Deformations of minimal curves in $\mathbb{C}^3$

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Abstract: We consider a representation formula for minimal curves, denoted by $\mathbf{Var}_{f,h}$, assigning to a pair (f,h) of meromorphic functions on an open subset $\mathcal{U}$ of $\mathbb{C}$ a minimal curve in $\mathbb{C}^3$, that is parameterized by rational functions of the derivatives of f and h of order not greater than 3. We prove some of its basic properties, among them a transformation formula, showing, that $\mathbf{Var}_{f,h}$ has a global analogue $\widetilde{\mathbf{Var}}_{f,\mathrm{h}}$, that acts as a nonlinear (rational) differential operator from the Cartesian product of the space of meromorphic functions f with the space of meromorphic vector fields h of a Riemann surfaces Σ onto the space of minimal meromorphic mappings of Σ into $\mathbb{C}^3$. We also prove an inversion formula for **Var** and show, how to represent **Var** by the Weierstrass formula.

1 Introduction

Minimal curves are understood as meromorphic mappings $\Phi : \mathcal{U} \subset \mathbb{C} \longrightarrow \mathbb{C}^3$ such that $\langle \Phi', \Phi' \rangle = 0$, where $\langle .,. \rangle$ is the bilinear extension of the ordinary Euclidean scalar product of $\mathbb{R}^3$ to $\mathbb{C}^3$. Basic invariants of minimal curves are the natural parameter of E. Vessiot, resembling usual arc length of real curves, and the invariant κ of E. Study, that resembles the curvature of curves in real Euclidean space.

We call the latter *minimal curvature*. An orthonormal moving frame for such curves can be defined in such a way, that the Frenet equations (3.3) are practically the same as in the case of curves in real three-dimensional space, only that the *minimal torsion*, i.e., the analogue of the usual torsion of curves in $\mathbb{R}^3$, depends on κ via $\tau = \kappa' \kappa^{-1}$. These invariants are introduced in Section 3. The minimal curvature is the basic invariant needed in the definition of the new representation formulas $\mathbf{Var}_{f,h}$ and $\mathbf{Def}_{f,h}$ for minimal curves.

Let primes denote derivatives with respect to the natural parameter. The variation $\Delta_{\Phi,h}$ of a minimal curve Φ is defined as a linear combination $\Delta_{\Phi,h} = v_1 \Phi' + v_2 \Phi'' + v_3 \Phi'''$, where the functions v_i are determined so that both $\Delta_{\Phi,h}$ and the deformation $\Psi_{\Phi,h} = \Phi + \Delta_{\Phi,h}$ are new minimal curves. It turns out that the coefficients v_i depend on the Study invariant of Φ and an arbitrary meromorphic function h (see (4.5)).

If Φ is given by the Weierstrass representation formula $\mathbf{Wei}_{f,g}$ (see (2.4)) in terms of two meromorphic functions f and g then Φ is parameterized by the natural parameter if and only if $f g' = i$. Replacing f by i/f' and g by f one obtains a special version $\mathbf{Wei}^*_f$ of the Weierstrass representation formula for all minimal curves Φ in natural parameterization.

Combining the variation $\Delta_{\Phi,h}$ with $\mathbf{Wei}^*_f$, i.e., putting $\Phi(z) = \mathbf{Wei}^*_f(z)$, and observing that the minimal curvature of $\mathbf{Wei}^*_f$ is the Schwarzian derivative of f, an explicit algebraic expression depending on the functions f and h and their derivatives is obtained.

This is a representation formula $\mathbf{Var}_{f,h}$ for minimal curves, called *variation* and displayed below by (4.12).

The *deformation* $\mathbf{Def}$ is a related formula defined by $\mathbf{Def}_{f,h} = \mathbf{Wei}^*_f + \mathbf{Var}_{f,h}$. Although somewhat involved, as a linear combination of Φ', Φ'' and Φ''' only, $\mathbf{Var}_{f,h}$ has the advantage to be free of integrations and linear in h. Other interesting properties can be proved.

We show in Section 4, how minimal curves given by the operators $\mathbf{Var}$ and $\mathbf{Def}$ are represented by the Weierstrass formula. In Section 5 it is shown that $\mathbf{Var}_{f,h}$ is equivariant with respect to a group homomorphism μ of $\mathrm{Gl}(2,\mathbb{C})$ into the complex conformal group $\mathrm{CO}(3,\mathbb{C})$, induced by the Weierstrass mapping $(z,w) \longrightarrow \frac{1}{2}z\left(1-w^2, i\,(1+w^2), 2w\right)^\top$ of $\mathbb{C}\times\mathbb{C}$ into the isotropic cone of $\mathbb{C}^3$. Moreover, under changes of the independent variable, it obeys the following transformation rule: If $t(z)$ is a holomorphic parameter transformation, then $\mathbf{Var}_{f,h}(t(z)) = \mathbf{Var}_{f_1,h_1}(z)$, where $f_1(z) = f(t(z))$ and $h_1(z) = h(t(z))/t'(z)$. Consequently, $\mathbf{Var}$ is only the local version of a global invariant of Riemann surfaces. Namely, for any Riemann surface Σ there exists a differential operator $\widetilde{\mathbf{Var}}$, acting from the space of pairs $(\widetilde{f}, \mathrm{h})$, where $\widetilde{f}$ is a meromorphic function and h a meromorphic vector field on Σ, onto the set of minimal curves defined on Σ. If z is a local chart on Σ and f and h are the local expressions of $\widetilde{f}$ and h respectively, the local expression of $\widetilde{\mathbf{Var}}_{\widetilde{f},\mathrm{h}}$ is given by $\mathbf{Var}_{f,h}$ as in (4.12).

$\widetilde{\mathbf{Var}}$ is linear, of order 2 in h and nonlinear, of order 3 in f. Like $\mathbf{Var}$, it is equivariant with respect to the group homomorphism μ.

Most of the results need elementary operations with algebraic expressions involving several holomorphic functions of one variable as well as their derivatives up to orders 4 or 5 and have been proved first with the help of a computer algebra system. Afterwards we worked out computer-free proofs for the most important theorems. Nevertheless, not all proofs are provided. The reader interested in details of proofs or in other computations should consult a computer. For this purpose we have written in the *Mathematica* system a package for various operations and functions.

Together with a tutorial text containing examples and graphics it may be ordered under the authors e-mail address.

2 Weierstrass formula and the natural parameter

We will denote by $\langle\cdot,\cdot\rangle$ the usual scalar product of $\mathbb{R}^3$ as well as its bilinear extension to $\mathbb{C}^3$ and by $(\cdot\ \cdot\ \cdot)$ the associated vector triple product. The norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ is defined with respect to some fixed branch of the square root function on $\mathbb{C}$.

Let $\mathcal{U} \subset \mathbb{C}$ be open. By $\mathbf{A}$ we denote the algebra of meromorphic functions on $\mathcal{U}$. Minimal curves in $\mathbb{C}^3$ are defined as meromorphic mappings $\Psi : \mathcal{U} \subset \mathbb{C} \longrightarrow \mathbb{C}^3$ such that $\langle\Psi',\Psi'\rangle = 0$. Similar to arc length of curves in the real Euclidean space, on minimal curves $\mathcal{C} \subset \mathbb{C}^3$ there exists a natural parameter p defined by the condition

$\left|\mathbf{d}^2\Psi/\mathbf{d}p^2\right|^2 = -1$. Namely, $\langle \Psi', \Psi' \rangle = 0$ implies $\langle \Psi', \Psi'' \rangle = 0$ and therefore, if $z = z(p)$ is a reparameterization of Ψ, then $\langle \Psi_{zz}, \Psi_{zz} \rangle = \left\langle \Psi_{pp}\, p_z^2 + \Psi_p\, p_{zz}, \Psi_{pp}\, p_z^2 + \Psi_p\, p_{zz} \right\rangle = \langle \Psi_{pp}, \Psi_{pp} \rangle\, p_z^4$, where subscripts denote derivatives with respect to the corresponding parameters. Therefore, $\langle \Psi'', \Psi'' \rangle$ defines a biquadratic differential form and its 4-th root defines locally a meromorphic differential form on the domain of Ψ.

