Gallery of pseudospherical surfaces

A. Ovchinninkov

3-6 Batyreva str., Podolsk, 142103 Moscow, Russia
e-mail: popov@ap.phys.msu.su

1 Introduction

Pseudospherical surfaces are known to play a significant role in the investigations of nonlinear processes. The aim of this lecture is to give a brief review of the theory of pseudospherical surfaces.

The investigations of pseudospherical surfaces are deeply connected with hyperbolic geometry, which was discovered by Nikolai Lobachevsky, who published his results in 1826. The same results were also obtained by K. Gauss and J. Bolyai, but Gauss did not publish them, and Bolyai did not develop his main ideas.

The main investigations in hyperbolic geometry (which is also called the Lobachevsky geometry) were carried out in Kazan University by Lobachevsky himself and his successors. After the Second World War a new scientific school began to form in Moscow University. Problems connected with the immersions of hyperbolic plane into Euclidean space came into the center of attention.

The main investigations connected with the hyperbolic geometry and surfaces of negative curvature were carried out by initiators of the Moscow geometrical school N. V. Efimov and E. G. Poznjak and their disciples and collaborators, I. H. Sabitov, E. R. Rosendorn, E. V. Shikin, S. B. Kadomtsev, D. D. Sokolov, A. G. Popov and others.

As it follows from the name, pseudospherical surfaces are surfaces of constant Gaussian curvature, which can be taken equal to -1. Internal geometry of such a surface locally coincides with the geometry of hyperbolic plane, or Lobachevsky plane, and most of the invariant properties of these surfaces can be naturally expressed in terms of Lobachevsky geometry.

Historically Lobachevsky geometry appeared as the first non-Euclidean geometry which was understood as such. Note that it was not the first, but the second non-Euclidean geometry. As long ago as at the Hellenic age spherical geometry was known, and it was developed as well as Euclidean one. But spherical geometry could not play the role of realized non-Euclidean geometry; may be because it can be interpreted "in the large" on a surface in Euclidean space.

The question regarding the same interpretation for the hyperbolic geometry was raised and partially solved by E. Beltrami [3]. He constructed the pseudospherical surface which was the immersion of the horocycle of Lobachevsky plane into the 3-dimensional Euclidean space. Earlier the surfaces of this type were investigated by F. Minding [12], but he did not connect them with hyperbolic geometry.

At present time the Lobachevsky geometry in the narrow sense of the word, i.e., the science analogous to the geometry of Euclid's "Elements" and Menelaus' "Spherics",

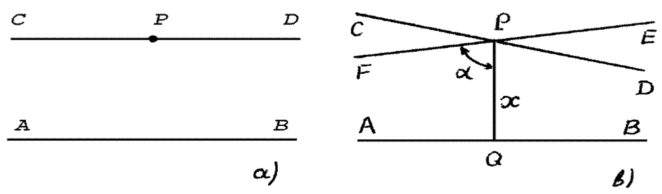

Fig. 1: Parallel straight lines in Euclidean (a) and Lobachevsky (b) geometries.

is exhausted from the point of view of idea content. But the complexity of problems connected with this geometry is developing intensively. Now the researches of applied branches of mathematics by means of Lobachevsky geometry are in the focus of attention. Lobachevsky geometry is regarded now as a typical representative of the geometries of the spaces of negative curvature, which is sufficiently easy to study. The number of physical applications of the hyperbolic geometry is connected with this fact. There are applications in general relativity, nuclear physics, physics of elementary particles etc. These applications are connected with geometrical interpretation of nonlinear equations which appear in these sciences.

2 Main principles of Lobachevsky geometry

Lobachevsky geometry is based on the same axioms as Euclidean one, except for the axiom of parallels. In Euclidean geometry there exists a unique straight line CD passing through a point P outside of straight line AB and not intersecting it. CD is called parallel to AB (see Fig. 1a).

In the Lobachevsky geometry the axiom of parallels requires the existence of more than one of such straight lines. All such lines form a pair of vertical angles with the vertex P. Lines forming the sides of these vertical angles separate the intersecting lines from nonintersecting and nonintersect themselves. These boundary straight lines are called parallels to the line AB in the point P: CD is parallel to AB in direction $\overrightarrow{AB}$, EF is parallel to AB in direction $\overrightarrow{BA}$. Other nonintersecting lines are called divergent with AB, or hyperparallels to AB (see Fig. 1b).

The angle α between the parallel in the point P and the perpendicular PQ is called the angle of parallelism of intercept $x = PQ$ and is denoted by $\alpha = \Pi(x)$. The function $\Pi(x)$ is called the Lobachevsky function, it was found by Lobachevsky himself:

$$\Pi(x) = 2 \arctan \exp \left(-\frac{x}{a} \right) .$$

Here a is a constant called the curvature radius of Lobachevsky plane. There exists an infinite number of different hyperbolic planes with different curvature radii.

Note that Euclidean geometry can be obtained as the limit case of Lobachevsky geometry when both the parallels merge into the same line. In this case the angle of

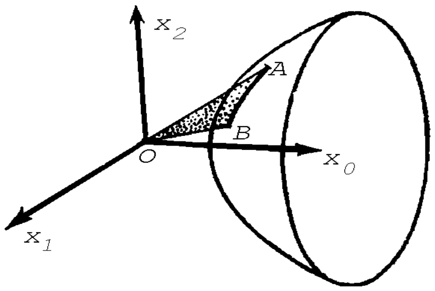

Fig. 2: Sphere in pseudoeuclidean space.

parallelism is equal to $\frac{\pi}{2}$ for all x. This condition is equivalent to the condition $a = \infty$. In small domains of hyperbolic plane, when the size of figures is small compared with the curvature radius, all the formulae of Lobachevsky geometry are approximated by the corresponding formulae of Euclidean geometry. This property means that Lobachevsky geometry is locally Euclidean; it is the feature of all Riemannian manifolds.

