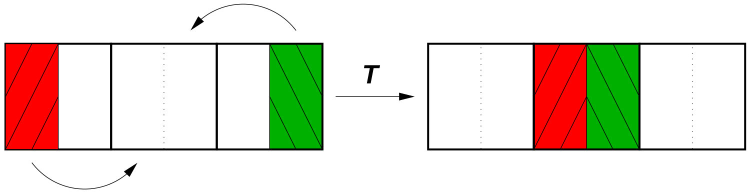


Deterministic model of a simple **random walk**

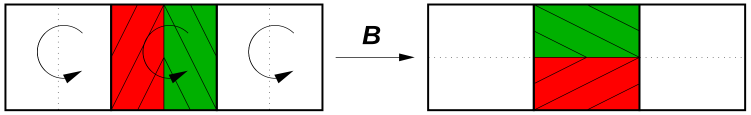


Classical multibakers: new perspective

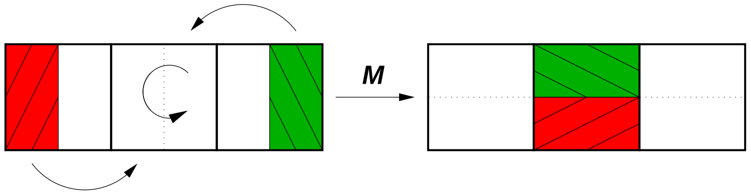
Kick mixes phase space of neighbors



Locally apply **baker** map

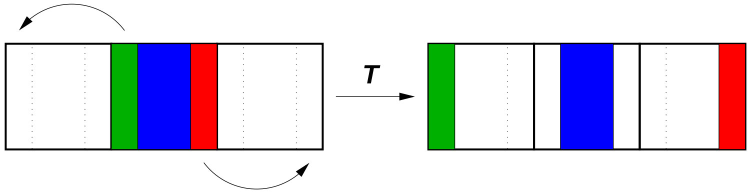


And you **get multibaker** map

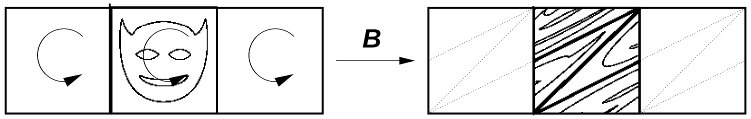


Classical multiplexer maps

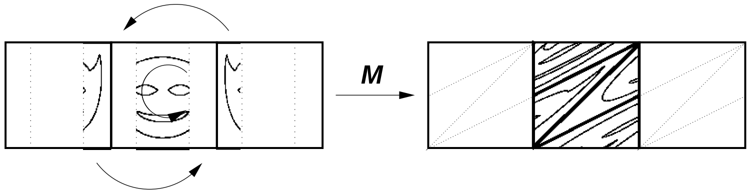
Kick mixes phase space of a number of neighbors



Locally apply **arbitrary map**



And you **get multiplexer** map



Quantum maps on torus

Quantum maps:

$$
\begin{aligned}
|\Psi\rangle &\to |\Psi'\rangle = \hat{U}|\Psi\rangle \\
A &\to A' = \hat{U}^\dagger A \hat{U}
\end{aligned}
$$

Example: stroboscopic map of a Hamiltonian system

$$\hat{U} = e^{-iT\hat{H}/\hbar}$$

where T is the sampling period.

Quantum kinematics on torus $[0,a] \times [0,b]$:

$$
\begin{aligned}
\Psi(x+a) &= \Psi(x)e^{i2\pi\varphi_q} \\
\tilde{\Psi}(p+b) &= \tilde{\Psi}(p)e^{i2\pi\varphi_p}
\end{aligned}
$$

These conditions imply finite N-dimensional Hilbert space

$$N = \frac{ab}{2\pi\hbar} = \frac{1}{h}$$

Quantum multiplexer maps

Local Hilbert space at each site is a simple sum

$$H(n) = \bigoplus_{\varepsilon} H_\varepsilon(n)$$

Dynamics consists of two steps:

- Mixing the Hilbert spaces of neighbors:

  states from $H_\varepsilon(n)$ move to $H_\varepsilon(n+\varepsilon)$

- Local action:

  Copies of map B act each on a single cell $H(n)$

Mean square displacement

We study mean square displacement

$$\langle(\Delta r)^2(t)\rangle = \langle(M^{\dagger t} r M^t - r)^2\rangle_{\rm eq}$$

Define velocity operator $v := M^\dagger r M - r$. Then

$$\langle(\Delta r)^2(t)\rangle = t\langle v^2\rangle + \sum_{n=1}^{t-1}(t-n)C_n$$

where $C_n = \langle\{v_n, v_0\}\rangle$ is the velocity autocorrelation function.