An other consequence of the condition $\langle \Psi'(z), \Psi'(z) \rangle = 0$ (see for instance [4]) is the following. If $\Psi: \mathcal{U} \subset \mathbb{C} \longrightarrow \mathbb{C}^3$ is a regular minimal curve and $\mathbf{p} \in \mathbb{C}^3$ an arbitrary vector, we have

$$\langle \Psi''(z), \Psi''(z) \rangle = \sqrt[3]{-\left(\Psi'(z)\ \Psi''(z)\ \Psi'''(z)\right)^2} = -\left(\frac{\left(\Psi'(z)\ \Psi''(z)\ \mathbf{p}\right)}{\langle \Psi'(z), \mathbf{p} \rangle}\right)^2. \tag{2.1}$$

We infer from (2.1) that $\langle \Psi''(z), \Psi''(z) \rangle \neq 0$ if and only if Ψ', Ψ'' and Ψ''' are linearly independent, that is, if Ψ is nowhere planar. Therefore, for nonplanar minimal curves, a new parameter can be defined as follows:

Definition 2.1. The *natural parameter* of a regular, nonplanar, minimal curve $\Psi : \mathcal{U} \subset \mathbb{C} \longrightarrow \mathbb{C}^3$ is the function $p(z)$ given by

$$p'(z) = \sqrt{-\frac{\left(\Psi'(z)\ \Psi''(z)\ \mathbf{p}(z)\right)}{\langle \Psi'(z), \mathbf{p}(z) \rangle}} \tag{2.2}$$

where $\mathbf{p}(z)$ is an arbitrary complex vector function (see [4]).

As a consequence of (2.1) and (2.2) we have $p'(z) \neq 0$ for any nonplanar minimal curve Ψ. Ψ can be reparameterized to the natural parameter and one obtains

$$\langle \Psi'', \Psi'' \rangle = -1. \tag{2.3}$$

The *Weierstrass representation formula* (see [5] or [6]) assigns to any pair f and g of meromorphic functions a minimal curve $\mathcal{C}$ given by the parameterization

$$\Phi(z) = \mathbf{Wei}_{f,g}(z) = \frac{1}{2}\int_{z_0}^{z} \left(f\left(1-g^2\right), i\, f\ \left(1+g^2\right), 2\, f\, g\right)^\top \mathbf{d}\zeta, \tag{2.4}$$

where $z_0 \in \mathbb{C}$ is arbitrary.

A direct computation with (2.2) gives the derivative p'_Φ of the natural parameter of the minimal curve $\Phi = \mathbf{Wei}_{f,g}$:

$$p'_\Phi = \sqrt{i\, f\, g'}. \tag{2.5}$$

By (2.5), putting $f = i/g'$, the Weierstrass formula produces minimal curves that are parameterized by their natural parameter.

DEFINITION 2.2. We define *the natural Weierstrass representation* $\mathbf{Wei}_g^*$ by

$$\mathbf{Wei}_g^*(z) = \mathbf{Wei}_{i/g',g}(z) = \int_{z_0}^{z} \frac{1}{2g'(\zeta)} \begin{pmatrix} i\,(1-g(\zeta)^2) \\ -\,(1+g(\zeta)^2) \\ 2\,i\,g(\zeta) \end{pmatrix} \mathbf{d}\,\zeta. \tag{2.6}$$

Let us denote by $\mathcal{IC}^*$ the set of parameterized, by $\mathcal{IC}$ the set of unparameterized minimal curves, that can be parameterized by a unique coordinate patch, i.e., $\mathcal{IC}$ is the set of all curves obtained from parameterized curves by forgetting the parameterization. The operator $\mathbf{Wei}$, considered as acting from $\mathbf{A} \times \mathbf{A}$ into $\mathcal{IC}^*$ is a surjection onto $\mathcal{IC}^*$ minus a very small subset of flat isotropic curves. The same is true for its compositions with the map $\mathcal{IC}^* \longrightarrow \mathcal{IC}$, given by forgetting the parameterization. Even more, the operator $\mathbf{Wei} : \mathbf{A} \times \mathbf{A} \longrightarrow \mathcal{IC}^* \setminus \{\Psi \mid \Psi_1' - i\,\Psi_2' \neq 0\}$ is bijective. For later use let us quote the simple formulae for its inverse: $\mathbf{Wei}^{-1}(\Psi) = (f, g) \in \mathbf{A} \times \mathbf{A}$, where

$$f(z) = \varphi_1(z) - i\,\varphi_2(z), \qquad g(z) = \frac{\varphi_3(z)}{\varphi_1(z) - i\,\varphi_2(z)} \tag{2.7}$$

and $\Psi'(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))^\top$.

Minimal surfaces $\mathcal{M}$ in the 3-dimensional Euclidean space $\mathbb{R}^3$ are in one-to-one correspondence with minimal curves $\mathcal{C}$ in the complex Euclidean space $\mathbb{C}^3$ (see [5] or [6]).

To establish this correspondence in the one direction, the easiest way is assigning to a parameterized minimal curve $\Psi(z)$ the parameterized real surface $\mathbf{x}(u, v) = \Re\,(\Psi(u + i\,v))$ — the real part of $\Psi(u + i\,v)$.

If $\Phi = \mathbf{Wei}_{f,g}$, the Gauss map $\mathbf{U} : \mathbb{C} \longrightarrow S^2$ of $\mathbf{x}$ is

$$\mathbf{U} = \frac{1}{|g|^2 + 1}\,\left(2\,\Re(g), 2\,\Im(g), |g|^2 - 1\right)^\top. \tag{2.8}$$

Consequently, if $\mathrm{st} : S^2 \longrightarrow \mathbb{R}^2 = \mathbb{C}$ denotes the stereographic projection, then $g = \mathrm{st} \circ \mathbf{U}$.

Basic geometric quantities of the minimal surface $\mathbf{x}(u, v)$ defined as real part of $\mathbf{Wei}_{f,g}(u + i\,v)$ can be easily expressed by f and g: The coefficients of the first and second fundamental form $\mathbf{I}$ and $\mathbf{II}$ of $\mathbf{x}$ are given by the matrices

$$\mathbf{I} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}, \quad \mathbf{II} = \begin{pmatrix} -\Re\,(f\,g') & \Im\,(f\,g') \\ \Im\,(f\,g') & \Re\,(f\,g') \end{pmatrix}, \tag{2.9}$$

where $\lambda^2 = \dfrac{|f|^2\,\left(1 + |g|^2\right)^2}{4}$. The Gauss curvature k of $\mathbf{x}(u, v)$ is the function

$$k = -\left(\frac{4|g'|}{|f|\,\left(1 + |g^2|\right)^2}\right)^2. \tag{2.10}$$

From (2.9) it is evident that in the case of a parameterized minimal surface $\mathbf{x}(u, v)$, given as real part of $\mathbf{Wei}_f^*(u + i\,v)$, the coordinate lines of $\mathbf{x}(u, v)$ are asymptotic lines and that, in addition to this, $\mathbf{II} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3 Frenet equations for minimal curves

The Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of a curve $\alpha(t)$ in $\mathbb{R}^3$ is an orthonormal frame of vector fields $\mathbf{T}$ (tangent vector), $\mathbf{N}$ (normal vector) and $\mathbf{B}$ (binormal vector) moving along the curve. The linear hull of $\mathbf{T}$ is the tangent line, the linear hull of $\mathbf{T}$, $\mathbf{N}$ is the osculating plane spanned by the vectors $\alpha'(t)$ and $\alpha''(t)$.

For a minimal curve $\Psi(z)$ we can try to find a Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in a similar way. But since $\langle \Psi'(z), \Psi'(z)\rangle = 0$, in order to have orthonormality, we can not choose $\mathbf{T}$ in the linear hull of $\Psi'(z)$. We take $\Psi''(z)$ instead.

Definition 3.1. Let $\Psi(z)$ be a minimal curve such that z is the natural parameter of Ψ and therefore $\langle \Psi''(z), \Psi''(z)\rangle = -1$. The functions

$$\kappa(z) = ||\Psi'''(z)|| \quad \text{and} \quad \tau(z) = \frac{\kappa'(z)}{\kappa(z)} \tag{3.1}$$

are called *the minimal curvature* and *the minimal torsion* of $\Psi(z)$ respectively. We define the vector fields $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ on $\Psi(z)$ by

$$\mathbf{T}(z) = i\,\Psi''(z), \quad \mathbf{N}(z) = \frac{\Psi'''(z)}{\kappa(z)}, \quad \mathbf{B}(z) = \mathbf{T}(z) \times \mathbf{N}(z). \tag{3.2}$$

An other name for the minimal curvature, used for instance in [11], is *Study parameter.*

Theorem 3.1. *The Frenet frame of a minimal curve is orthonormal. For the derivatives of its vectors we have the following analogue of the Frenet formulae.*

$$\left\{ \begin{array}{rcl} \mathbf{T}'(z) & = & i\,\kappa(z)\mathbf{N}(z) \\ \mathbf{N}'(z) & = & -i\,\kappa(z)\mathbf{T}(z) + \qquad -i\,\tau(z)\mathbf{B}(z) \\ \mathbf{B}'(z) & = & i\,\tau(z)\mathbf{N}(z). \end{array} \right. \tag{3.3}$$

Proof: It is clear from the construction that $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ are an orthonormal frame. Consequently, expressing these derivatives $\mathbf{T}'$, $\mathbf{N}'$, $\mathbf{B}'$ as linear combinations of $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, the matrix of coefficients so obtained must be skew symmetric.

