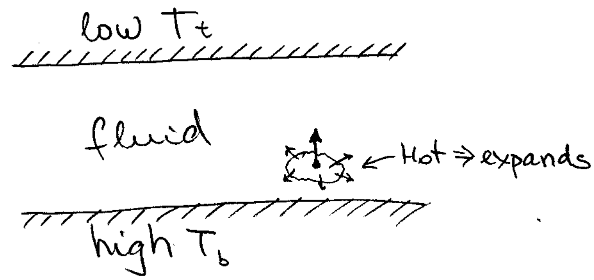
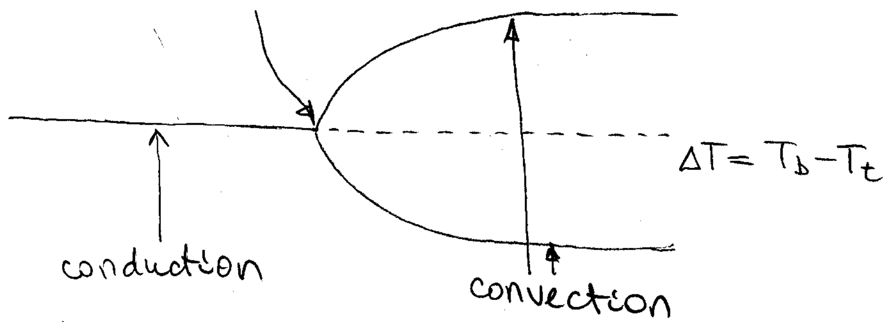
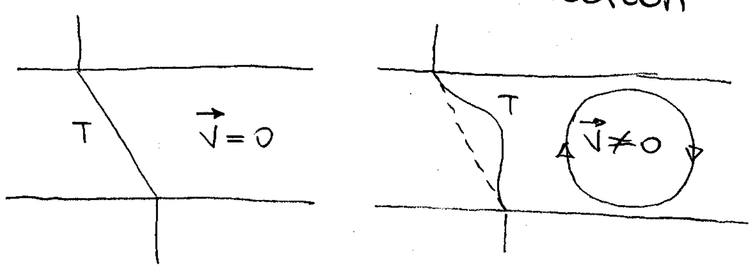
Convection



up-down symmetry ⇒ pitchfork bifurcation





Equations:

Navier-Stokes: $\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \sigma\left[-\vec{\nabla}p + rT\vec{e}_z + \vec{\nabla}^2\vec{v}\right]$

Heat Equation: $\partial_t T + (\vec{v}\cdot\vec{\nabla})T = \vec{\nabla}^2 T$

Continuity: $\vec{\nabla}\cdot\vec{v} = 0$

Stream function: $\Psi(x,z)$: $\quad V_x = -\partial_z\Psi, \quad V_z = \partial_x\Psi, \quad V_y = 0$

Take $\Psi(x,z,t) = \underline{\underline{0}} + 2\sqrt{6}\,\underline{\underline{X(t)}}\cos(\pi z)\sin(\pi a x)$

$$T(x,y,z) = \underline{\underline{-rz}} + 9\pi^3\sqrt{3}\,\underline{\underline{Y(t)}}\cos(\pi z)\cos(\pi a x) +$$
$$+\frac{27}{4}\pi^3\,\underline{\underline{Z(t)}}\sin(2\pi z), \qquad -\frac{1}{2} < z < \frac{1}{2}$$

Lorenz Equations

First 3-d system:
$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

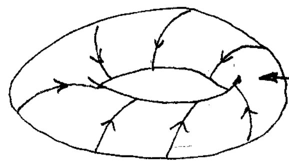
σ – Prandtl number, r – Rayleigh number, b – aspect ratio
$\sigma = \nu/\kappa$ $\quad r = g\alpha h^3 \Delta T/(\kappa\nu)$ $\quad b = 4/(1+a^2)$

<u>Symmetry</u>: $(x,y,z) \to (-x,-y,z)$

Volume Contraction:

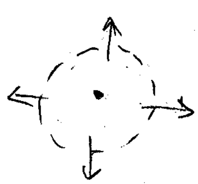
$$\vec{\nabla}\cdot\vec{f} = \partial_x(\sigma(y-x)) + \partial_y(rx - y - xz) + \partial_z(xy - bz) =$$
$$= -\sigma - 1 - b < 0 \qquad (\sigma, r, b > 0)$$

<u>Corollary 1</u>: No quasiperiodic orbits

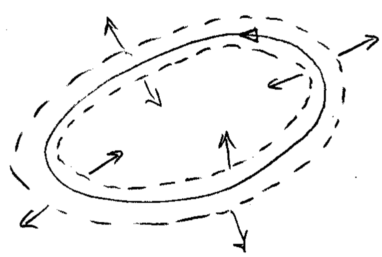


$V = \text{const} \Rightarrow \dot{V} = 0$
(invariant)

<u>Corollary 2</u>: No repelling f.p. (sources) or limit cycles



$\dot{V} > 0$



$\dot{V} > 0$

Fixed Points:

$$\begin{cases} x = y \\ rx - y = xz \\ xy = bz \end{cases} \Rightarrow \begin{cases} y = x \\ xz = x(r-1) \\ z = \frac{1}{b}x^2 \end{cases} \Rightarrow$$

O: $x^* = y^* = z^* = 0$
(conduction)

$C^{\pm}$: $x^* = y^* = \pm\sqrt{b(r-1)}$
$z^* = r - 1$
(convection)

Linear Stability of 0:

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y \\ \dot{z} = -bz \end{cases}$$

$\dot{x}, \dot{y}$ equations $\Rightarrow A = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$

$\dot{z} = -bz \Rightarrow z \sim \exp(-bt) \to 0.$

$\Delta = \sigma(1-r), \ \tau = -\sigma - 1 < 0$

$\underline{r<1:}$ $\Delta > 0, \ \tau^2 - 4\Delta = (\sigma+1)^2 - 4\sigma(1-r) = (\sigma-1)^2 + 4\sigma r > 0$

$\Rightarrow$ Stable node (sink)

$\underline{r>1:}$ $\Delta < 0 \Rightarrow$ saddle (2 stable, 1 unstable direction)

Nonlinear (Global) Stability of 0:

Lyapunov function: $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$

$$\frac{1}{2}\dot{V} = \frac{1}{\sigma}x\dot{x} + y\dot{y} + z\dot{z} = (yx - x^2) + (ryx - y^2 - xyz) + (xyz - bz^2)$$

$$= (r+1)xy - x^2 - y^2 - bz^2 = -\left[x - \frac{r+1}{2}y\right]^2 - \left[1 - \left(\frac{r+1}{2}\right)^2\right]y^2 - bz^2$$

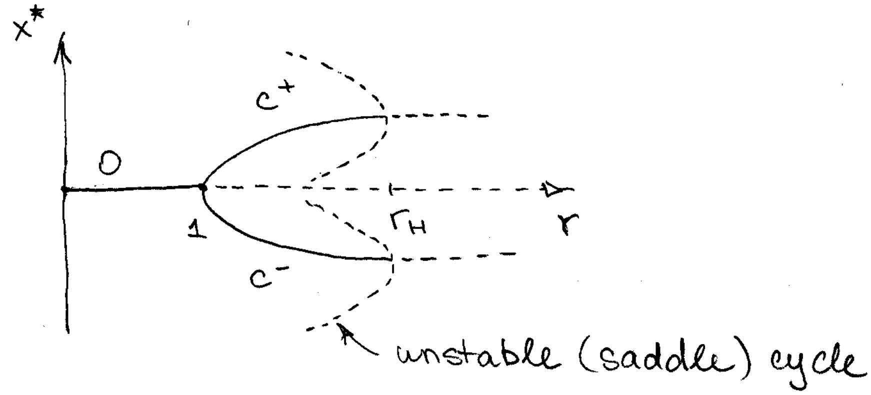
$\dot{V} < 0 \Leftrightarrow \left(\frac{r+1}{2}\right)^2 < 1 \Leftrightarrow r < 1$ (when linearly stable)