Due to translational invariance we can write

$$\begin{aligned} C_n &= \frac{2}{LN}\mathrm{Tr}\,[M^{\dagger n} v M^n v] \\ &= \frac{2}{N}\mathrm{Tr}\,[B^{\dagger n} J B^n J] \end{aligned}$$

where J is the velocity operator reduced to one cell. Similar result holds for the classical version of the map.

MSD: general formula

In terms of the spectrum and eigenvectors of the local dynamics

$$B|k\rangle = e^{i\varphi_k}|k\rangle$$

the **velocity autocorrelation** function takes form

$$\begin{aligned} C_n &= \frac{2}{N}\sum_{j,k}|J_{jk}|^2 e^{i(\varphi_j-\varphi_k)n} \\ &= \frac{2}{N}\mathrm{Tr}J^2 - \frac{8}{N}\sum_{j>k}|J_{jk}|^2\sin^2\frac{(\varphi_j-\varphi_k)n}{2} \end{aligned}$$

Thus the formula for the **mean square displacement** becomes

$$\boxed{\langle(\Delta r)^2(t)\rangle = \langle|J_{jj}|^2\rangle t^2 + \frac{1}{N}\sum_{j\neq k}|J_{jk}|^2\frac{\sin^2\frac{(\varphi_j-\varphi_k)t}{2}}{\sin^2\frac{\varphi_j-\varphi_k}{2}}}$$

It does **not** imply that there is **always** a **ballistic** contribution.

MSD: extremal cases

Properties of MSD (classical + quantum)

$$\begin{aligned}\langle(\Delta r)^2\rangle(0) &= 0\\ \langle(\Delta r)^2\rangle(1) &= \langle v_0^2\rangle = \mathrm{Tr}J^2\\ 0 \le \langle(\Delta r)^2\rangle(t) &\le t^2\end{aligned}$$

The slowest motion

Dyadic system interchanging two halves of a cell keeps the wave function localized:

$$\langle(\Delta r)^2\rangle(t) \approx t^0$$

The fastest motion

Trivial dynamics $B = \mathrm{Id}$ leads to ballistic motion

$$\langle(\Delta r)^2\rangle(t) = t^2$$

The typical motion

Average behavior in an ensemble of evolution operators:

Random Matrix Theory

t^0? t^1? t^2? t^α?

Random Matrix Theory estimates

Draw B randomly from some **ensemble of random matrices**.

Assume the distribution of eigenvalues **independent** from the distribution of elements of eigenvectors.

For dyadic maps $\mathrm{Tr}J^2 = N$. Ensemble average of the **mean square displacement** becomes

$$\begin{aligned}\langle\langle(\Delta r)^2\rangle\rangle &= t + t(t-1)\langle|J_{jj}|^2\rangle\\ &\quad +2(N-1)\langle|J_{j\neq k}|^2\rangle \sum_{n=1}^{t-1}(t-n)\,\langle e^{i\,\Delta\varphi\,n}\rangle\end{aligned}$$

Straightforward calculation gives

$$\begin{aligned}\langle|J_{jj}|^2\rangle &= k/(N+k)\\ (N-1)\langle|J_{j\neq k}|^2\rangle &= N/(N+k)\end{aligned}$$

where $k = 1$ for CUE, and 2 for COE.

Simple average

On the average, the **eigenphases** of a typical random matrix in any circular ensemble **are equally spaced** ("spectral rigidity"). Assuming $\varphi_j - \varphi_k = 2\pi m/N$ we obtain a naive estimate

$$\langle e^{i(\varphi_j - \varphi_k)n} \rangle = \begin{cases} 1 & \text{if } N|n \\ -1/(N-1) & \text{otherwise} \end{cases}$$

This assumption leads to

$$\langle\langle(\Delta r)^2\rangle\rangle = \frac{k}{N+k}t^2 - \frac{N}{N+k}\frac{l(l-N)}{N-1}$$

where l is the remainder from the integer division of t by N.