Therefore, since the first of the equations (3.3) follows immediately from (3.2), we have only to prove that the last equation of (3.3) is true. But $\langle \mathbf{B}', \mathbf{B}\rangle = 0$ and $\langle \mathbf{B}', \mathbf{T}\rangle = -\langle \mathbf{B}, \mathbf{T}'\rangle = i\,\kappa\,\langle \mathbf{B}, \mathbf{N}\rangle = 0$ imply that $\mathbf{B}' = \langle \mathbf{B}', \mathbf{N}\rangle\,\mathbf{N}$ and, therefore, it is sufficient to show that $\langle \mathbf{B}', \mathbf{N}\rangle = -\langle \mathbf{B}, \mathbf{N}'\rangle = -i\,\kappa'\,\kappa^{-1}$. Now

$$\left\{ \begin{array}{rcl} -\langle \mathbf{B}, \mathbf{N}'\rangle & = & -\left\langle (i\,\Psi'') \times (\kappa^{-1}\,\Psi'''), (\kappa^{-1}\,\Psi''')' \right\rangle \\ & = & -\left\langle (i\,\Psi'') \times (\kappa^{-1}\,\Psi'''), (\kappa^{-1})'\,\Psi''' + \kappa^{-1}\,\Psi^{(4)} \right\rangle \\ & = & -i\,\kappa^{-2}\left\langle \Psi'' \times \Psi''', \Psi^{(4)} \right\rangle. \end{array} \right. \tag{3.4}$$

Next we prove

LEMMA 3.1. *For any minimal curve Ψ in natural parameterization one has*

$$\Psi^{(4)} = \kappa\,\kappa'\,\Psi' + \kappa^2\Psi''. \tag{3.5}$$

Proof: In fact, the equations $\langle\Psi',\Psi'\rangle = 0$, $\langle\Psi'',\Psi''\rangle = -1$ and $\langle\Psi''',\Psi'''\rangle = \kappa^2$ give by successive differentiation $\langle\Psi',\Psi''\rangle = 0$, $\langle\Psi',\Psi'''\rangle = 1$, $\langle\Psi'',\Psi'''\rangle = 0$ and finally

$$\left\langle\Psi^{(4)},\Psi'\right\rangle = 0, \quad \left\langle\Psi^{(4)},\Psi''\right\rangle = -\kappa^2, \quad \left\langle\Psi^{(4)},\Psi'''\right\rangle = \kappa\,\kappa'. \tag{3.6}$$

Taking the scalar product of a linear combination $\Psi^{(4)} = a_1\,\Psi' + a_2\,\Psi'' + a_3\,\Psi'''$ with Ψ', Ψ'', Ψ''' respectively and using the identities just mentioned we obtain $a_3 = 0$, $a_2 = \kappa^2$ and $a_1 = \kappa\,\kappa'$. Inserting (3.5) in (3.4) we get

$$\begin{aligned}
-\langle\mathbf{B},\mathbf{N}'\rangle &= \langle\mathbf{B}',\mathbf{N}\rangle = -i\,\kappa^{-2}\left\langle\Psi''\times\Psi''',\Psi^{(4)}\right\rangle \\
&= -i\,\kappa^{-2}\left\langle\Psi''\times\Psi''',\kappa\,\kappa'\,\Psi' + \kappa^2\Psi''\right\rangle \\
&= -i\,\kappa^{-1}\,\kappa'\,\langle\Psi''\times\Psi''',\Psi'\rangle = -i\,\tau\,\langle\Psi''\times\Psi''',\Psi'\rangle\,.
\end{aligned}$$

Since Ψ has the natural parameterization, we infer from (2.2) (putting $\mathbf{p} = \Psi'''$), that $\langle\Psi''\times\Psi''',\Psi'\rangle = \langle\Psi'\times\Psi'',\Psi'''\rangle = \langle\Psi',\Psi'''\rangle = 1$, therefore, theorem 3.1 is proved. $\square$

The minimal curvature of a minimal curve Φ that is described by one of the representation formulae of above in terms of a number of meromorphic functions can be expressed by these functions and their derivatives. In the case of the natural Weierstrass representation, the minimal curvature of $\Phi = \mathbf{Wei}^*_f$ comes out as the square root of the *Schwarzian derivative* $\mathcal{S}(f)$, namely

$$\kappa(z) = (\mathcal{S}(f))^{1/2} = \frac{\sqrt{3\,f''(z)^2 - 2\,f'(z)\,f^{(3)}(z)}}{f'(z)}. \tag{3.7}$$

We mention the following "chain rule" (see [7]): If $f, g \in \mathbf{A}$, then

$$\mathcal{S}(f\circ g)(z) - \mathcal{S}(g)(z) = (g'(z))^2\;\mathcal{S}(f)(g(z)). \tag{3.8}$$

Let the group $\mathrm{Sl}(2,\mathbb{C})$ act on the algebra $\mathbf{A}$ by linear fractional transformations

$$(\mathrm{g}, f) \in \mathrm{Sl}(2,\mathbb{C})\times\mathbf{A} \;\longrightarrow\; l_{\mathrm{g}}(f) = \frac{a\,f + b}{c\,f + d} \in \mathbf{A}, \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.9}$$

The Schwarzian derivative is an invariant of this group action, i.e., $\mathcal{S}(f) = \mathcal{S}\left(l_{\mathrm{g}}(f)\right)$ for $\mathrm{g} \in \mathrm{Sl}(2,\mathbb{C})$.

The Frenet equations show that it is possible to reconstruct a minimal curve in natural parameterization from its minimal curvature by solving a system of ordinary differential equations. Up to a motion by the complex orthogonal group $\mathrm{SO}(3,\mathbb{C})$, this reconstruction is unique.

By this, we have proved:

THEOREM 3.2. *For any function* $f \in \mathbf{A}$ *and any* $\mathrm{g} \in \mathrm{Sl}(2,\mathbb{C})$ *the derivatives* $\left(\mathbf{Wei}_f^*\right)'$ *and* $\left(\mathbf{Wei}_{l_{\mathrm{g}}(f)}^*\right)'$ *differ only by a transformation with an element of* $\mathrm{SO}(3,\mathbb{C})$□

Equation (3.7) shows a way to reduce the system (3.3) to some integrations and solving a Riccati equation. Namely, under the substitution $y(z) = f''(z)/f'(z)$, (3.7) turns into

$$y'(z) = \left(y(z)^2 - \kappa(z)^2\right)/2. \tag{3.10}$$

Solving this Riccati equation for $y(z)$, integrating twice to obtain $f(z)$ and, finally, applying the operator $\mathbf{Wei}^*$ to f, gives the minimal curve whose moving frame solves the system (3.3).

Let us determine the minimal curves of constant minimal curvature. If $\kappa(z) = a =$ const, the solution of (3.10) is $y(z) = a\tan\left(\dfrac{a\,(z+2\,c_1)}{2}\right)$ and in turn

$$f(z) = \frac{2\,i\,e^{c_2} + a\,c_3 + \exp\left(i\,a\,(z+2\,c_1)\right)\left(-2\,i\,e^{c_2} + a\,c_3\right)}{a\,\left(1+\exp\left(i\,a\,(z+2\,c_1)\right)\right)}.$$

This function belongs to the $\mathrm{Sl}(2,\mathbb{C})$-orbit of the exponential function e^{iaz} as well as $f(z) = \tan(az/2)$ that is one of the most symmetric functions in $\mathbf{A}$ with respect to $\mathrm{Gl}(2,\mathbb{C})$. It has $\mathrm{O}(2,\mathbb{R})$ as 1-dimensional symmetry group, where *symmetry* is understood as follows: If $r \in \mathbb{C}$ and $\mathrm{g} = \begin{pmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{pmatrix}$, then $(\mathrm{g}\cdot f)(z) = f(z + 2\,r/a)$. The minimal curve $\Psi(z)$ associated to f via $\mathbf{Wei}^*$ is

$$\Psi(z) = \left(\sin(2\,a\,z)/2\,a^2, -z/2\,a, -i\cos(a\,z)^2/2\,a^2\right)^\top.$$

All curves corresponding to functions in the orbit of $\exp(a\,z)$ have $\kappa(z) = -a =$ const and $\tau(z) = 0$. Among the minimal surfaces corresponding to the curves of this orbit are the helicoid and the catenoid.

Some power functions give well known surfaces. So for instance, the minimal curves $\left(z - \dfrac{z^{1+2\,n}}{1+2\,n}, -i\left(z + \dfrac{z^{1+2\,n}}{1+2\,n}\right), \dfrac{2\,z^{1+n}}{1+n}\right)^\top$ of the Enneper surfaces are generated by the function $f(z) = -\left(\dfrac{-i}{8\,n}\right)^{\frac{n}{1+n}} (z\,(1+n))^{\frac{2\,n}{1+n}}$. The minimal curvature and torsion of the minimal curve generated by $f(z) = z^n$ are $\kappa(z) = -\sqrt{-1+n^2}/z$ and $\tau(z) = -1/z$ resp.