3 Models of Lobachevsky geometry

The proof of the consistency of Lobachevsky geometry is carried out by construction of an interpretation or a model. We consider some examples of models in inverse chronological order.

In 1887 H. Poincaré proposed the model of Lobachevsky geometry as the geometry of planar diametrical sections on the one of the connected components of two-sheeted hyperboloid, which can also be treated as a sphere in pseudoeuclidean space.

Let M be the 3-dimensional pseudoeuclidean space of signature $(+ - -)$. The sphere of radius a is defined by the equation

$$x_0^2 - x_1^2 - x_2^2 = a^2. \tag{1}$$

It looks like a two-sheeted hyperboloid with a symmetry axis Ox_0 (see Fig. 2).

Denote by Λ^2 one of its connected components, on which $x_0 \geq a$. Λ^2 is called the Lobachevsky plane, a being its curvature radius. Intersections of Λ^2 with the planes of space M passing through the origin of coordinates are called straight lines (in the sense of Lobachevsky).

One can verify that these notions satisfy all the axioms of Lobachevsky geometry.

Let A, B be points on Λ^2. Then the angle $\theta = \angle AOB$ is defined. Hyperbolic distance between them is the product $\theta \cdot a = |AB|_{\Lambda^2} = \rho(A, B)$. The pair (Λ^2, ρ) is a complete metrical space.

Quantities x_0, x_1, x_2 satisfying equation (1) are called Weierstrass coordinates of the point $P \in \Lambda^2$. In Weierstrass coordinates straight lines of the space Λ^2 are defined by

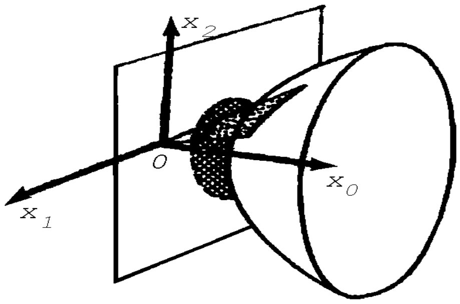

Fig. 3: Beltrami–Klein model.

the system of equations

$$A_0x_0 + a_1x_1 + A_2x_2 = 0, \qquad x_0^2 - x_1^2 - x_2^2 = a^2,$$

and the formula expressing the length of a curve Γ is the following:

$$s = \int_\Gamma \sqrt{-dx_0^2 + dx_1^2 + dx_2^2},$$

or $ds^2 = -dx_0^2 + dx_1^2 + dx_2^2$. The sign in these formulae differs from the sign in equation (1) because for the time-like vector $\mathbf{r}(t)$, which defines the curve Γ in the space Λ^2, its derivative, is a space-like vector, i.e. $|\mathbf{r}'(t)| < 0$.

To represent the geometry of the space Λ^2 more visually, we can project it orthogonally onto the coordinate plane $x_0 = 0$. This projection is bijective. But such a model is not convenient because straight lines of the space Λ^2 are depicted by the hyperbolae in it.

A more convenient model can be produced by the central projection of Λ^2 from the point O onto the plane $x_0 = a$, which is tangent to the hyperboloid Λ^2 in the point $(a, 0, 0)$. This projection is also bijective and the image of the hyperboloid Λ^2 is the open disk of radius a, which we denote by Λ^2 too (see Fig. 3). This model is called Beltrami–Klein model. It appeared in 1871 [10].

Originally this model was not connected with the projection described. Klein used the projective Cayley metric to construct this model. In fact, hyperbolic geometry can be realized on some subset U of the projective space, if we select the projective transformations which generate in U the group of transformations isomorphic to the motion group of Lobachevsky plane. We can take the interior of a real nondegenerate oval on the real projective space as the subset U. The boundary ∂U of the subset U is called the absolute of the model.

Projective properties of the Klein model imply that the distance between points is expressed by anharmonic ratio of quadruple of points, and the angle between straight lines is expressed by anharmonic ratio of quadruple of lines.

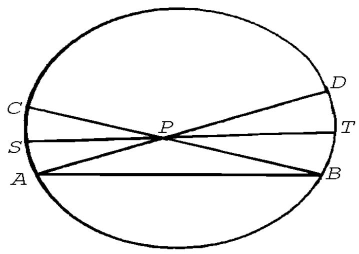

Fig. 4: Klein model in disc.

In the considered case all the metric relations of the Klein model can be obtained by reduction of the metric relations of the Poincaré model on a hyperboloid.

The coordinates y_1, y_2 of the image of the point $A \in \Lambda^2$ are called Beltrami coordinates of the point A. They are connected with the Weierstrass coordinates by formulae

$$y_1 = \frac{ax_1}{x_0}, \qquad y_2 = \frac{ax_2}{x_0},$$
$$x_0 = \frac{a^2}{\sqrt{a^2 - y_1^2 - y_2^2}}, \qquad x_1 = \frac{ay_1}{\sqrt{a^2 - y_1^2 - y_2^2}}, \qquad x_2 = \frac{ay_2}{\sqrt{a^2 - y_1^2 - y_2^2}}.$$

In Beltrami coordinates the straight lines of Lobachevsky plane are defined by the linear equations:

$$A_0 a + A_1 y_1 + A_2 y_2 = 0.$$

Therefore, hyperbolic straight lines are depicted by the usual Euclidean straight lines in the Beltrami–Klein model.

In the picture (see Fig. 4) we can see line CB which is parallel to the line AB in the direction $\overrightarrow{AB}$; line DA is parallel to AB in the direction $\overrightarrow{BA}$; all the lines filling the vertical angles with the vertex P are hyperparallel to AB.