Taking $l = t$ for $t < N$ and $l = 0$ for $t \gg N$ we get

$$\begin{aligned} \langle\langle(\Delta r)^2\rangle\rangle &= t + \frac{N(k-1)-k}{(N+k)(N-1)}t(t-1) & t \leq N \\ &\approx \frac{k}{N+k}t^2 & t \gg N \end{aligned}$$

Ensemble average

Taking the **average with the pair correlation function** for CUE

$$R_2(\varphi, \vartheta) = \frac{N^2}{4\pi^2}\left[1 - \frac{\sin^2 \frac{N(\varphi-\vartheta)}{2}}{N^2 \sin^2 \frac{(\varphi-\vartheta)}{2}}\right]$$

we obtain 0 for $n \geq N$ and

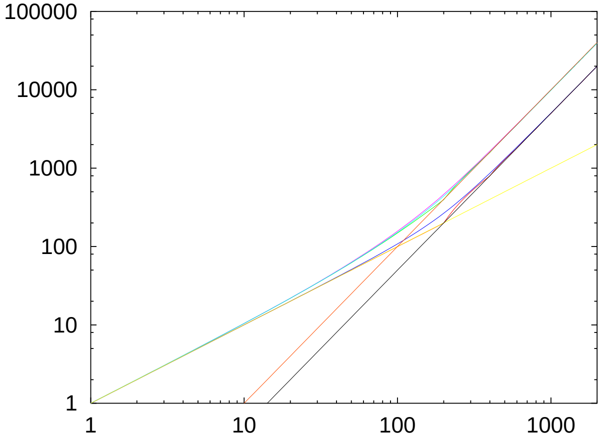
$$\langle e^{[i(\varphi-\vartheta)n]} \rangle = \frac{n-N}{N(N-1)} \quad \text{for } n < N,$$

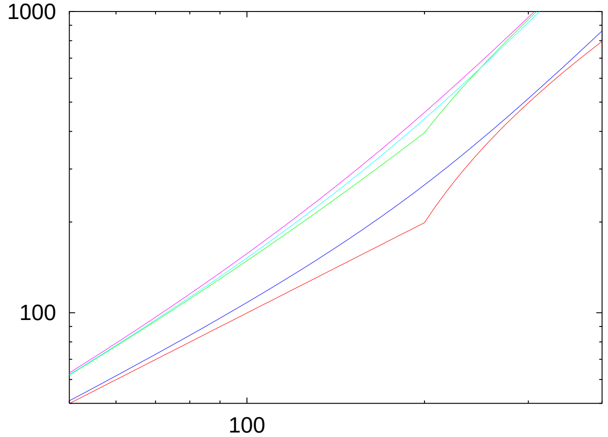
Then the mean square displacement takes form

$$\begin{aligned} \langle(\Delta r)^2(t)\rangle &= t + \frac{t(t-1)}{N+k}\left[k - 1 + \frac{t-2}{3(N-1)}\right] & t \leq N \\ &= \frac{k}{N+k}t^2 + \frac{N}{3} - \frac{N(k-1)}{3(N+k)} & t > N \end{aligned}$$

(plus a negligible correction in COE case)