PROPOSITION 3.1. *If* $\Psi(z)$ *is a minimal curve parameterized by its natural parameter and in general position, the derivatives* $\Psi'(z)$, $\Psi''(z)$ *and* $\Psi''(z)$ *form a (nonorthonormal) moving frame that is expressed as a linear combination of the Frenet frame in the* *following way:*

$$\begin{cases} \Psi'(z) & = & & \dfrac{1}{\kappa(z)}\,\mathbf{N}(z) & +\dfrac{i}{\kappa(z)}\,\mathbf{B}(z), \\ \Psi''(z) & = & -i\,\mathbf{T}(z), & & \\ \Psi'''(z) & = & & \kappa(z)\,\mathbf{N}(z). & \end{cases} \tag{3.11}$$

Proof: The last two equations of (3.11) are consequences of (3.2). For the proof of the first equation, a tentative ansatz

$$\Psi'(z) = a_1(z)\,\mathbf{T}(z) + a_2(z)\,\mathbf{N}(z) + a_3(z)\,\mathbf{B}(z) \tag{3.12}$$

is multiplied with the vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$. As in the proof of Theorem 3.1, differentiating the equation $\langle\Psi', \Psi'\rangle = 0$ gives $a_1 = \langle\Psi', \mathbf{T}\rangle = -i\,\langle\Psi', \Psi''\rangle = 0$. The equation $\langle\Psi'', \Psi''\rangle = -1$ gives $a_2 = \langle\Psi', \mathbf{N}\rangle = \frac{1}{\kappa}\,\langle\Psi', \Psi'''\rangle = -\frac{1}{\kappa}\,\langle\Psi'', \Psi''\rangle = \frac{1}{\kappa}$ and finally, since we are dealing with the natural parameter, by (2.2) $a_3 = \langle\Psi', \mathbf{B}\rangle = \langle\Psi', \mathbf{T}\times\mathbf{N}\rangle = \frac{i}{\kappa}\,(\Psi'\Psi''\Psi''') = \frac{i}{\kappa}\sqrt{-\langle\Psi'', \Psi''\rangle^3} = \frac{i}{\kappa}$. □

As a consequence of Proposition 3.1 and Theorem 3.1, we see, that any derivative $\Psi^{(i)}(z)$ can be expressed as a linear combination of the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ with coefficients that are given in terms of the minimal curvature $\kappa(z)$ of Ψ and its derivatives only. Consequently, the scalar products $\langle\Psi^{(i)}(z), \Psi^{(j)}(z)\rangle$ of the higher-order derivatives of Ψ depend only on the function $\kappa(z)$ and its derivatives. If we confine to the case $i, j \leq 5$, the matrix of these scalar products is the following one:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & \kappa^2(z) \\ 0 & -1 & 0 & -\kappa^2(z) & m_{25} \\ 1 & 0 & \kappa^2(z) & \kappa(z)\,\kappa'(z) & m_{35} \\ 0 & -\kappa^2(z) & \kappa(z)\,\kappa'(z) & -\kappa^4(z) & -2\,\kappa^3(z)\,\kappa'(z) \\ \kappa^2(z) & m_{52} & m_{53} & -2\,\kappa^3(z)\,\kappa'(z) & m_{55} \end{pmatrix}, \tag{3.13}$$

where $m_{25} = m_{52} = -3\,\kappa(z)\,\kappa'(z)$, $m_{35} = m_{53} = \kappa^4(z) + (\kappa'(z))^2 + \kappa(z)\,\kappa''(z)$ and $m_{55} = \kappa^2(z)\,\left(\kappa^4(z) - 7\,(\kappa'(z))^2 + 2\,\kappa(z)\,\kappa''(z)\right)$.

To compute this matrix by hand could be somewhat tedious. However computer algebra systems like *'Mathematica'* solve the task without problems even for higher orders of derivatives.

Let us see, how the minimal curvature of a minimal curve Φ in arbitrary parameterization can be found. Denote by p the natural, by z an arbitrary parameter of Φ and by $\Phi_{,ppp}$ and $\Phi_{,zzz}$ the corresponding derivatives. Then, by the chain rule

$$\Phi_{,zzz} = \Phi_{,ppp}\,p_{,z}^3 + 3\,\Phi_{,pp}\,p_{,z}\,p_{,zz} + \Phi_{,p}\,p_{,zzz}. \tag{3.14}$$

We have already seen that $\langle \Phi_{,p}, \Phi_{,p} \rangle = \langle \Phi_{,p}, \Phi_{,pp} \rangle = \langle \Phi_{,ppp}, \Phi_{,pp} \rangle = 0$, $\langle \Phi_{,pp}, \Phi_{,pp} \rangle = -\langle \Phi_{,p}, \Phi_{,ppp} \rangle = -1$, $\langle \Phi_{,ppp}, \Phi_{,ppp} \rangle = \kappa_\Phi^2$ and so the scalar product of (3.14) with itself becomes

$$\langle \Phi_{,zzz}, \Phi_{,zzz} \rangle = {p'}^6 \, \kappa_\Phi^2 + 2\, {p'}^3 \, p''' - 9\, {p'}^2 \, {p''}^2. \tag{3.15}$$

Therefore, we have proved

PROPOSITION 3.2. *Let $\Phi(z)$ be a minimal curve in arbitrary parameterization, $p(z)$ being the natural parameter of $\Phi(z)$.*

Then the minimal curvature $\kappa_\Phi(z)$ of Φ is

$$\kappa_\Phi^2(z) = \frac{\langle \Phi_{,zzz}(z), \Phi_{,zzz}(z) \rangle + 9\,(p'(z))^2\,(p''(z))^2 - 2\,(p'(z))^3\,p'''(z)}{(p'(z))^6}. \tag{3.16}$$

If $\Phi = \mathbf{Wei}_{f,g}$, then $p'(z) = \sqrt{-i\, f(z)\, g'(z)}$ (see (2.5)) and as a consequence of Proposition 3.2 we get

COROLLARY 1. *The square of the minimal curvature of a minimal curve that is given as $\mathbf{Wei}_{f,g}(z)$ by the Weierstrass representation formula, is the following*

$$\left\{ \begin{array}{rcl} \kappa_{f,g}^2(z) & = & \dfrac{i}{4\, f^3(z)\, {g'}^3(z)} \Big(5\, {f'}^2(z)\, {g'}^2(z) - 4\, f(z)\, {g'}^2(z)\, f''(z) \\ & & +2\, f(z)\, f'(z)\, g'(z)\, g''(z) - 7\, f^2(z)\, g''(z)^2 \\ & & +4\, f^2(z)\, g'(z)\, g^{(3)}(z) \Big). \end{array} \right. \tag{3.17}$$

4 Deformations of minimal curves

We are going to develop some operations on $\mathcal{IC}^*$ that lead to two new representation formulae of minimal curves. Let us begin with a general observation on deformations of minimal curves and surfaces.

PROPOSITION 4.1. *Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{MS}^*$ be two conformally parameterized minimal surfaces defined on some open set $\mathcal{U} \subset \mathbb{R}^2$. Suppose that the spherical maps $\mathbf{U}_1, \mathbf{U}_2 : \mathcal{U} \longrightarrow S^2 \subset \mathbb{R}^3$ of $\mathbf{x}_1, \mathbf{x}_2$ are equal. Then, $\mathbf{x}(u,v) = \mathbf{x}_1(u,v) + \mathbf{x}_2(u,v) : \mathcal{U} \longrightarrow \mathbb{R}^3$ is another conformally parameterized minimal surface.*

Under similar assumptions, parameterized minimal curves can be added to obtain new minimal curves.

