$F = -k_B T \ln Z$

$$Z = \int \mathcal{D}\phi \; e^{-\int \mathcal{H}(\phi)\, d^d x}$$

$$\int \mathcal{H}(\phi) = \frac{1}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2 + u\phi^4 \Big] .$$

Separate $\phi(\vec{k}) = \phi_> + \phi_<$

$\phi_<(\vec{k}) \; (0 \le k \le b^{-1})$

$\phi_>(\vec{k}) \; (b^{-1} \le k \le 1)$ $(b>1)$

Rules for diagrammatic expansion.
external legs have $|\vec{k}| \le b^{-1}$, internal legs have $|\vec{k}| > b^{-1}$ and are integrated out.

Also must rescale the momenta & fields.

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{2}(k^2+r)\phi(\vec{k})\phi(-\vec{k}) \rightarrow \int_< \frac{d^d k}{(2\pi)^d} \frac{1}{2}(k^2+r)\phi(\vec{k})\phi(-\vec{k}) + \int_> \rightarrow \text{this is a constant added to the free energy}$$

Rescale: $\phi(\vec{k}) = \zeta\,\phi'(\vec{k}')$; $\vec{k}' = bk$.

$$\therefore \int_< \frac{d^d k}{(2\pi)^d} \frac{1}{2}(k^2+r)|\phi(\vec{k})|^2 = b^{-d} \underbrace{\int \frac{d^d k'}{(2\pi)^d}}_{0<|k'|\le 1} \frac{1}{2}(b^{-2}k'^2 + r)\zeta^2|\phi'(\vec{k}')|^2$$

choose ζ^2 such that $\zeta^2 b^{-d} b^{-2} = 1$ keep coefficient of $(\nabla\phi)^2$ fixed. To this order $\boxed{\zeta^2 = b^{d+2}}$

$$\therefore \text{We have} \int \frac{d^d k'}{(2\pi)^d} \frac{1}{2}(k'^2 + r')|\phi'(\vec{k}')|^2$$

where $\boxed{r' = b^2 r}$

Similarly $\displaystyle\int_< \frac{d^d k_1}{(2\pi)^d} \int_< \frac{d^d k_2}{(2\pi)^d} \int_< \frac{d^d k_3}{(2\pi)^d} \; u\, \phi(\vec{k}_1)\phi(\vec{k}_2)\phi(\vec{k}_3)\phi(-\vec{k}_1-\vec{k}_2-\vec{k}_3)$

upon rescaling yields $\zeta^4 b^{-3d} = b^{4-d}$

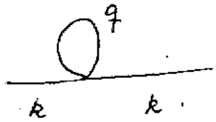
i.e., $\boxed{u' = b^{4-d} u}$ $\qquad 4-d \equiv \epsilon$.

Now expand the exponential, integrate out large k-modes re-exponentiate to find $\mathcal{H}'$, i.e., r', u'; in principle generate other terms (IRRELEVANT)

$$\int \mathcal{D}\phi \, e^{-\int \mathcal{H}_0(\phi)} \left[1 - u \int d^dx \, \phi^4 + \frac{u^2}{2!} \int d^dx \, \phi^4 \int d^dy \, \phi^4 + \dots \right]$$

$k < b^{-1}$ $\quad q > b^{-1}$.

$u\phi^4$ gives



This yields a term $\phi_k \phi_{-k} \displaystyle\int_{b^{-1}<q<1} \frac{d^dq}{(2\pi)^d} \frac{1}{q^2+r} \; C$

HARTREE NO k-dependence!

In the limit $b \mapsto 1$

$$\int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2+r} = \frac{\Omega_d}{(2\pi)^d} \int_{b^{-1}}^{1} \frac{q^{d-1}dq}{q^2+r}$$

$$= \underbrace{\frac{\Omega_d}{(2\pi)^d}}_{= K_d} \frac{1}{1+r} \underbrace{(1-b^{-1})}_{\delta\ell}$$

C = combinatoric factor = **6**.