Predictions of Random Matrix Theory

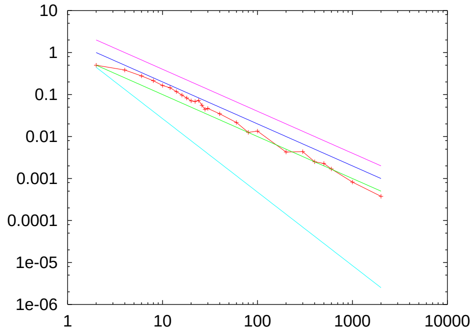


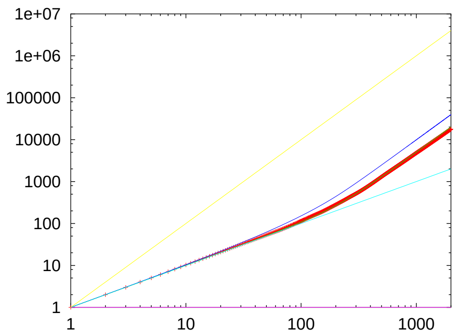


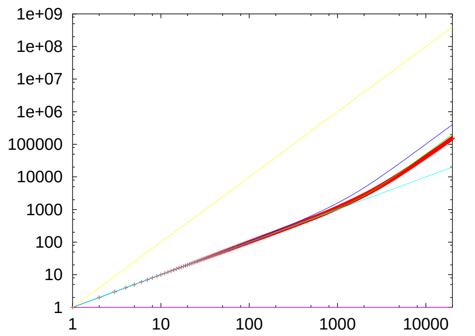
Numerical results: baker, $\varphi_q = 0.0,\ \varphi_p = 0.0$

$$(G_N(\varphi_q, \varphi_p))_{k,l} = \langle p_k | q_l \rangle = \frac{1}{\sqrt{N}} e^{-i2\pi(l+\varphi_q)(k+\varphi_p)/N}$$

$$B = G_N^{-1} \begin{bmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{bmatrix}$$



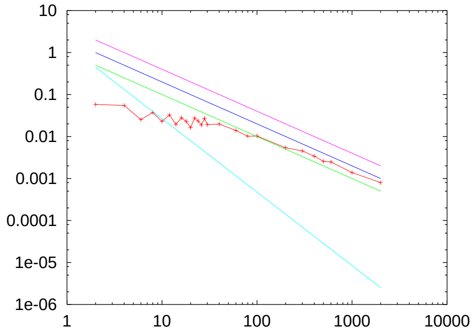


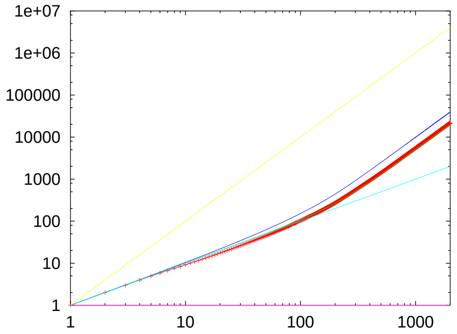


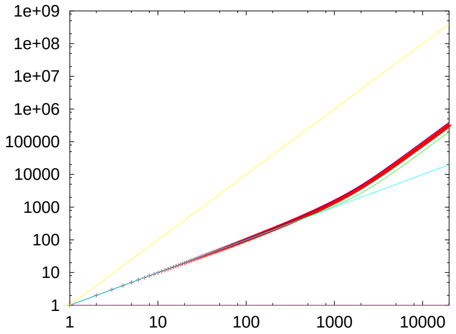
Numerical results: baker, $\varphi_q = 0.73$, $\varphi_p = 0.11$

$$(G_N(\varphi_q,\varphi_p))_{k,l} = \langle p_k|q_l\rangle = \frac{1}{\sqrt{N}} e^{-i2\pi(l+\varphi_q)(k+\varphi_p)/N}$$

$$B = G_N^{-1} \begin{bmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{bmatrix}$$



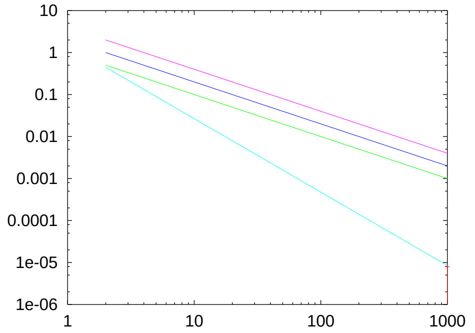


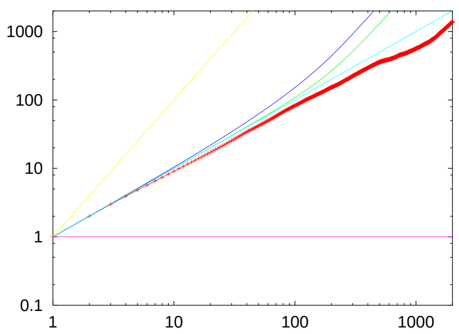


Numerical results: baker, $\varphi_q = 0.5$, $\varphi_p = 0.5$

$$(G_N(\varphi_q,\varphi_p))_{k,l} = \langle p_k|q_l\rangle = \frac{1}{\sqrt{N}} e^{-i2\pi(l+\varphi_q)(k+\varphi_p)/N}$$