Proof: The two surfaces $\mathbf{x}_1$ and $\mathbf{x}_2$ are the real parts of two minimal curves $\Phi_1(z)$ and $\Phi_2(z)$. These can be represented by the Weierstrass formula as $\Phi_1(z) = \mathbf{Wei}_{f_1,g_1}(z)$ and $\Phi_2(z) = \mathbf{Wei}_{f_2,g_2}(z)$ with certain functions $f_1, g_1, f_2, g_2 \in \mathbf{A}$ that are uniquely determined by Φ_1 and Φ_2. The spherical maps $\mathbf{U}_1$ and $\mathbf{U}_2$ are the unit normal vectors and by (2.8), they are uniquely determined by $g_1(z)$ and $g_2(z)$ respectively. In fact (see[5]), their compositions with the stereographic projection of the two-dimensional sphere S^2 into $\mathbb{R}^2 = \mathbb{C}$ are g_1 and g_2 respectively. Since $\mathbf{U}_1$ and $\mathbf{U}_2$ agree, $g_1(z) \equiv g_2(z)$. Since the Weierstrass formula (2.4) is linear in $f(z)$, putting $g_1(z) = g_2(z) = g(z)$, we obtain $\Phi_1(z) + \Phi_2(z) = \mathbf{Wei}_{f_1,g}(z) + \mathbf{Wei}_{f_2,g}(z) = \mathbf{Wei}_{f_1+f_2,g}(z)$. This implies that, up to some constants of integration, $\mathbf{x}(u,v) = \mathbf{x}_1(u,v) + \mathbf{x}_2(u,v) = \Re\left(\mathbf{Wei}_{f_1+f_2,g}(x+i\,y)\right)$ and consequently, $\mathbf{x}$ is a conformal minimal patch. □

Suppose that $\Phi(z)$ is a nonplanar minimal curve and z the natural parameter on $\Phi(z)$. Then, $\Phi'(z)$, $\Phi''(z)$ and $\Phi^{(3)}(z)$ are linearly independent and any other curve $\Delta(z)$ in $\mathbb{C}^3$ can be represented as a linear combination

$$\Delta(z) = v_1(z)\Phi'(z) + v_2(z)\Phi''(z) + v_3(z)\Phi^{(3)}(z) \tag{4.1}$$

with certain functions $v_1, v_2, v_3 \in \mathbf{A}$. We want to determine them so that the two new curves $\Delta(z)$ and $\Psi(z) = \Phi(z) + \Delta(z)$ are again minimal. At first we observe, that Δ' and Ψ' have equal length if and only if $v_2 + v_3' = 0$. In fact, Lemma 3.1 or the matrix (3.13) show that

$$\begin{aligned} |\Psi'|^2 - |\Delta'|^2 &= 2\left\langle \Phi', \Delta' \right\rangle \\ &= 2\left\langle \Phi', v_1\Phi'' + v_2\Phi''' + v_3\Phi^{(4)} + v_1'\Phi' + v_2'\Phi'' + v_3'\Phi''' \right\rangle \\ &= 2\,v_2\left\langle \Phi', \Phi''' \right\rangle + 2\,v_3\left\langle \Phi', \kappa_\Phi\kappa_\Phi'\Phi' + \kappa_\Phi^2\Phi'' \right\rangle + 2\,v_3'\left\langle \Phi', \Phi''' \right\rangle \\ &= 2\,v_2 + 2\,v_3', \end{aligned} \tag{4.2}$$

i.e., $\Delta(z)$ and $\Psi(z)$ have equal length if and only if, $v_2 = -v_3'$. Next, we put $\Delta = v_1\,\Phi' - v_3'\,\Phi'' + v_3\,\Phi'''$ and obtain with Lemma 3.1

$$\begin{aligned} \Delta' &= v_1'\,\Phi' + v_1\,\Phi'' - v_3''\,\Phi'' + v_3\,\left(\kappa_\Phi\kappa_\Phi'\Phi' + \kappa_\Phi^2\Phi''\right) \\ &= \left(v_1' + v_3\,\kappa_\Phi\,\kappa_\Phi'\right)\Phi' + \left(v_1 - v_3'' + v_3\,\kappa_\Phi^2\right)\Phi''. \end{aligned} \tag{4.3}$$

This yields

$$\left\langle \Delta', \Delta' \right\rangle = -\left(v_1 - v_3'' + v_3\,\kappa_\Phi^2\right)^2. \tag{4.4}$$

Therefore, the general solution to the system of ordinary differential equations $\langle \Psi', \Psi' \rangle = 0$, $\langle \Delta', \Delta' \rangle = 0$ is obtained by choosing $v_3(z) = h(z)$ as an arbitrary meromorphic function and putting $v_2(z) = -h'(z)$ and $v_1(z) = h(z)\,\kappa_\Phi^2(z) - h''(z)$. In this way, two operators $\Delta_{\Phi,h}$ and $\Psi_{\Phi,h}$ are obtained that assign to a minimal curve Φ and a meromorphic function h new minimal curves. They are given by

$$\Delta_{\Phi,h}(z) = \left(-h(z)\,\kappa_\Phi^2(z) + h''(z)\right)\Phi'(z) - h'(z)\Phi''(z) + h(z)\Phi^{(3)}(z). \tag{4.5}$$

and

$$\Psi_{\Phi,h}(z) = \Phi(z) + \Delta_{\Phi,h}(z). \tag{4.6}$$

It is immediately seen from definition (4.5), that for fixed Φ the mapping $h \in \mathbf{A} \longrightarrow \Delta_{\Phi,h} \in \mathcal{IC}^*$ is $\mathbb{C}$-linear. The following proposition lists some other properties of these operators.

Proposition 4.2. *For any minimal curve $\Phi(z)$, parameterized by its natural parameter, and for any meromorphic function $h(z)$, the derivatives $p_\Delta(z)$ and $p_\Psi(z)$ of the natural parameters of $\Delta_{\Phi,h}$ and $\Psi_{\Phi,h}(z)$ respectively are given by*

$$\left\{\begin{array}{lcl} p'_\Delta(z) & = & \sqrt{-\kappa_\Phi(z)\,\kappa'_\Phi(z)\,h(z) - \kappa^2_\Phi(z)\,h'(z) + h^{(3)}(z)}, \\ p'_\Psi(z) & = & \sqrt{1 - \kappa_\Phi(z)\,\kappa'_\Phi(z)\,h(z) - \kappa_\Phi(z)^2\,h'(z) + h^{(3)}(z)}, \end{array}\right. \tag{4.7}$$

where κ_Φ is the minimal curvature of Φ. We have in particular $(p'_\Psi(z))^2 - (p'_\Delta(z))^2 = 1$.

The derivatives $\Delta'_{\Phi,h}(z)$ and $\Psi'_{\Phi,h}(z)$ both are multiples of $\Phi'(z)$, namely, we have

$$\left\{\begin{array}{l} \Delta'_{\Phi,h}(z) = (p'_\Delta(z))^2\ \Phi'(z), \\ \Psi'_{\Phi,h}(z) = (p'_\Psi(z))^2\ \Phi'(z). \end{array}\right. \tag{4.8}$$

If $f(z)$ is a meromorphic function with $\Phi(z) = \mathbf{Wei}^*_f(z)$ then functions f_Ψ and g_Ψ with $\Psi_{\Phi,h}(z) = \mathbf{Wei}_{f_\Psi,g_\Psi}(z)$ and functions $f_\Delta(z)$ and $g_\Delta(z)$ with $\Delta_{\Phi,h}(z) = \mathbf{Wei}_{f_\Delta,g_\Delta}(z)$ are given by

$$f_\Psi(z) = i\,\frac{(p'_\Psi(z))^2}{f'(z)}, \quad f_\Delta(z) = i\,\frac{(p'_\Delta(z))^2}{f'(z)}, \quad g_\Psi(z) = g_\Delta(z) = f(z). \tag{4.9}$$

Proof: Differentiating (4.5) and using (3.5) we get

$$\left\{\begin{array}{lcl} \Delta'_{\Phi,h} & = & \left(-h'\kappa^2 - 2\,h\,\kappa\,\kappa' + h'''\right)\,\Phi' + \left(-h\,k^2 + h''\right)\,\Phi'' \\ & & -h''\,\Phi'' - h'\,\Phi''' + h'\,\Phi''' + h\,\left(\kappa\,\kappa'\,\Phi' + \kappa^2\,\Phi''\right) \\ & = & \left(-h'\,\kappa^2 - h\,\kappa\,\kappa' + h'''\right)\,\Phi'. \end{array}\right. \tag{4.10}$$

This gives the proof of the first of the equations (4.8). The second follows in the same way. Assuming that $\Phi = \mathbf{Wei}_{f,g}$, the inversion formulae (2.7) together with (4.8) give immediately (4.9). Finally, since we know the Weierstrass functions of Δ and Ψ, we can compute the natural parameters of these curves with (2.5), obtaining the expressions (4.7). □

Definition 4.1. We define the operators $\mathbf{Var} : \mathbf{A}\times\mathbf{A} \longrightarrow \mathcal{C}^*$ and $\mathbf{Def} : \mathbf{A}\times\mathbf{A} \longrightarrow \mathcal{C}^*$ by

$$\mathbf{Var}_{f,h} = \Delta_{\mathbf{Wei}^*_f,\,h} \text{ and } \mathbf{Def}_{f,h} = \Psi_{\mathbf{Wei}^*_f,\,h}. \tag{4.11}$$

