Phys 4267/6268 zipped!

World Wide Quest to Tame Chaos

`PHYS-4267-16`, rev. 65: last edit by Predrag Cvitanović, 12/09/2016

Predrag Cvitanović

December 15, 2016

chaos course - week 13

Divide & count

Georgia Tech PHYS-4267

Homework HW13 due Thursday, December 1, 2016

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 13.1 *Full tent map periodic points*	4 points + 3 points
Exercise 13.2 *"Golden mean" pruned map*	4 points + 1 point
Exercise 13.3 *Transition matrix and cycle counting*	4 points

13.1 Is the geometry of nature fractal?

Christopher Marcotte (Nov 15, 2016) writes: Box-counting is dumb but it's a very intuitive way to show numerically that you can generate fractal sets from simple laws. Grassberger–Procaccia [3] is who comes to mind. It also gives them a chance to "build" and "see" and "check" fractals using their own methods.

Predrag response: Why would one teach this in a physics course? It's misleadingly easy to explain, and then it turns out to be good for nothing. It is totally useless in dynamical systems theory –unless you miraculously come back to me with Grassberger & Procaccia dimension of your cardiac system, and even then– which cardiologist would want to to know that number? For what purpose? We do care about the physical dimensions of inertial manifolds [2] but those calculations are dynamically informed, painstaking calculations, in no sense mindless algorithm crunching through a set of points.

Avnir, Biham, Lidar, and Malcai [1] write: "the majority of the data that was interpreted in terms of fractality in the surveyed Physical Review journals does not seem to be linked (at least in an obvious way) to existing models and, in fact, does not have theoretical backing. Most of the data represent results from nonequilibrium processes. The common situation is this: An experimentalist performs a resolution analysis and finds a limited-range power law with a value of D smaller than the embedding dimension. Without necessarily resorting to special underlying mechanistic arguments, the experimentalist then often chooses to label the object for which she or he finds this power law a 'fractal.' This is the fractal geometry of nature".

In *Deterministic chaos: the science and the fiction* Ruelle [4] writes: "In conclusion, one should not believe dimension estimates that are not well below $2\log_{10} N$. [...] claim to find a dimension 3.1 for a 'climatic attractor' with $N = 500$ data points. [...] The 'dimensions' of the order 6 that are obtained are very close to the upper bound $2\log_{10} N$ permitted by the Grassberger-Procaccia algorithm (N is the length of the time series used, of the order of 103). The 'proof' that one has low dimensional dynamics is therefore inconclusive, and the suspicion is that the time evolutions under discussion do not correspond to low-dimensional dynamics. It is possible that interesting information can nevertheless be extracted from the time series examined, but this would probably require new ideas".

"Readers of *The hitchhiker's guide to the galaxy*, that masterpiece of British literature by D. Adams, know that a huge supercomputer has answered 'the great problem of life, the universe, and everything'. The answer obtained after many years of computation is 42. Unfortunately, one does not know to what precise question this is the answer, and what to make of it. It think that what happened is this. The supercomputer took a very long time series describing all it knew about 'life, the universe, and everything' and proceeded to compute the correlation dimension of the corresponding dynamics, using the Grassberger–Procaccia algorithm. This time series had a length N somewhat larger than 10^{21}. And you can imagine what happened. After many years of computation the answer came: dimension is approximately $2\log_{10} \approx 42$."

13.2 Temporal ordering: Itineraries

Copied here are a few snippets from this week's lecture notes, needed here just because exercises refer to them - do read the full lecture notes.

For $1d$ maps the *critical value* denotes either the maximum or the minimum value of $f(x)$ on the defining interval; we assume here that it is a maximum, $f(x_c) \geq f(x)$ for all $x \in \mathcal{M}$. The critical point x_c that yields the critical value $f(x_c)$ belongs to neither the left nor the right partition $\mathcal{M}_i$ and is instead denoted by its own symbol $s = C$. As we shall see, its images and preimages serve as partition boundary points.