Stability of C^+ & C^-

C^+, C^- exist for $r > 1 \Rightarrow$ Supercritical Pitchfork at $r=1$

C^+, C^- stable for $1 < r < r_H$

At $r_H = \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$ - Hopf bifurcation occurs

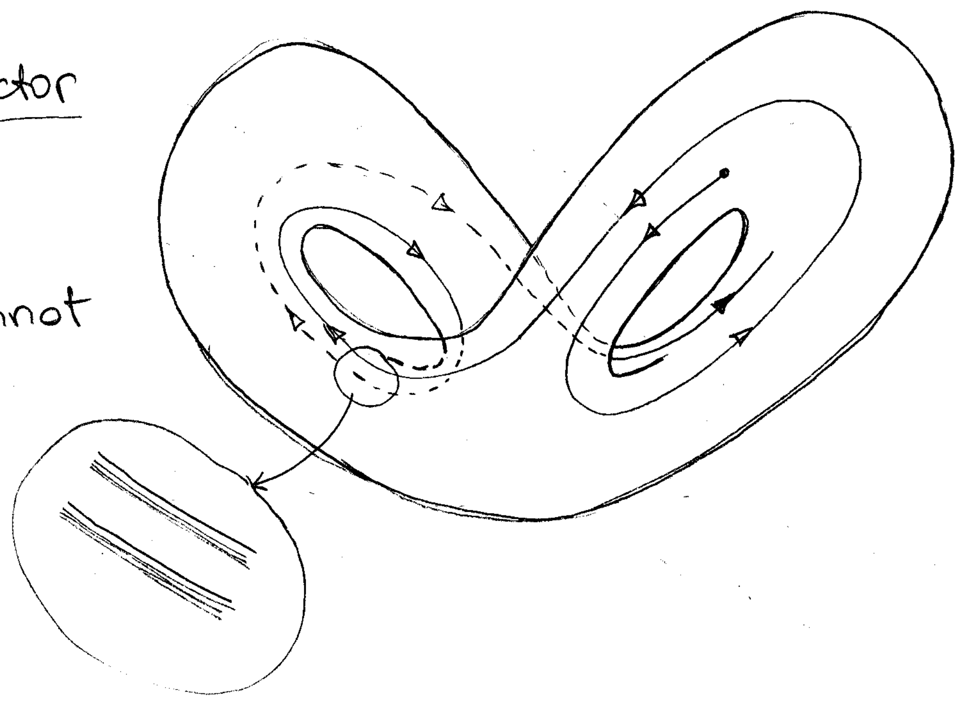


What Happens for $r > r_H$?

Take $\sigma = 10$, $b = 8/3$, $r = 28$ ($r_H = 24.74$)

$\longrightarrow$ Computer experiment

Strange Attractor



Trajectories cannot intersect $\Rightarrow$

Increasing resolution discover that each "leaf" of the attractor consists of infinitely many surfaces very close to each other.

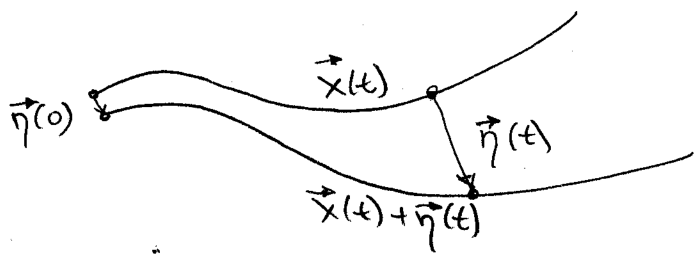
"Fractal" dimension $= 2.05$

2d surface in 3d

not a continuum of surfaces, but infinite number

Exponential Divergence of Nearby Trajectories

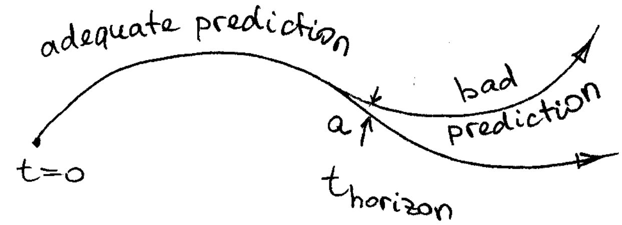
$\rightarrow$ Computer Experiment



$$\|\vec{\eta}(t)\| \sim \underbrace{\|\vec{\eta}(0)\|}_{\eta_0} e^{\lambda t} \Rightarrow \text{Sensitive dependence on initial conditions (for } \lambda>0)$$