The components Φ_1, Φ_2, Φ_3 of the curve $\Phi = \mathbf{Var}_{f,h}$ are

$$\left\{\begin{array}{rcl}
\Phi_1 &=& -h'\left(-i\,f + \dfrac{\frac{i}{2}\,(f^2-1)\,f''}{f'^2}\right) \\
&& +h\left(-i\,f' + \dfrac{i\,f\,f''}{f'} - \dfrac{i\,(f^2-1)\,f''^2}{f'^3} + \dfrac{\frac{i}{2}\,(f^2-1)\,f^{(3)}}{f'^2}\right) \\
&& -\dfrac{\frac{i}{2}\,(f^2-1)\,\left(-3\,h\,f''^2 + f'^2\,h'' + 2\,h\,f'\,f^{(3)}\right)}{f'^3}, \\
\Phi_2 &=& -h'\left(\dfrac{(f^2+1)\,f''}{2\,f'^2} - f\right) \\
&& +h\left(-f' + \dfrac{f\,f''}{f'} - \dfrac{(f^2+1)\,f''^2}{f'^3} + \dfrac{(f^2+1)\,f^{(3)}}{2\,f'^2}\right) \\
&& +\dfrac{(f^2+1)\,\left(3\,h\,f''^2 - f'^2\,h'' - 2\,h\,f'\,f^{(3)}\right)}{2\,f'^3}, \\
\Phi_3 &=& \dfrac{i}{f'^3}\Big(-f'^3\,h' - h\,f'^2\,f'' + f\,f'\,h'\,f'' - f\,h\,f''^2 \\
&& +f\,f'^2\,h'' + f\,h\,f'\,f^{(3)}\Big).
\end{array}\right. \tag{4.12}$$

The expression (4.7) for $p'_\Delta(z)$ will be of importance for studying the behaviour of **Var** under transformations of the independent variable.

If $\Phi(z) = \mathbf{Wei}^*_f(z)$, its explicit form is given by

$$\left\{\begin{array}{rcl}
(p'_\Delta(z))^2 &=& \dfrac{-3\,h'(z)\,f''(z)^2}{f'(z)^2} + \dfrac{3\,h(z)\,f''(z)^3}{f'(z)^3} + \dfrac{2\,h'(z)\,f^{(3)}(z)}{f'(z)} \\
&& -\dfrac{4\,h(z)\,f''(z)\,f^{(3)}(z)}{f'(z)^2} + h^{(3)}(z) + \dfrac{h(z)\,f^{(4)}(z)}{f'(z)}.
\end{array}\right. \tag{4.13}$$

Definition 4.2. We define *the differential operator* $\mathcal{D}(f,h)$ as the right-hand side of the expression (4.13).

$\mathcal{D}$ is obviously linear in h. Less obvious are the following properties.

Proposition 4.3. *Let $\mathcal{D}(f,h)$ be the operator given by the right-hand side of (4.13). Then, for arbitrary meromorphic functions $f, h, t \in \mathbf{A}$ we have*

$$\mathcal{D}\left(f\circ t, \frac{h\circ t}{t'}\right) = t'^2\left(\mathcal{D}(f,h)\circ t\right). \tag{4.14}$$

The kernel $\mathrm{Ker}\mathcal{D}(f,.)$ *is spanned by the functions*

$$u_1(z) = \frac{1}{f'(z)}, \quad u_2(z) = \frac{f(z)}{f'(z)}, \quad u_3(z) = \frac{f^2(z)}{f'(z)}. \tag{4.15}$$

Proof: From (3.7) and (4.7) we infer that

$$\mathcal{D}(f,h) = -\tfrac{1}{2}\mathcal{S}(f)'\,h - \mathcal{S}(f)\,h' + h^{(3)}. \tag{4.16}$$

Putting $h_1 = (h\circ t)/t'$ and $f_1 = f\circ t$ we get from (3.8)

$$\left\{\begin{array}{rcl} \mathcal{D}(f_1,h_1) & = & -\tfrac{1}{2}\left({t'}^2\,(\mathcal{S}(f)\circ t) + \mathcal{S}(t)\right)'\,h_1 \\ & & -\left({t'}^2\,(\mathcal{S}(f)\circ t) + \mathcal{S}(t)\right)\,h_1' + h_1^{(3)}. \end{array}\right. \tag{4.17}$$

A separate computation gives

$$h_1^{(3)} - \mathcal{S}(t)'\,h_1/2 - \mathcal{S}(t)\,h_1' = {t'}^2\,\left(h^{(3)}\circ t\right). \tag{4.18}$$

By this, (4.17) becomes

$$\left\{\begin{array}{rcl} \mathcal{D}(f_1,h_1) & = & -\dfrac{1}{2}\left(2\,t'\,t''\,(\mathcal{S}(f)\circ t) + {t'}^2\,(\mathcal{S}(f)'\circ t)\,t'\right)\,\dfrac{h\circ t}{t'} \\ & & -{t'}^2\,(\mathcal{S}(f)\circ t)\left(h'\circ t - \dfrac{(h\circ t)\,t''}{{t'}^2}\right) + {t'}^2\,(h^{(3)}\circ t) \\ & = & {t'}^2\left(\left(-\dfrac{1}{2}\mathcal{S}(f)'\,h - \mathcal{S}(f)\,h' + h^{(3)}\right)\circ t\right) \\ & = & {t'}^2\,(\mathcal{D}(f,h)\circ t)\,. \end{array}\right. \tag{4.19}$$

Therefore (4.14) is proved. In order to see that the functions (4.15) are a fundamental system of the ordinary linear differential equation $\mathcal{D}(f,u) = 0$ one could try to verify the equations $\mathcal{D}\,(f,u_1) = \mathcal{D}\,(f,u_2) = \mathcal{D}\,(f,u_3) = 0$ directly. But a look at Lemma 3.1 tells us that $\mathcal{D}(f,\varphi_i) = 0$ for each of the three components φ_i, $(i = 1,2,3)$ of the derivative of the minimal curve $\Phi = \mathbf{Wei}_f^*$. Since the functions φ_i are linear combinations of u_1, u_2, u_3 with a regular matrix of coefficients we see that $\mathcal{D}\,(f,u_i) = 0$. Moreover, one shows that the Wronskian of u_1, u_2, u_3 is equal to 2. □

We observe that (4.18) can be put into the form

$$\mathcal{D}\left(t, \frac{h\circ t}{t'}\right) = {t'}^2\,\left(h^{(3)}\circ t\right). \tag{4.20}$$

As a consequence of Propositions 4.2 and 4.3 we have

Proposition 4.4. *For any fixed $f \in \mathbf{A}$ the mapping $h \in \mathbf{A} \longrightarrow \dfrac{\mathbf{dVar}_{f,h}}{\mathbf{d}z}$ is $\mathbb{C}$-linear and its kernel is spanned by the functions* (4.15).

One can show even more, namely that **Var** is a bijection of $\mathbf{A}\times\mathbf{A}$ onto the set of minimal curves for which the inversion formula (2.8) works, i.e., for minimal curves $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ such that $\Delta_1' - i\,\Delta_2' \neq 0$. If $\Delta(z) = \mathbf{Var}_{f,h}(z)$ then the first equation of (4.8) shows, that $\Delta' = \left(p_\Delta'(z)\right)^2\,\mathbf{Wei}_f^* = \mathcal{D}(f,h)\,\mathbf{Wei}_f^*$. Therefore, applying the second formula of (2.8) to Δ we obtain the function f from Δ. Having determined f, consider the definition (4.5) of $\Delta_{\Phi,h}$ with $\mathbf{Wei}^*_f$ in place of Φ. The scalar product of the left-hand side of this with $\dfrac{\mathbf{dWei}^*_f(z)}{\mathbf{d}z}$ yields $h(z)$.

Therefore an elementary inversion formula for the operator **Var** is proved:

PROPOSITION 4.5. *For a minimal curve $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ functions f and h such that $\Delta = \mathbf{Var}_{f,h}$ are given by*

$$f(z) = \frac{\Delta_3'(z)}{\Delta_1'(z) - i\,\Delta_2'(z)} \text{ and } h(z) = \left\langle \Delta(z), \frac{\mathbf{dWei}^*_f(z)}{\mathbf{d}z} \right\rangle. \tag{4.21}$$

In Proposition 4.2 it was shown how the Weierstrass data of a minimal curve are obtained from its variation data. With the help of Proposition 4.5 we can solve the inverse task.