The trajectory $x_1, x_2, x_3, \ldots$ of the initial point x_0 is given by the iteration $x_{n+1} = f(x_n)$. Iterating f and checking whether the point lands to the left or to the right of x_c generates a *temporally* ordered topological itinerary for a given trajectory,

$$s_n = \begin{cases} 1 & \text{if } x_n > x_c \\ C & \text{if } x_n = x_c \\ 0 & \text{if } x_n < x_c \end{cases} . \tag{13.1}$$

We refer to $S^+(x_0) = .s_1 s_2 s_3 \cdots$ as the *future itinerary*. Our next task is to answer the reverse problem: given an itinerary, what is the *spatial* ordering of points that belong to the corresponding state space trajectory?

13.3 Spatial ordering

A well-known theorem states that combinatorial factors are impossible to explain.

—G. 't Hooft and M. Veltman, DIAGRAMMAR

The tent map point $\gamma(S^+)$ with future itinerary S^+ is given by converting the itinerary of s_n's into a binary number γ by the following algorithm:

$$\begin{aligned} w_{n+1} &= \begin{cases} w_n & \text{if} \quad s_{n+1} = 0 \\ 1 - w_n & \text{if} \quad s_{n+1} = 1 \end{cases} , \qquad w_1 = s_1 \\ \gamma(S^+) &= 0.w_1 w_2 w_3 \ldots = \sum_{n=1}^{\infty} w_n / 2^n . \end{aligned} \tag{13.2}$$

This follows by inspection from the the way a unimodal map partitions its 1-dimensional state space (the unit interval) Once you figure this out, feel free to complain that the way the rule is stated here is incomprehensible, and show us how you did it better.

We refer to $\gamma(S^+)$ as the *(future) topological coordinate*. The w_t's are the digits in the binary expansion of the starting point γ for the full tent map. In the left half-interval the map $f(x)$ acts by multiplication by 2, while in the right half-interval the map acts as a flip as well as multiplication by 2, reversing the ordering, and generating in the process the sequence of s_n's from the binary digits w_n.

13.4 Full tent map

The simplest example of unimodal maps with complete binary symbolic dynamics is the *full tent map,*

$$f(\gamma) = 1 - 2|\gamma - 1/2|\,, \qquad \gamma \in \mathcal{M} = [0,1]\,. \tag{13.3}$$

For unimodal maps the Markov partition of the unit interval $\mathcal{M}$ is given by intervals $\{\mathcal{M}_0, \mathcal{M}_1\}$. We refer to (13.3) as the *complete* tent map because its symbolic dynamics is completely binary: as both $f(\mathcal{M}_0)$ and $f(\mathcal{M}_1)$ fully cover $\mathcal{M} = \{\mathcal{M}_0, \mathcal{M}_1\}$, all binary sequences are realized as admissible itineraries.

Periodic points of the full tent map.

Each cycle p is a set of n_p rational-valued full tent map periodic points γ. It follows from (13.2) that if the repeating string $s_1 s_2 \ldots s_n$ contains an odd number of '1's, the string of well ordered symbols $w_1 w_2 \ldots w_{2n}$ has to be of the double length before it repeats itself. The cycle-point γ is a geometrical sum which we can rewrite as the odd-denominator fraction

$$\begin{aligned}\gamma(\overline{s_1 s_2 \ldots s_n}) &= \sum_{t=1}^{2n} \frac{w_t}{2^t} + \frac{1}{2^{-2n}} \sum_{t=1}^{2n} \frac{w_t}{2^t} + \cdots \\ &= \frac{2^{2n}}{2^{2n}-1} \sum_{t=1}^{2n} \frac{w_t}{2^t} \end{aligned} \tag{13.4}$$

References

[1] D. Avnir, O. Biham, D. Lidar, and O. Malcai, "Is the geometry of nature fractal?", Science **279**, 39–40 (1998).

[2] X. Ding, H. Chaté, P. Cvitanović, E. Siminos, and K. A. Takeuchi, "Estimating the dimension of the inertial manifold from unstable periodic orbits", Phys. Rev. Lett. **117**, 024101 (2016).