The metric in Beltrami coordinates has the form

$$ds^2 = a^2 \frac{(a^2 - y_2^2)dy_1^2 + 2y_1y_2dy_1dy_2 + (a^2 - y_1^2)dy_2^2}{(a^2 - y_1^2 - y_2^2)^2}.$$

Note that the hyperbolic angle between the two lines differs from the Euclidean one between them.

We can obtain another model by means of a stereographic projection of the hyperboloid onto the coordinate plane $x_0 = 0$ from the point N having coordinates $(-a, 0, 0)$ (see Fig. 5). Lobachevsky plane is depicted by an open disc of radius a; the boundary of this disc is called the absolute of the model, and the model itself is called the Poincaré model in a disc (see Fig. 6a).

In this case straight lines of Lobachevsky plane are depicted by arcs of circles, orthogonal to the absolute and by diameters of the absolute. In the picture you can see the straight line AB, lines CB and AD parallel to AB and the line ST hyperparallel to AB.

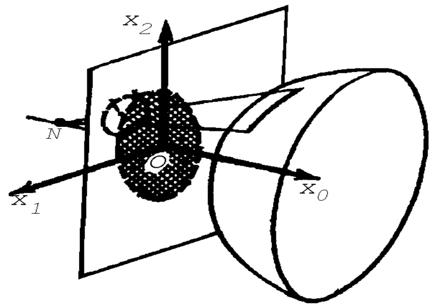

Fig. 5: Poincaré model.

Direct calculation gives the following formula for the metric of the Poincaré model:

$$ds^2 = 4a^2 \frac{dx_1^2 + dx_2^2}{(a^2 - x_1^2 - x_2^2)^2}.$$

If we identify the coordinates x_1, x_2 of the Poincaré model with the Cartesian coordinates on the Euclidean plane, we obtain the conformal mapping from the Lobachevsky plane onto an open disc of Euclidean plane. That is why the angle between hyperbolic straight lines in the Poincaré model is equal to the Euclidean angle between the circles depicting these lines. This property is known as the conformality of the Poincaré model.

It is convenient to use complex coordinates $w = x_1 + ix_2$. These coordinates allow one to write down the following formula for the metric:

$$ds^2 = 4a^4 \frac{dw\, d\overline{w}}{(a^2 - |w|^2)^2}.$$

Let us consider the linear-fractional (or bilinear) transformation

$$\begin{aligned} w \quad &\mapsto z = i\frac{a-w}{a+w}, \\ \text{disc } |w| < a \quad &\to \text{half-plane } \operatorname{Im} z > 0, \end{aligned}$$

which maps the Poincaré disc onto the upper half-plane. This half-plane can be regarded as the model of the Lobachevsky plane, too. This model is called the Poincaré model on the half-plane (see Fig. 6b). In this model the hyperbolic straight lines are depicted by semicircles and half-lines orthogonal to the real axis. This fact follows from the properties of bilinear mappings, which map circles and straight lines into circles and straight lines and preserve the angles between them.

All the given examples of the metrics on Lobachevsky plane demonstrate that it can be considered as the Riemannian manifold, and hyperbolic straight lines are the geodesic lines relative to the corresponding Riemannian structure.

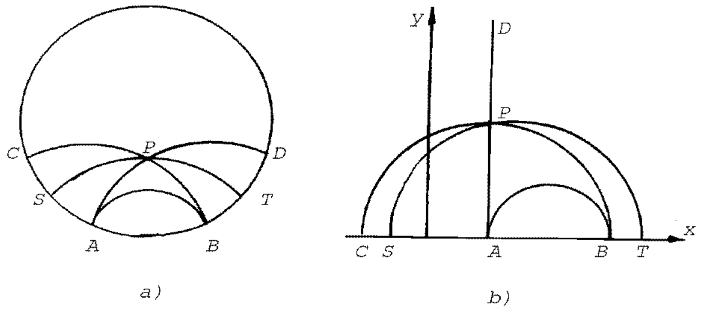

Fig. 6: Poincaré models in disc (a) and on upper half-plane (b).

The calculation shows that the scalar curvature of Lobachevsky plane regarded as the Riemannian manifold is negative constant. It is expressed by the formula

$$R = -\tfrac{2}{a^2},$$

where a is the curvature radius of the Lobachevsky plane. Recall that scalar curvature is the complete contraction of the curvature tensor $R = g^{ml} R^i_{mil}$. The Gaussian curvature K of the 2-dimensional surface is known to be connected with its scalar curvature R by the relation $K = \dfrac{1}{2}R$. So, if we consider a local isometric immersion of the Lobachevsky plane into Euclidean space E^3, the resulting surface will have constant negative Gaussian curvature, which can be chosen to be equal -1.

4 Examples of pseudospherical surfaces

Now we consider some examples of surfaces of constant negative curvature in 3-dimensional Euclidean space.

Let us write down the metric of Lobachevsky plane in isothermic coordinates u, v:

$$ds^2 = \frac{e^\Phi}{2}(du^2 + dv^2),$$

where $\Phi = \Phi(u, v)$ is the unknown function. The Gaussian curvature of such a metric is given by the expression

$$K = -\frac{1}{e^\Phi}\Delta\Phi,$$

where Δ denotes the Laplace operator. For the pseudospherical metric we obtain the equation $\Delta\Phi = e^\Phi$, which is known as the elliptic Liouville equation. Some particular

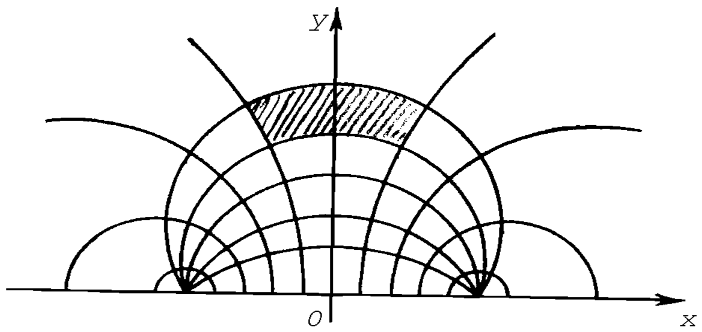

Fig. 7: First kind geodesic bundle.