$$a \sim \eta_0 e^{\lambda t}$$

$$\Rightarrow t_{horizon} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{\eta_0}\right)$$



$\boxed{\text{Only log-dependence on } \eta_0!}$

Example: $a = 10^{-3}, \lambda = 2$

$$\eta_0 = 10^{-7}: \quad t^{(1)}_{horizon} \approx \frac{1}{2} \ln \frac{10^{-3}}{10^{-7}} = 2 \ln 10$$

$$\eta_0 = 10^{-13}: \quad t^{(2)}_{horizon} \approx \frac{1}{2} \ln \frac{10^{-3}}{10^{-13}} = 5 \ln 10 = 2.5 \cdot t^{(1)}_{horizon}$$

Chaos:

"Operational" definition: Chaos is an aperiodic long term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

- Aperiodic - complex
- Deterministic - nonlinearity, not noise

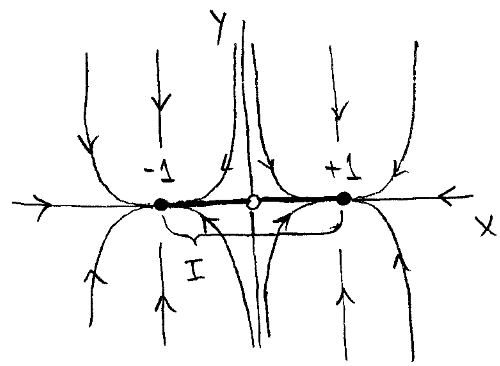
Attractors

Definition: An attractor A is a minimal (indecomposable) attracting invariant set

1) Invariant: any trajectory that starts in A, stays in A
2) Attracting: for any trajectory that starts close to A, the distance from $\vec{x}(t)$ to $A \to 0$, $t \to \infty$
3) Minimal: no proper subset of A satisfies both 1), 2)

Examples:
a) stable fixed point
b) stable limit cycle
c) strange attractor

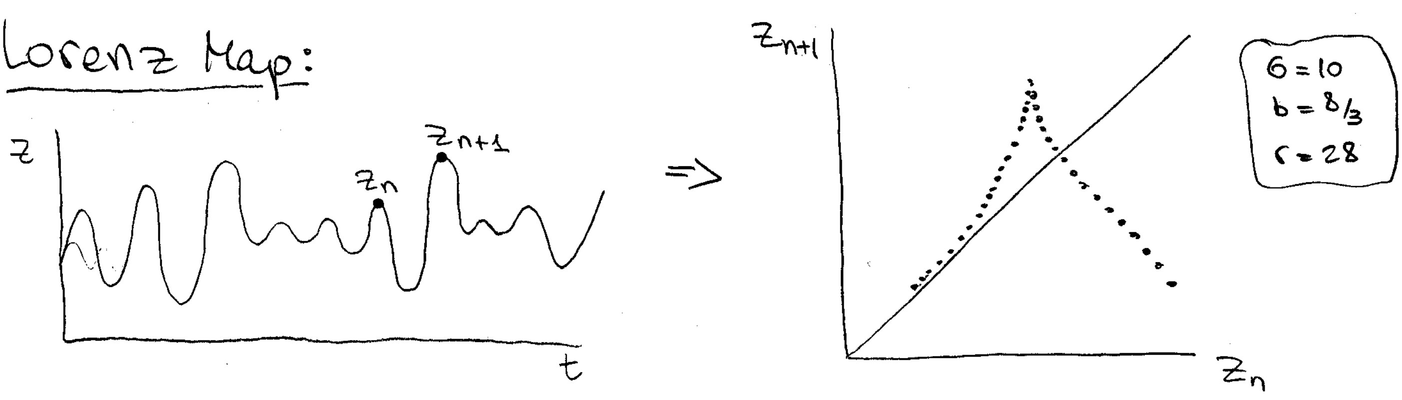
Example:
$$\begin{cases} \dot{x} = x - x^3 \\ \dot{y} = -y \end{cases}$$