Kiddies can think of this as coming from $(\phi_> + \phi_<)^4 = \dots + 6\,\phi_>^2 \phi_<^2$

This yields $\quad \zeta^2 b^{-d} \int d^dk' \, \phi'(\vec{k}') \, \phi'(-\vec{k}') \left(-\frac{6u}{(1+r)} K_d \, \delta\ell \right)$

Now reexponentiate and add this to r'

$$\therefore \quad r' = \zeta^2 b^{-d} \left[r + \frac{12 u K_d \delta\ell}{(1+r)} \right]$$

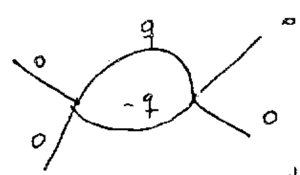
[absorb K_d into u later]

$$\therefore \quad r' = (1+2\delta\ell) r + \frac{12 u K_d \cdot \delta\ell}{(1+r)}$$

$b \equiv e^{\ell} \approx 1 + \delta\ell$

$$\Rightarrow \quad \boxed{\frac{dr}{d\ell} = 2r + \frac{12 u K_d}{(1+r)}}$$

from the second-order term



CHOOSE EXTERNAL LEGS to have $k_i' \equiv 0$

in principle $u \equiv u(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$

but $u(0,0,0,0)$ is all we need the rest are irrelevant; in a Taylor expansion each k comes with a factor of b^{-1}

$$6 \times 6 \times 2 \, \frac{u^2}{2!} \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2+r)^2}$$

$$\frac{36 u^2}{(1+r)^2} K_d$$

$K_d = \dfrac{\Omega_d}{(2\pi)^d}$

∴ re-exponentiating

$$u' = b^{-3d} \zeta^{4} \left[u - \frac{36\, u^{2} K_d\, \delta}{(1+r)^{2}} \right]$$

$$\Rightarrow \quad \boxed{\frac{du}{dl} = \epsilon u - \frac{36\, u^{2} K_d}{(1+r)^{2}}}$$

Together we have

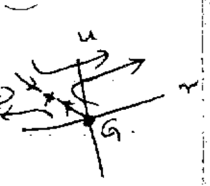
$$\boxed{\begin{aligned} \frac{dr}{dl} &= 2r + \frac{12u}{(1+r)} \\ \frac{du}{dl} &= \epsilon u - \frac{36u^{2}}{(1+r)^{2}} \end{aligned}} \quad (A)$$

absorb K_d into u

Fixed points: $r=0,\ u=0$ Gaussian; doubly unstable.

Wilson Fisher: $u^{*} \approx \epsilon/36 \quad r^{*} \approx -\epsilon/6 \left[+ O(\epsilon^{2})\right]$

Linearize (A) around W-F fixed point



$r = r^{*} + \Delta r$

$u = u^{*} + \Delta u$

$$\frac{d}{dl}\begin{pmatrix} \Delta r \\ \Delta u \end{pmatrix} = \begin{pmatrix} 2-\epsilon/3 & \frac{12}{1-\epsilon/6} \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta u \end{pmatrix}$$

Eigenvalues: $\lambda_1 \equiv 2 - \epsilon/3$ (relevant)

$\lambda_2 \equiv -\epsilon$ (irrelevant)

ONE RELEVANT "EVEN" DIRECTION (TEMPERATURE)
ONE RELEVANT "ODD" DIRECTION (MAG. FIELD)

RG preserves Z the partition function.

CORRELATION LENGTH $\quad \dfrac{\xi(r,u)}{b} = \xi'(r', u' \dots)$

i.e., $\xi(r,u) = b\, \xi(b^{\lambda_1} g_1, b^{\lambda_2} g_2)$ near critical point

g_1, g_2 scaling variables

$= (g_1)^{-1/\lambda_1}\, \xi(1, g_2\, g_1^{-\lambda_2/\lambda_1})$

$g_1 \equiv \Delta r$ to this order in ϵ.

$g_1 \equiv$ relevant variable; identify with $t \equiv T - T_c / T_c$

$\therefore \quad \xi \sim t^{-1/\lambda_1} \qquad \xi \sim t^{-\nu}$ definition of ν.

$$\therefore \quad \boxed{\nu = \frac{1}{\lambda_1}} = \frac{1}{2-\epsilon/3} \approx \frac{1}{2} + \frac{\epsilon}{12}.$$

$\epsilon = 1 \quad \nu \approx 0.583$

3-d ISING. $\nu \approx 0.63..$

GOT BETTER THAN MFT