solutions of this equation, which depend on one of the variables, are

$$\begin{aligned} &\text{I.} \quad \ln\frac{2}{\sin^2 v}, \\ &\text{II.} \quad \ln\frac{2}{\sinh^2 v}, \\ &\text{III.} \quad \ln\frac{2}{v^2}, \end{aligned}$$

and the corresponding pseudospherical metrics are

$$\begin{aligned} &\text{I.} \quad ds^2 = \frac{du^2+dv^2}{\sin^2 v}, \\ &\text{II.} \quad ds^2 = \frac{du^2+dv^2}{\sinh^2 v}, \\ &\text{III.} \quad ds^2 = \frac{du^2+dv^2}{v^2}. \end{aligned}$$

The former case corresponds to the geodesic bundle of the first kind. It is formed by straight lines orthogonal to a given straight line which is the base of the bundle. Orthogonal trajectories of the bundle are hypercycles, or equidistant curves. The shaded domain can be immersed into E^3 as a rotation surface, if we identify the corresponding geodesic lines (see Figs. 7, 8).

The whole equidistant strip can be immersed into E^3 as a universal covering of this surface.

Consider a number of copies of Lobachevsky plane, each of them containing the same equidistant strip. Let's identify corresponding hypercycles by means of the natural isometries. This way we obtain multisheet domain which is immersed into E^3 as a

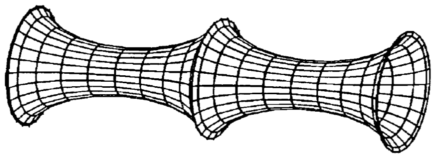

Fig. 8: Pseudospherical surface corresponding to the 1-st kind geodesic bundle.

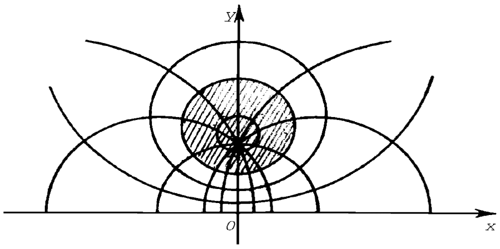

Fig. 9: Second kind geodesic bundle.

periodic surface. This surface was found by Ferdinand Minding in the middle of last century. This surface has singularities — edges, which break its smoothness. These edges correspond to the sewing lines.

In the second case (Fig. 9) we obtain a geodesic bundle of the second type, which is formed by straight lines passing through a fixed point on the Lobachevsky plane. Orthogonal trajectories of the bundle are circles. A geodesic disc of the Lobachevsky plane can be immersed into E^3 as a rotation surface, which you can see on Fig. 10. This surface has a cusp corresponding to the center of disc.

Considering several copies of such a disc and identifying them we obtain a periodic surface which has edges and cusps. This surface was discovered by Minding, too.

In the third case we have a geodesic bundle of the third type, which is formed by straight lines parallel in some direction (Fig. 11). Orthogonal trajectories of the bundle are called horocycles.

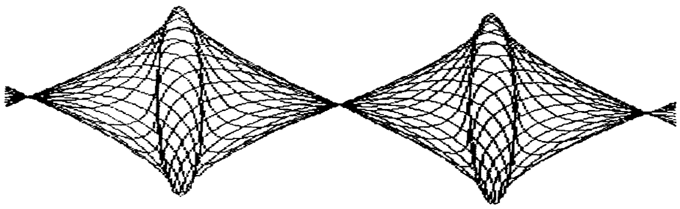

Fig. 10: Pseudospherical surface corresponding to the 2-nd kind geodesic bundle.

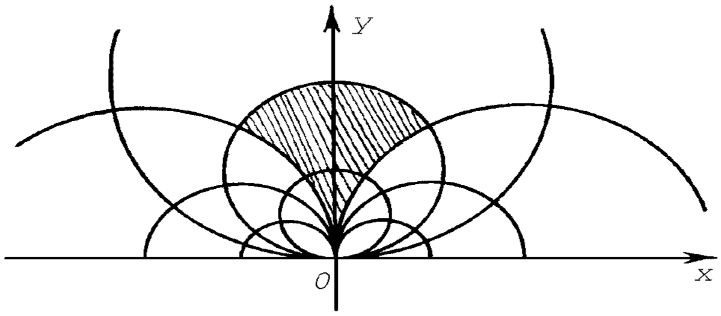

Fig. 11: Third kind geodesic bundle.

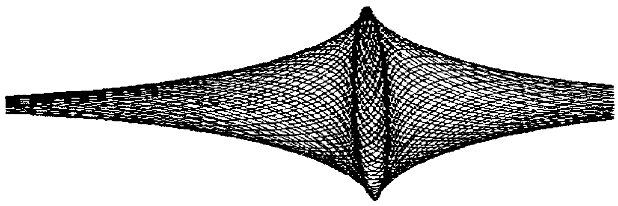

Fig. 12: Beltrami's pseudospere.

Every horocyclic sector can be immersed into $\mathbb{E}^3$ and so it looks like the surface shown in Fig. 12; here we have identified the boundaries of the sector. The whole interior of the horocycle is immersed into $\mathbb{E}^3$ as the universal covering of this surface.

Sewing two copies of horocycles, we can immerse the obtained two-sheet domain into $\mathbb{E}^3$. So we obtain the universal covering of the pseudosphere, which was discovered by Beltrami. Beltrami was the first who found that Lobachevsky geometry can be locally realized on a surface of Euclidean space. The pseudosphere was the first interpretation of hyperbolic geometry, and this fact led to wide recognition of noneuclidean geometry.