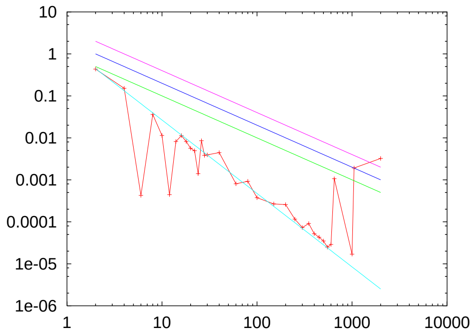
$$B = G_N^{-1} \begin{bmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{bmatrix}$$

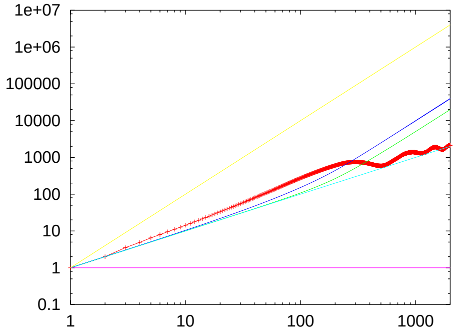


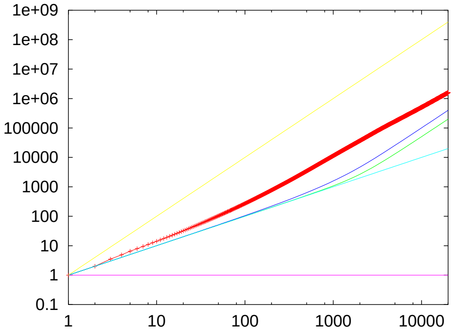


Numerical results: standard map, $\kappa = 6.5$

$$B_{k,l} = \frac{1}{\sqrt{N}} \exp\left[\frac{i\pi}{N}(k-l)^2 + \frac{i\kappa N}{2\pi}\cos\left(\frac{2\pi l}{N}\right)\right]$$