PROPOSITION 4.6. *If $\Phi = \mathbf{Wei}_{f,g}$ then $\Phi = \mathbf{Var}_{f_1,h}$, where $f_1(z) = g(z)$ and*

$$\begin{cases} h(z) = \dfrac{i}{4\,g'(z)}\Bigg(4\,g(z) \displaystyle\int f(\zeta)\,g(\zeta)\,\mathbf{d}\zeta - \int f(\zeta)\,\left(-1 + g^2(\zeta)\right)\,\mathbf{d}\zeta \\[2ex] \qquad - 2\,g^2(z) \displaystyle\int f(\zeta)\,\mathbf{d}\zeta - \int f(\zeta)\,\left(1 + g^2(\zeta)\right)\,\mathbf{d}\zeta\Bigg). \end{cases} \tag{4.22}$$

A more general variation operator starting from the Weierstrass formula and depending on three function f, g, h can be established by computing the variation of the minimal curve $\Phi = \mathbf{Wei}_{f,g}$. The natural parameter and the minimal curvature of Φ are given by (2.5) and (3.17) respectively. Then (4.5) can be used to determine with the chain rule of order 3 a highly involved expression for the variation $\Delta_{\Phi,h}$ of Φ. With the help of Proposition 4.5, this expression can be represented by the operator $\mathbf{Var}_{f_1,h_1}$. The result is comparatively simple. We used a computer to obtain

PROPOSITION 4.7. *Under the assumptions of above we have $\Delta_{\Phi,h} = \mathbf{Var}_{f_1,h_1}$ with*

$$f_1 = g \text{ and } h_1 = \frac{i\,h\,\sqrt{-i\,f\,g'}}{f\,g'}. \tag{4.23}$$

For the last application of Proposition 4.5 let us consider the question of iterated variations, i.e., given functions f, h_1 and h_2, determine the function h_3 so that

$$\Delta_{\mathbf{Var}_{f,h_1},\,h_2} = \mathbf{Var}_{f,h_3}. \tag{4.24}$$

Since by Proposition 4.2 $\mathbf{Var}_{f,h_1} = \mathbf{Wei}_{f_0,g_0}$ where $f_0 = \dfrac{\mathcal{D}(f,h_1)}{f'(z)}$ and $g_0 = f$ the last results can be used to obtain

$$h_3 = \frac{h_2}{\sqrt{-\mathcal{D}(f,h_1)}}. \tag{4.25}$$

5 Transformations and the variation operator

We start this section with explaining a group homomorphism μ induced by an algebraic version of the Weierstrass formula and prove the equivariance of the variation operator with respect to μ. After this, we prove that the variation operator can be globalized to a nonlinear differential operator $\widetilde{\mathbf{Var}}_{f,\mathbf{h}}$ assigning to a meromorphic function f and a holomorphic vector field $\mathbf{h}$ on a Riemann surface Σ a minimal map Φ of Σ in $\mathbb{C}^3$, i. e., a meromorphic map with $\langle \mathbf{d}\Phi, \mathbf{d}\Phi\rangle = 0$.

Let $\mathcal{I} \subset \mathbb{C}^3$ denote the isotropic cone of $\mathbb{C}^3$. We define the Weierstrass map $\mathbf{Wei}_\star :$ $\mathbb{C} \times \mathbb{C} \longrightarrow \mathcal{I}$ by $\mathbf{Wei}_\star(z, w) = \frac{1}{2} z \left(1 - w^2, i\,(1 + w^2), 2\,w\right)^\top$. $\mathcal{I}$ is a homogeneous $\mathrm{CO}(3, \mathbb{C})$-space.

Looking for the transformations of $\mathbb{C} \times \mathbb{C}$ induced by pulling back the $\mathrm{CO}(3, \mathbb{C})$-transformations with $\mathbf{Wei}_\star$, one finds that $\mathbf{Wei}_\star$ induces the group homomorphism $\mu : \mathrm{Gl}(2, \mathbb{C}) \longrightarrow \mathrm{SO}(3, \mathbb{C})$ given by

$$\left\{\begin{array}{l} \mu : \mathrm{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}(2, \mathbb{C}) \longrightarrow \mu(\mathrm{g}) = \dfrac{1}{2(ad - bc)} \times \\[2ex] \begin{pmatrix} a^2 - b^2 - c^2 + d^2 & i\,\left(a^2 + b^2 - c^2 - d^2\right) & 2\,(c\,d - a\,b) \\ -i\,\left(a^2 - b^2 + c^2 - d^2\right) & a^2 + b^2 + c^2 + d^2 & 2\,i\,(a\,b + c\,d) \\ 2\,(b\,d - a\,c) & -2\,i\,(a\,c + b\,d) & 2\,(b\,c + a\,d) \end{pmatrix} . \end{array}\right. \tag{5.1}$$

The homomorphism μ is the one that makes the Weierstrass map equivariant in the following way: prolong $\mathbb{C}$ to the Riemann sphere S^2 and consider the usual action (3.9) of $\mathrm{Gl}(2, \mathbb{C})$ on S^2 by linear fractional transformations and extend it to an action of $\mathrm{Gl}(2, \mathbb{C})$ on $\mathbb{C} \times \mathrm{S}^2$ by

$$\left\{\begin{array}{r} (\mathrm{g}, (z, w)) \in \mathrm{Gl}(2, \mathbb{C}) \times \mathbb{C} \times \mathrm{S}^2 \longrightarrow \\ l_{\mathrm{g}}(z, w) = \left(\dfrac{(c\,w + d)^2}{a\,d - b\,c}\,z, \dfrac{a\,w + b}{c\,w + d}\right) \in \mathbb{C} \times \mathrm{S}^2. \end{array}\right. \tag{5.2}$$

Proposition 5.1. *The mapping* (5.2) *defines a group action and the mapping* μ *is a group homomorphism. Both commute with the algebraic Weierstrass map in the sense that for all* $\mathrm{g} \in \mathrm{Gl}(2, \mathbb{C})$ *and* $(z, w) \in \mathbb{C} \times \mathrm{S}^2$ *such that* $\mathbf{Wei}_\star(z, w)$ *and* $\mathbf{Wei}_\star \circ l_{\mathrm{g}}(z, w)$ *are defined, we have*

$$\mu(\mathrm{g}) \circ \mathbf{Wei}_\star(z, w) = \mathbf{Wei}_\star \circ l_{\mathrm{g}}(z, w). \tag{5.3}$$

The proof is omitted. But a consequence of this equivariance is the equivariance of the variation formula. We are using the notation l_{g} not only for the action (5.2) but also for the usual action (3.9) of $\mathrm{Gl}(2, \mathbb{C})$ on S^2 and also for the following action of $\mathrm{Gl}(2, \mathbb{C})$ on $\mathbf{A} \times \mathbf{A}$:

$$l_{\mathrm{g}} : (f, h) \in \mathbf{A} \times \mathbf{A} \longrightarrow (l_{\mathrm{g}}(f), \det(\mathrm{g})\,h) \in \mathbf{A} \times \mathbf{A}, \tag{5.4}$$

where $\mathrm{g} \in \mathrm{Gl}(2,\mathbb{C})$.

We extend the homomorphism μ by

$$\hat{\mu} : \mathrm{g} \in \mathrm{Gl}(2,\mathbb{C}) \longrightarrow \det(\mathrm{g})\,\mu(\mathrm{g}) \in \mathrm{CO}(3,\mathbb{C}) \tag{5.5}$$

to a homomorphism into the complex conformal group. μ and $\hat{\mu}$ agree on $\mathrm{Sl}(2,\mathbb{C})$.

PROPOSITION 5.2. *Let* $\mathrm{CO}(3,\mathbb{C})$ *act on the space* $\mathcal{IC}^*$ *in the usual way and let* $\mathrm{Gl}(2,\mathbb{C})$ *act on* $\mathbf{A}\times\mathbf{A}$ *by* (5.4). *Then the variation operator is equivariant with respect to these two actions, i.e.,*

$$\hat{\mu}(\mathrm{g}) \circ \mathbf{Var}_{f,h} = \mathbf{Var}_{l_{\mathrm{g}}(f,h)} \text{ for } \mathrm{g} \in \mathrm{Gl}(2,\mathbb{C}). \tag{5.6}$$

Proof: By the definition (4.5) and by the linearity of **Var** in its second argument, it is sufficient to show, that the derivative of the natural Weierstrass representation, $f \in \mathbf{A} \longrightarrow \left(\mathbf{Wei}^*_f\right)'$, is equivariant with respect to μ.