5 Hilbert's theorem

All these examples of pseudospherical surfaces possess two features:

— there are edges and/or cusps on them;

— they are immersions of a part of Lobachevsky plane, not of the whole plane.

Considering this David Hilbert raised a question about a possibility of finding a regular isometric immersion of the whole Lobachevsky plane into Euclidean space. This problem was solved by himself in 1901. He proved that in the 3-dimensional Euclidean space there exists no analytical complete surface of constant negative Gaussian curvature.

Here is the sketch of the proof.

Suppose that the whole Lobachevsky plane is immersed into $\mathbb{E}^3$ and the resulting surface is F. It can be deduced from the condition $K = \text{const} < 0$ that asymptotic net on the surface is a Chebyshev net, i.e. in each infinitesimal net quadrangle the values of length of opposite intercepts are the same. The metric in the coordinates of Chebyshev net is:

$$ds^2 = dx^2 + 2\cos\phi\, dx\, dy + dy^2,$$

but it can be proposed only locally. Here ϕ denotes the angle between the lines of the net on the surface. This angle must satisfy the condition

$$0 < \phi < \pi. \tag{2}$$

It is sufficient to prove that the whole surface F is covered by the Chebyshev net, i.e. the asymptotic net on the surface is homeomorphic to the Cartesian net on the Euclidean plane.

The curvature of this metric is given by the expression

$$K = -\frac{\phi_{xy}}{\sin\phi}.$$

The surface F is pseudospherical, so we have the equation for unknown function ϕ:

$$\phi_{xy} = \sin\phi.$$

This equation is called the sine-Gordon equation. Hilbert had shown that this equation has no solutions $\phi(x, y)$ defined on the whole plane of variables (x, y) satisfying condition (2). This fact proves the theorem.

This proof is valid not only in the smoothness class C^∞, but in an arbitrary class C^r where $r \geq 2$. Therefore, the following theorem is true: *There is no C^2-smooth surface in E^3 isometric to Lobachevsky plane in the large.* However, this theorem is not valid in the smoothness class C^1. I shall touch upon this subject later.

6 Special solutions of sine–Gordon equation and corresponding pseudospherical surfaces

Let us consider the Goursat problem for the sine-Gordon equation:

$$\begin{aligned}
\frac{\partial^2\phi}{\partial x\partial y} &= \sin\phi,\\
\phi(x,0) &= f(x), \quad f(x) \in C^r,\\
\phi(0,y) &= g(y), \quad g(y) \in C^r, \quad f(0) = g(0).
\end{aligned}$$

Solutions of this problem defined on the whole plane of variables (x, y) are known to exist, but they do not satisfy condition (2).

E. Poznjak proved that one can construct a vector function $\mathbf{r} = \mathbf{r}(x, y)$ from such a solution $\phi(x, y)$ of the sine-Gordon equation. This vector function defines a surface in $\mathbb{E}^3$. The curvature of this surface is equal to -1 in every point (x, y) where $\phi(x, y) \neq \pi n$, $n \in \mathbb{Z}$. The points where $\phi(x, y) = \pi n$ form the singularities of the surface. Coordinate lines x, y form an asymptotic net on the surface with the net angle $\phi(x, y)$.

Let us consider examples of surfaces which correspond to the travelling waves-like solutions of sine-Gordon equation. Such solutions have the form $\phi(x, y) = \psi(ax+by)$, $a, b =$ const. If we denote $ax + by = \xi$, then the sine-Gordon equation can be reduced to the ordinary differential equation

$$ab\frac{d^2\psi}{d\xi^2} = \sin\psi.$$

Solutions of this equation are expressed through elliptic functions. There exist three types of solutions:

1) $\psi(\xi) = 4\arctan\exp\dfrac{\xi}{\sqrt{ab}}$,

2) $\psi(\xi) = 2\arccos\operatorname{sn}\left(-\dfrac{\xi}{k\sqrt{ab}}, k\right)$,

where $k = \dfrac{1}{\sqrt{1 + \dfrac{ab}{4}\left(C - \dfrac{2}{ab}\right)}}$, $C > \dfrac{2}{ab}$, is the module of elliptic function,

3) $\psi(\xi) = 2\arcsin\operatorname{dn}\left(\dfrac{\xi}{\sqrt{ab}}, k\right)$,

where $k = \sqrt{1 + \dfrac{ab}{4}\left(C - \dfrac{2}{ab}\right)}$, $|C| < \dfrac{2}{ab}$, is the module of elliptic function.

As follows from Poznjak's theorem, we can construct three types of pseudospherical surfaces corresponding to these solutions. This construction was made by S. Zadadaev. Pseudospherical surfaces corresponding to the cases 2 and 3 are Minding surfaces. Case 1 gives so-called "generalized pseudosphere", which is depicted in Fig. 13.

Let us consider one more example of pseudospherical surface which was constructed by Amsler in 1955. This surface corresponds to an automodelling solution of the sine-Gordon equation which depends on the product of variables:

$$\phi(x, y) = \psi(xy) = \psi(t), \quad t = xy.$$

In this case the sine-Gordon equation turns into equation

$$t\psi'' + \psi' = \sin\psi,$$

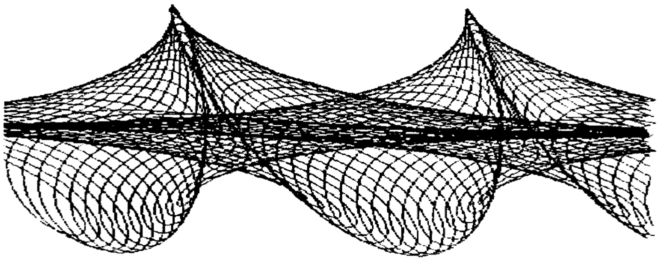

Fig. 13: "Generalized pseudospere".