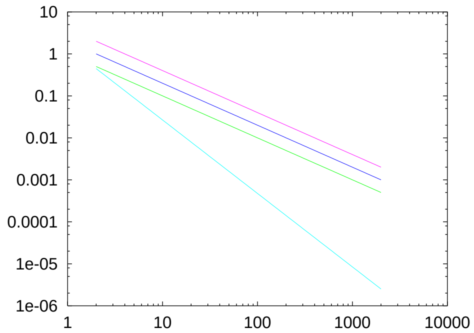


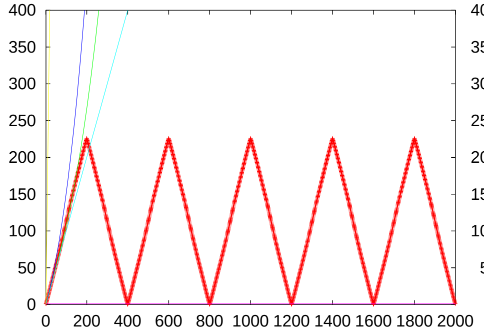


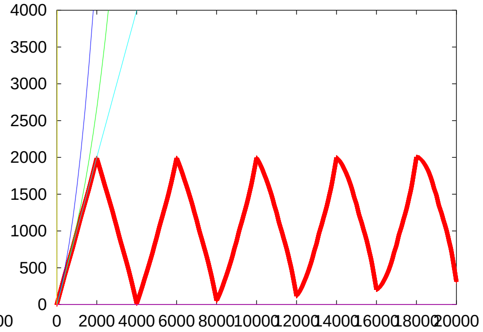


Numerical results: standard map, $\kappa = 0$

$$B_{k,l} = \frac{1}{\sqrt{N}} \exp\left[\frac{i\pi}{N}(k-l)^2\right]$$

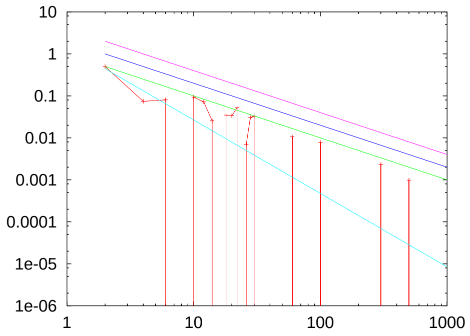


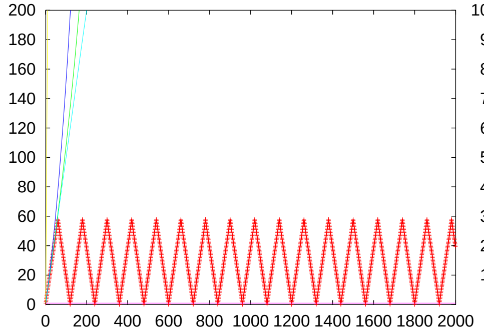


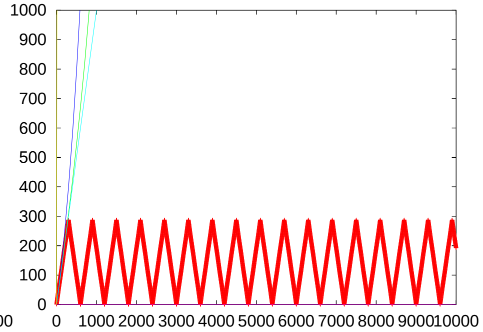


Numerical results: cat map

$$B_{k,l} = \frac{1}{\sqrt{N}} \exp\left[\frac{2\pi i}{N}(k^2 - kl + l^2)\right]$$

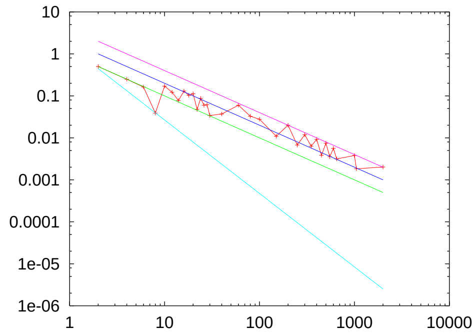


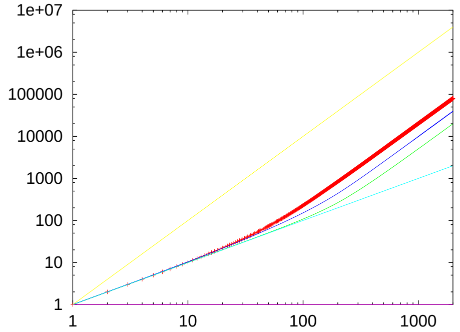


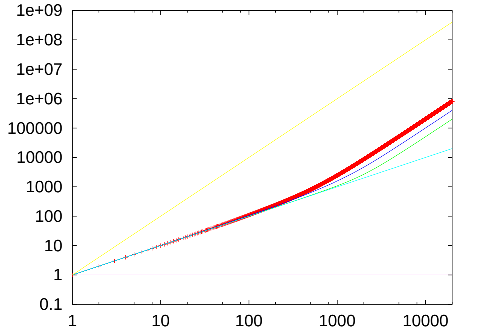


Numerical results: perturbed cat map, $\kappa = 0.3$

$$B_{k,l} = \frac{1}{\sqrt{N}} \exp\left[\frac{2\pi i}{N}(k^2 - kl + l^2) + \frac{i\kappa N}{2\pi} \sin\left(\frac{2\pi l}{N}\right)\right]$$







Summary

- New class of systems: **multiplexer maps**
- General formula for the mean square displacement in classically chaotic quantum multiplexer maps.
- Numerical evaluation + RMT predictions: good agreement.
- Typical behavior of quantum multiplexer maps is diffusive for times of the order of h^{-1} and ballistic for longer times.
- Classical behavior persists up to times on the order of h^{-1}.
- “Ballistic coefficient” is proportional to Planck constant
- Semiclassical transition is smooth: There is no need for decoherence is establishing diffusive behavior in semiclassical limit in 1D systems.
- If RMT predictions are generically connected with classical chaos, then classical chaoticity enforces diffusion.