[3] P. Grassberger and I. Procaccia, "Characterization of strange attractors", Phys. Rev. Lett. **50**, 346–349 (1983).

[4] D. Ruelle, "The Deterministic chaos: the science and the fiction", Proc. R. Soc. London A **427**, 241–248 (1990).

Exercises

13.1. **Full tent map periodic points.** This exercise is easy: just making sure you know how to go back and forth between spatial and temporal ordering of trajectory points.

(a) compute the two periodic points of cycle $\overline{01}$ "by hand," by solving the fixed-point condition for the second iterate $f_1 \circ f_0$

(b) compute the periodic points of two 3-cycles $\overline{001}$ and $\overline{011}$ by solving the fixed-point condition for the third iterates

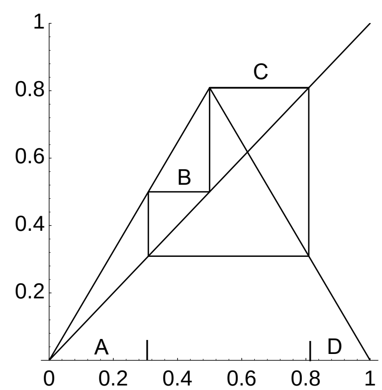
(c) compute the five periodic points of cycle $\overline{10011}$ using (13.4)

(d) compute the five periodic points of cycle $\overline{10000}$

(e) derive (13.4)

(f) plot the above two 5-cycles on the graph of the full tent map, and as many others as you find interesting. Why? Because you can start appreciating the power of kneading theory–while the state space orbits get more and more complicated and impenetrable, the kneading sequence pruning rule is as simple and as sharp as a knife.

13.2. **"Golden mean" pruned map.** Consider a symmetric tent map on the unit interval such that its highest point belongs to a 3-cycle:



(a) Find the value $|\Lambda|$ for the slope (the two different slopes $\pm\Lambda$ just differ by a sign) where the maximum at $1/2$ is a periodic point in a 3-cycle, as depicted in the figure. Partition the state space $\mathcal{M}$ (i.e., the unit interval) into four intervals
$\mathcal{M}_A = [0, (\sqrt{5}-1)/4)$ $\mathcal{M}_B = ((\sqrt{5}-1)/4, 1/2)$
$\mathcal{M}_C = (1/2, (\sqrt{5}+1)/4)$ $\mathcal{M}_D = ((\sqrt{5}+1)/4, 1]$.

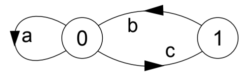
(b) Show that no orbit of this map can visit the interval $\mathcal{M}_D$ more than once. Verify also that once an orbit is outside the interval $\mathcal{M}_A$, it cannot reenter it.

(c) If an orbit is in the interval $\mathcal{M}_B$, where will it be on the next iteration?

(d) If the symbolic dynamics is such that for $x < 1/2$ we use the symbol 0, for $x = 1/2$ we use the symbol C, and for $x > 1/2$ we use the symbol 1, show that no periodic orbit will contain the substring _00_.

(e) On a second thought, are there periodic orbits that violate the _00_ pruning rule?

13.3. **Transition matrix and cycle counting.** Suppose you are given the transition graph



This diagram can be encoded by a matrix T, where the entry T_{ij} means that there is a link connecting node i to node j. The value of the entry is the weight of the link.

(a) Walks on the graph are given a weight that is the product of the weights of all links crossed by the walk. Check that the transition matrix for this graph is:

$$T = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} .$$

(b) Enumerate all the walks of length three on the transition graph. Now compute T^3 and look at the entries. Is there any relation between the terms in T^3 and all the walks?

(c) Show that T^n_{ij} is the number of walks from point i to point j in n steps. (Hint: one might use the method of induction.)

(d) Estimate the number K_n of walks of length n for this simple transition graph.

(e) The topological entropy h measures the rate of exponential growth of the total number of walks K_n as a function of n. What is the topological entropy for this transition graph?