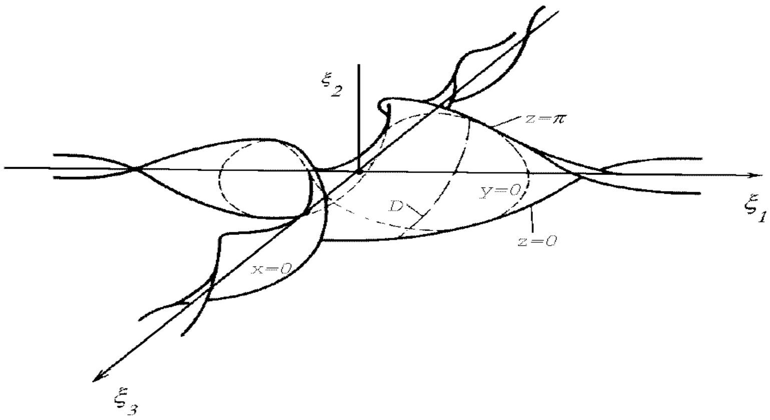

Fig. 14: Amsler's surface.

where prime denotes the derivative. After a change of the function $w = e^{i\psi}$ this equation becomes so-called third Painlevé equation

$$w'' + \frac{w'^2}{w} + \frac{2w' - w^2 + 1}{2t} = 0.$$

Solution $\phi(x, y) = \psi(xy)$ oscillates near π when $t > 0$ and near 0 when $t < 0$. Therefore this solution has denumerable set of level lines $\phi = \pi$ and $\phi = 0$, and corresponding surface has denumerable set of singular edges.

Amsler investigated this surface for values $\phi \in [0, \pi]$. He constructed this surface and proved it to have two cuspidal edges corresponding to the lines of values $\phi = \pi$ and $\phi = 0$.

Consider a part of the solution between the first and the second values $\phi = \pi$. This part corresponds to the regular part of the surface which can be sewn to the first part.

Note that there exist 2 whole straight lines in the sense of hyperbolic geometry on this surface. In the preceding examples there were no whole straight lines on the surface (as in examples of pseudosphere or top-like surface) or there was only one straight line on the universal covering of a spool.

The existence of a surface with such a property was predicted by L. Bianchi.

7 Generalizations of Hilbert's theorem

From the point of view of internal geometry Hilbert's theorem means that there exists a Chebyshev net on the whole hyperbolic plane. Every global solution of sine-Gordon equation generates a Chebyshev net on the Lobachevsky plane, but this net covers only a part of the plane, probably by a few sheets.

For example, the one-soliton solution of the sine-Gordon equation generates the Chebyshev net which covers the interiority of horocycle by two sheets.

From the point of view of immersions this fact means that the whole Lobachevsky plane cannot be immersed into 3-dimensional Euclidean space even as piecewise smooth surface. This result was found by T. Bortsova and A. Ovchinnikov and independently by A. Badjin. It is a generalization of Hilbert's theorem.

Different generalizations of this theorem are also possible. Namely, the following questions arise:

1. Can a p-dimensional Lobachevsky space be immersed into an N-dimensional Euclidean space?
2. Can the given domain of Lobachevsky plane be immersed into the 3-dimensional Euclidean space?
3. Is Hilbert theorem valid in the case of nonconstant curvature?
4. Are isometric immersions of Lobachevsky plane into Euclidean space possible under decreased conditions of smoothness?

The positive answer to the first question follows from results of Nash: a p-dimensional Lobachevsky space can be isometrically immersed and even isometrically imbedded into an N-dimensional Euclidean space if the codimension $N-p$ is sufficiently large.

Nash obtained several relations between p and N.

In the case $p=2$ the first global result was obtained by Bieberbach in 1932. He constructed explicitly the analytic imbedding of Lobachevsky plane into Hilbert space and proved that the resulting surface does not lie in any finite-dimensional subspace and its group of motion is induced by the motions of ambient space.

In 1953 Blanuša proved that Lobachevsky plane can be isometrically imbedded into 6-dimensional Euclidean space. This imbedding belongs to the smoothness class C^∞ and is given by explicit formulae.

Later Rosendorn proved that Lobachevsky plane can be isometrically immersed into 5-dimensional Euclidean space.

It is still unknown whether Lobachevsky plane can be immersed into E^4, although a piecewise smooth immersion $\Lambda^2 \to E^4$ has been constructed by Sabitov.

Let us consider the third question. We know that the following domains of Lobachevsky plane admit the isometric immersion into E^3:

— the interiority of the horocycle;

— the equidistant strip;

— the quadrangle with vertices on the absolute.

We cite a few other examples.

In 1975 Efimov proved that Lobachevsky half-plane cannot be immersed into E^3.

Poznjak considered isometric immersions of polygons with finite or infinite numbers of sides, which vertices could lie on the absolute. He proved that a wide class of such polygons admits immersions into E^3. For example, every polygon with a finite number of vertices can be immersed into E^3.

Recently A. Badjin have generalized this result. He proved that every amoeba-like domain can be immersed into E^3 if every stalk has only one point lying on the absolute and it is sufficiently narrow.

In 1986 Kajdasov and Shikin established that there exists a smooth surface of constant negative curvature on which two nonintersecting horocycles can be placed. This surface looks like a long narrow rouléau.