The set I is:
1) Invariant (the whole x-axis is)
2) Attracting (globally)
3) Not minimal (fixed points $(\pm 1, 0)$ are)

$\Rightarrow (\pm 1, 0)$ – attractors, I is not!

Lorenz Map:



$z_{n+1} = f(z_n)$

Comments:

a) $f(z)$ is not a curve (infinitely many close curves)

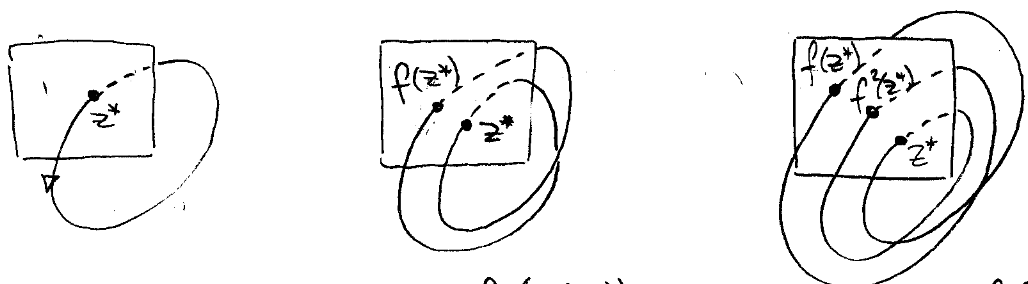
Reason: 3-d dynamical system

$\Rightarrow$ 2-d Poincaré map, say $(x_{n+1}, z_{n+1}) = F(x_n, z_n)$

b) $|f'(z)| > 1$, everywhere

Corollary: No stable limit cycles:

Recall: limit cycle — fixed point of the map



$z^* = f(z^*)$ $\qquad$ $z^* = f(f(z^*))$ $\qquad$ $z^* = f(f(f(z^*)))$

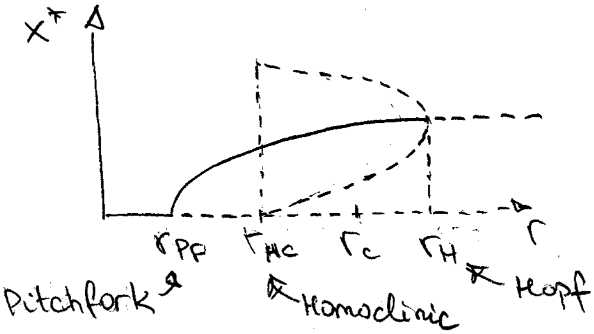
In general: $z^* = \underbrace{f^n}_{\text{composition, not power!}}(z^*) \Leftrightarrow z_n = z_0$

Stability: $z_n = f^n(z^*) + \eta_n \Rightarrow$

$$\eta_n = f'(z_{n-1})\eta_{n-1} = f'(z_{n-1})f'(z_{n-2})\eta_{n-2} = \left[\prod_{k=0}^{n-1} f'(z_k)\right]\eta_0$$

$$|f'(z)| > 1 \Rightarrow \left|\prod_{k=0}^{n-1} f'(z_k)\right| = \prod_{k=0}^{n-1}|f'(z_k)| > 1 \Rightarrow \text{Unstable } \forall n$$

Parameter Space



$0 < r < r_{PF}$: stable 0

$r_{PF} < r < r_{HC}$: stable $C^{\pm}$

$r_{HC} < r < r_c$: stable $C^{\pm}$, transient chaos

$r_c < r < r_H$: stable $C^{\pm}$, strange attractor

$r_H < r$: chaos w/periodic windows