For this purpose we observe at first that the mapping $q : f \in \mathbf{A} \longrightarrow \left(\frac{i}{f'}, f\right) \in \mathbf{A}\times\mathbf{A}$ is equivariant, namely we have $l_{\mathrm{g}} \circ q(f)(z) = q\,(l_{\mathrm{g}} f)\,(z)$ for all $f \in \mathbf{A}$. The operators $(\mathbf{Wei}^*)'$ and $\mathbf{Wei}_\star$ are related by

$$\left(\mathbf{Wei}^*_f\right)'(z) = \mathbf{Wei}_\star\left(\frac{i}{f'(z)}, f(z)\right) = \left(\mathbf{Wei}_\star \circ q(f)\right)(z). \tag{5.7}$$

Therefore, we infer from (5.3) that

$$\mu(\mathrm{g}) \circ \left(\mathbf{Wei}^*_f\right)' = \mathbf{Wei}_\star \circ l_{\mathrm{g}}\left(\frac{i}{f'}, f\right) = \mathbf{Wei}_\star\left(\frac{i}{(l_{\mathrm{g}} f)'}, l_{\mathrm{g}} f\right) = \left(\mathbf{Wei}^*_{l_{\mathrm{g}} f}\right)'. \tag{5.8}$$

i.e., we have $\mu(\mathrm{g}) \circ (\mathbf{Wei}^*)' = (\mathbf{Wei}^*)' \circ l_{\mathrm{g}}$. □

Next let us look for transformations $t : \mathbb{C} \longrightarrow \mathbb{C}$ of the independent variable. At first, for the operator **Wei** it is easy to see, that

$$\mathbf{Wei}_{f,g} \circ t = \mathbf{Wei}_{t'(f\circ t), g\circ t}. \tag{5.9}$$

From this property and from Proposition 4.3 we infer that

PROPOSITION 5.3. *If* $t, f, h \in \mathbf{A}$ *and*

$$f_1 = f \circ t \text{ and } h_1 = \frac{h \circ t}{t'} \tag{5.10}$$

then

$$\mathbf{Var}_{f,h} \circ t = \mathbf{Var}_{f_1,h_1}. \tag{5.11}$$

Proof: For the moment let us write $\mathbf{Wei}(f,g)$ and $\mathbf{Var}(f,h)$ instead of $\mathbf{Wei}_{f,g}$ and $\mathbf{Var}_{f,h}$. By Proposition 4.2 (see the Weierstrass functions (4.9) of $\Delta_{\Phi,h}$) we have

$$\mathbf{Var}\,(f,h) = \mathbf{Wei}\left(-i\,\frac{\mathcal{D}(f,h)}{f'}, f\right). \tag{5.12}$$

We infer from (4.14) and (5.9)

$$\left\{\begin{array}{lcl} \mathbf{Var}\,(f_1,h_1) & = & \mathbf{Wei}\left(-i\,\dfrac{t'^2\,\left(\mathcal{D}(f,h)\circ t\right)}{t'\,(f'\circ t)}, f\circ t\right) \\ & = & \mathbf{Wei}\left(-i\,t'\,\left(\left(\dfrac{\mathcal{D}(f,h)}{f'}\right)\circ t\right), f\circ t\right) \\ & = & \mathbf{Wei}\left(-i\,\dfrac{\mathcal{D}(f,h)}{f'}, f\right)\circ t = \mathbf{Var}\,(f,h)\circ t. \end{array}\right. \tag{5.13}$$

This completes the proof of (5.11). □

From Proposition 5.3 we get a global version $\widetilde{\mathbf{Var}}$ of the operator $\mathbf{Var}$. Let Σ be a Riemann surface and $\mathcal{A} = (\mathcal{V}_i, \phi_i)_{i\in I}$ an atlas on Σ, i.e., each ϕ_i is a biholomorphic mapping of some open subset $\mathcal{V}_i \subset \Sigma$ onto an open subset $\mathcal{U}_i \subset \mathbb{C}$. Let $C^\omega(\Sigma)$ denote the algebra of holomorphic functions, $C^\omega(T\Sigma)$ the space of holomorphic vector fields on Σ and $C^\omega_*(\Sigma) \subset C^\omega(\Sigma)$ the subset of functions f with nowhere vanishing differential $\mathbf{d}f$.

For $f \in C^\omega_*(\Sigma)$, $\mathbf{h} \in C^\omega(T\Sigma)$ and $(\mathcal{V}_i, \phi_i) \in \mathcal{A}$ let f_i, h_i be the local expressions of $f, \mathbf{h}$ with respect to $(\mathcal{V}_i, \phi_i)$. Then we define

$$\widetilde{\mathbf{Var}}_{f,\mathbf{h}}(p) = \mathbf{Var}_{f_i,h_i}(\phi_i(p)) \text{ for } p \in \Sigma. \tag{5.14}$$

By Proposition 5.3 this definition is independent of the choice of a chart. Moreover, the equivariance property of Proposition 5.2 holds for $\widetilde{\mathbf{Var}}$ in the same way. $\widetilde{\mathbf{Var}}_{f,\mathbf{h}}$ is linear in $\mathbf{h}$. In analogy to (4.15) the kernel of the differential of $\widetilde{\mathbf{Var}}_{f,\cdot}$ is spanned by the vector fields

$$\mathbf{h}_1 = \theta_f(\mathbf{d}f), \quad \mathbf{h}_2 = f\,\theta_f(\mathbf{d}f), \quad \mathbf{h}_3 = f^2\,\theta_f(\mathbf{d}f), \tag{5.15}$$

where $\theta_f : T^*\Sigma \longrightarrow T\Sigma$ is the isomorphism of the cotangent onto the tangent bundle of Σ given by $\mathbf{d}f$. Note that $\mathbf{d}f$ is a basis of $T^*\Sigma$.

Is it possible to deform any minimal curve into any other minimal curve?

Proposition 4.2 and the transformation formula (5.9) give the following answer to this question: let $\Phi = \mathbf{Wei}^*_f$ and $\Psi = \mathbf{Wei}_{f_1,g_1}$ be two minimal curves. Suppose that g_1 has an inverse function and put $t = g_1^{-1}\circ f$. By (5.9) $\Psi\circ t = \mathbf{Wei}_{t'(f_1\circ t),f}$. A variation of Φ with a function $h \in \mathbf{A}$ is given by (5.12). Therefore, if h satisfies the differential equation $-i\,\mathcal{D}(f,h)/f' = t'\,(f_1\circ t)$, then $\Psi\circ t = \Delta_{\Phi,h} = \mathbf{Var}_{f,h}$. As we have seen, this equation can be solved by integrations. For example, the minimal curve of the catenoid is $\Phi(z) = (\cos(z), \sin(z), -i\,z)^\top$ and its variation

$$\begin{array}{rcl} \Delta_{\Phi,h}(z) & = & (\cos(z)\,h'(z) - \sin(z)\,h''(z), \\ & & \sin(z)\,h'(z) + \cos(z)\,h''(z), -i\,\left(h(z) + h''(z)\right))^\top . \end{array}$$

Locally, for every minimal curve there exist a reparameterization and a function h such that this reparameterization is given by $\Delta_{\Phi,h}(z)$.

REFERENCES

[1] Robert D. M. Accola: *Topics in the Theory of Riemann Surfaces,* Lecture Notes in Mathematics No. 1595, Springer Verlag 1994.

[2] L. V. Ahlfors und L. Sario: *Riemann Surfaces,* Princeton University Press, Princeton, NJ, 1960.

[3] L. Bianchi: *Lezioni di Geometria Differenziale,* Terza edizione interamente rifatta, Nicola Zanichelli, Bologna 1927. German Translation by M. Lukat: *Vorlesungen über Differentialgeometrie,* Teubner, Leipzig 1899.

[4] W. Blaschke: *Vorlesungen über Differentialgeometrie. Vol. I: Elementare Differentialgeometrie,* 4-th edition, Springer, Berlin, 1945.

[5] U. Dierkes, S. Hildebrandt, A. Küster and O. Wohlrab: *Minimal Surfaces,* Vol. 1, Springer, Berlin-New York 1992.

[6] A. Gray: *Modern Differential Geometry of Curves and Surfaces,* CRC Press, 1995, 2-nd edition.

[7] A. R. Forsyth: *A Treatise on Differential Equations,* London 1956.

[8] H. Gollek: *Natural Equations of minimal surfaces and the Weierstrass-representation formula,* (unpublished tutorial notes, containing *Mathematica* programs, examples and graphics, Berlin 1995).

[9] H. Karcher: *Embedded minimal Surfaces derived from Scherk's Examples, Manuscripta Math.* **62** (1988), 83–114.

[10] H. Karcher: *The triply periodic minimal Surfaces of Alan Schoen and their constant mean Curvature Companions, Manuscripta Math.* **64** (1989), 291–357.

[11] H. Pabel: *Deformationen von Minimalflächen, Geometrie und ihre Anwendungen,* Carl Hanser Verlag, München, Wien 1994.

[12] H. A. Schwarz: *Zur Theorie der Minimalflächen, deren Begrenzung aus geradlinigen Strecken besteht, Sitz.-Ber. Königl. Preuss. Akad. Wiss Berlin, Phys. Math. Cl.* (1894), 1237–1266.

[13] G. Springer: *An Introduction to Riemann Surfaces,* Chelsea Publishing Co., New York 1981.

[14] E. Study: *Zur Differentialgeometrie der analytischen Kurven, Trans. Amer. Math. Soc.* **10** (1909), 1–49; (see also *Am. J. Math.* **32** (1910), 257–263, *Am. J. Math.* **32** (1910), 264–278).

[15] E. Vessiot: *Comptes Rendus* **140** (1905), 1381–1384.