8 Analytic apparatus of the theory of isometric immersions

The analytic apparatus of the theory of isometric immersions is the Gauss–Mainardi–Codazzi system. This system is the relation between the coefficients of the first and the second fundamental forms of surface:

$$\begin{aligned} \mathrm{I} &= ds^2 = E\,dx^2 + 2F\,dxdy + G\,dy^2, \\ \mathrm{II} &= L\,dx^2 + 2M\,dxdy + N\,dy^2. \end{aligned}$$

The first equation of this system is the Gauss equation, which is a nonlinear algebraic equation

$$\frac{LN - M^2}{EG - F^2} = K,$$

where K can be expressed by the coefficients of the metric form only and their derivatives:

$$K = -\frac{1}{4W^2}\det\begin{vmatrix} E & E_x & E_t \\ F & F_x & F_t \\ G & G_x & G_t \end{vmatrix} - \frac{1}{2\sqrt{W}}\left[\frac{\partial}{\partial t}\frac{E_t - F_x}{\sqrt{W}} - \frac{\partial}{\partial x}\frac{F_t - G_x}{\sqrt{W}}\right],$$

where $W = EG - F^2$.

The other two equations are the Mainardi–Codazzi equations which are partial differential equations linear in L, M, N:

$$\begin{aligned} L_t + \Gamma^1_{11}M + \Gamma^2_{11}N &= M_x + \Gamma^1_{12}L + \Gamma^2_{12}M, \\ M_t + \Gamma^1_{12}M + \Gamma^2_{12}N &= N_x + \Gamma^1_{22}L + \Gamma^2_{22}M, \end{aligned}$$

where Γ_{ij}^k are the Christoffel symbols which are expressed by the coefficients E, F, G of the metric form and its derivatives.

Let $k = \sqrt{-K}$, l, m, n are the reduced coefficients of the second fundamental form of the surface which, by definition, satisfy the relation

$$ln - m^2 = -K.$$

We can choose new unknown functions in the immersion problem, so called Riemannian invariants:

$$r = -\frac{m+k}{n}, \quad s = \frac{m-k}{n},$$

so that the immersion problem turns into a system of quasilinear equations

$$\begin{aligned} r_x + s r_y &= A_0 + A_1 r + A_2 s + A_3 r^2 + A_4 rs + A_5 r^2 s, \\ s_x + r s_y &= A_0 + A_1 s + A_2 r + A_3 s^2 + A_4 rs + A_5 s^2 r, \end{aligned}$$

which is called the system of equations in Riemannian invariants or the system of Rozhdestvensky–Poznjak. Coefficients of this system are expressed through E, F, G. This system is equivalent to the Gauss–Mainardi–Codazzi equations under the condition $r \neq s$. Riemannian invariants have simple geometrical meaning: they are the slopes of the images of asymptotic curves of the surface on the plane of variables (x, y). Asymptotic curves are the characteristic curves of the quasilinear system of Rozhdestvensky–Poznjak, and the condition $r \neq s$ means that the system is nondegenerate on the given solution. Note that systems of such kind are used in gas dynamics.

9 The case of nonconstant curvature

The generalization of the Hilbert theorem to the case of nonconstant curvature was given by Efimov. He proved that there exists no C^2-smooth surface F in E^3, with negative Gaussian curvature and the radius of the curvature $R = \dfrac{1}{\sqrt{|K|}}$ satisfying the condition $|R(P) - R(Q)| \leq C_1 \|P - Q\| + C_2$. It follows from this theorem that inequality $\sup K \geq 0$ holds on an arbitrary complete C^2-smooth surface in E^3. Thus the Hilbert theorem is a special case of the Efimov theorem.

Efimov's theorem is valid for surfaces of smoothness class C^2, but there exists a counterexample in the case $C^{(1,1)}$. The $C^{(1,1)}$-regularity means that the first derivatives of the radius-vector of the surface are continuous and noncollinear on the surface and they satisfy the Lipschitz condition in each compact domain of the surface.

This counterexample is the surface looking like a distorted tetrahedron, faces of which are warped and vertices moved to infinity. The C^2-smoothness is broken in four points of the surface which are the "centers of faces of tetrahedron". The curvature treated in generalized sense of Alexandrov is defined in any point of the surface and satisfies the condition $K \leq \text{const} < 0$.

Investigation of immersion problems shows that the slower the curvature changes the more difficult it is to immerse it into Euclidean space.

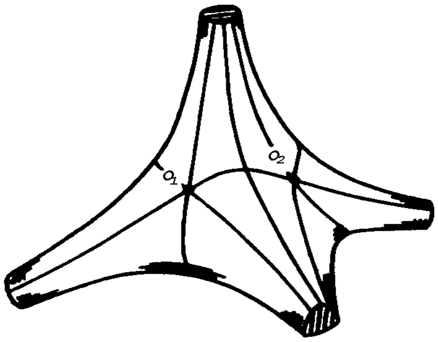

Fig. 15: $C^{(1,1)}$-smooth surface.

We have seen that decreasing of smoothness requirements allows one to immerse a metric of negative curvature into E^3. It is valid for the metrics of constant curvature. As shown by Kuiper, the Lobachevsky plane can be isometrically imbedded into E^3 under the condition that smoothness class of the resulting surface will be C^1. This surface may be represented as the goffered plane.

10 Construction of pseudospherical surfaces

Let us finally consider the problem of constructing pseudospherical surfaces. The first method was mentioned above. It is based on Poznjak theorem and associates a pseudospherical surface with each solution of the sine–Gordon equation.

Another method is the Bäcklund transform. The main idea of this method is construction of new solutions of the sine-Gordon equation from given ones. The Bäcklund transform connects two distinct solutions of the sine-Gordon equation by the formulae:

$$\frac{\partial \phi^*}{\partial x} = \frac{\partial \phi}{\partial x} + 2k \cdot \sin \frac{\phi^* + \phi}{2},$$
$$\frac{\partial \phi^*}{\partial y} = -\frac{\partial \phi}{\partial y} + \frac{2}{k} \cdot \sin \frac{\phi^* - \phi}{2},$$

where $k = \text{const}$. For example, having taken the trivial solution $\phi \equiv 0$ of the sine-Gordon equation, we can obtain one-soliton solution

$$\phi^* = 4 \arctan \exp \left(kx + \frac{y}{k} \right).$$

In geometry the Bäcklund transform allows one to construct one-parametric family of pseudospherical surfaces F^* defined by the radius-vector $\mathbf{r}^*$ provided that the surface F defined by the radius-vector $\mathbf{r}$ is given:

$$\mathbf{r}^* = \mathbf{r} + R \cdot \sin \sigma \cdot (\mathbf{p} \cos \phi + \mathbf{q} \sin \phi),$$

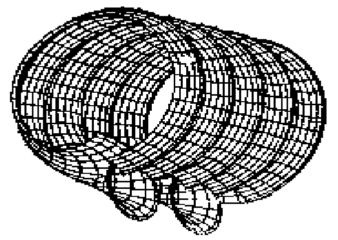

Fig. 16: Enneper surface.

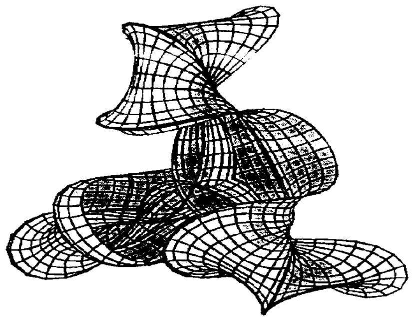

Fig. 17: Dobriner surface.

where σ is a parameter, $\mathbf{p}$ and $\mathbf{q}$ are unit vectors tangent to the curvature lines on the surface F, and ϕ is the solution of the sine-Gordon equation corresponding to the surface F. This method was developed by Bäcklund himself especially for the pseudospherical surfaces.

The method of separation of variables applied to the sine-Gordon equation allows to obtain some pseudospherical surfaces which belong to the class of surfaces of Enneper. One of the families of curvature lines of these surfaces is formed by plane curves. These surfaces were investigated by Steuerwald.

Another approach of constructing the pseudospherical surfaces was developed in works of A. Sym [14, 15]. These methods are based on the soliton theory.

A lot of pseudospherical surfaces can be obtained by computer calculations based on soliton simulations [16]. Some examples are given below.

A pseudospherical surface with one family of planar lines of curvature is referred to as an Enneper surface (see Fig. 16).

The second family is then automatically spherical.

A non-Enneper pseudospherical surface with one family of spherical lines of curvature is referred to as a Dobriner surface (see Fig. 17).

Hasimoto surfaces form another class of pseudospherical surfaces. They can be described as follows. Given a smooth curve in E^3 parameterized by arc length, let it evolve

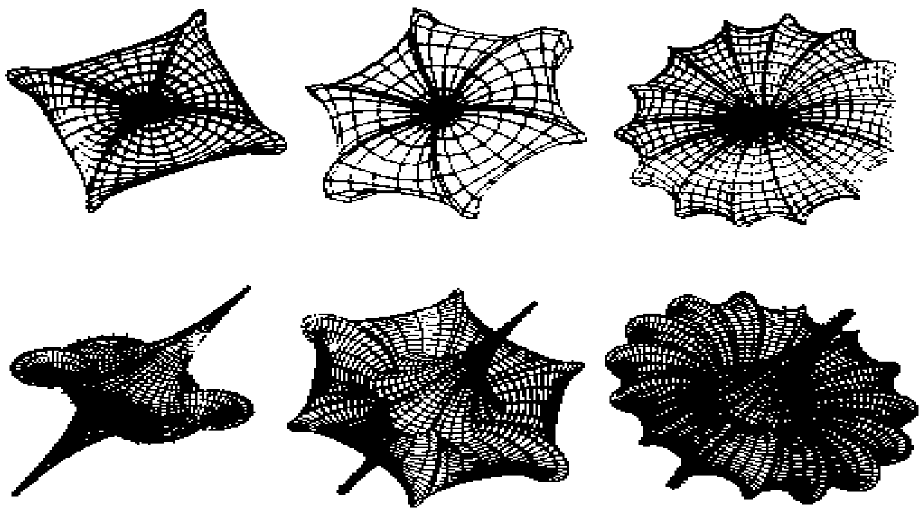

Fig. 18: Hasimoto pseudospherical surfaces.

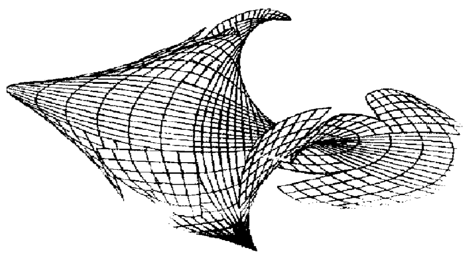

Fig. 19: Wissler surface.

in time by moving in the direction of its binormal, with speed proportional to its curvature. This is the way in which a smoke-ring evolves in time. A Hasimoto surface is, by definition, the surface swept out by such a smoke-ring. Hasimoto showed that the curvature function of an evolving smoke-ring satisfies the nonlinear Schrödinger equation. The existence of pseudospherical Hasimoto surfaces (see Fig. 18) indicates a relationship between sine-Gordon equation and nonlinear Schrödinger equation which deserves further study.

Another example is the Wissler surface (see [16]). Wissler's construction is based on work by Amsler.

11 Conclusion

We can see that pseudospherical surfaces are very important in soliton theory. Various procedures of construction of such surfaces are based on soliton techniques, but investigation of them is closely related to the Lobachevsky geometry. There is a lot of problems in both of these fields. In soliton theory: construction of new integrable equations, investigation of their properties and related geometrical structures. In geometry we have a very important problem of immersion of pseudospherical metrics into Riemannian spaces and related topics in the theory of partial differential equations. It is evident that the interaction of these branches of mathematics is very fruitful.

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