1 Faraday instability

The Faraday instability, produced by vertical oscillation of an incompressible liquid with a free upper surface, makes it possible to explore a variety of issues related to nonlinear pattern formation.

We consider the linear stability of the interface between two incompressible Newtonian fluids. The lower fluid is a liquid layer of large depth h. The upper layer is air whose density and hydrodynamics we neglect. We summarize the main points of the theoretical analysis, following ref. [1].

Elimination of the pressure in the linearized Navier- Stokes equations leads to

$$(\partial_t - \nu\nabla^2)\nabla^2 w = 0 \tag{1}$$

where $w(x, y, z, t)$ and ν are the vertical velocity and kinematic viscosity, respectively. In a reference frame that oscillates vertically with the container and whose origin $z = 0$ coincides with the flat interface, the boundary conditions are

$$w = \partial_z w = 0 \qquad \text{at} \qquad z = -h,\, \infty \tag{2}$$

The interface position is $z = \zeta(x, y, t)$ and is advected by the fluid

$$\partial_t \zeta = w\,. \tag{3}$$

Several conditions must be imposed at the interface. Continuity of velocity and of tangential stress imply that

$$\begin{aligned} w = \partial_z w &= 0 & (4)\\ \rho\nu(\nabla_H^2 - \partial_{zz})w &= 0\,, & (5) \end{aligned}$$

where ∇_H^2 is the horizontal Laplacian evaluated at the interface, and ρ is the fluid density.

Curvature of the interface yields a normal stress discontinuity of $-\sigma\nabla_H^2\zeta$, where σ is the surface tension. The total balance of forces normal to the interface is

$$\rho[\partial_t - \nu(3\nabla_H^2 + \partial_{zz})]\partial_z w = \rho[g + f(t)]\nabla_H^2\zeta - \sigma\nabla_H^4\zeta, \tag{6}$$

where g is the usual gravitational acceleration and $f(t)$ is its effective modulation by the imposed vertical oscillation.

The full hydrodynamic problem would require the interface conditions (4)-(6) to be applied at $z = z(x, y, t)$, but linearization justifies the more tractable use of $z = 0$. The linear system (1)-(5) is homogeneous in the horizontal directions and periodic in time. With the assumption of boundedness (as $x, y \to \infty$), the solutions are of the Floquet-Fourier form

$$w(x,y,z,t) = \sin(k_x x + k_y y) e^{(\mu + i\alpha)t} \sum_n w_n(z) e^{in\omega t} + c.c. \tag{7}$$

$$\zeta(x,y,t) = \sin(k_x x + k_y y) e^{(\mu + i\alpha)t} \sum_n \zeta_n(z) e^{in\omega t} + c.c., \tag{8}$$

where $2\pi/\omega$ is the period of the vertical acceleration $f(t)$, $k = \sqrt{k_x^2 + k_y^2}$ is the horizontal wave number, and $\mu + i\alpha$ the Floquet exponent, with $0 \leq \alpha \leq \omega/2$. The z dependence of w is determined by substituting (7) into (1):

$$w_n(z) = a_n e^{kz} + b_n e^{-kz} + c_n e^{q_n z} + d_n e^{-q_n z} \tag{9}$$

where

$$q_n^2 = k^2 + \frac{\mu + (\alpha + n\omega)t}{\nu}. \tag{10}$$

Equations (2) and (3), which do not couple the different temporal Fourier components, allow the four coefficients in (9) to be expressed in terms of ζ_n. The remaining equation (6) can then be written as

$$A_n \zeta_n = 2[f\zeta]_n, \tag{11}$$

where A_n is an algebraic function of the physical parameters and μ, α, k and $[fz]_n$ is the nth Fourier component of the product of $f(t)$ and $\sum_{n'} \zeta_{n'} e^{in'\omega t}$. With the vertical acceleration given by

$$f(t) = a\cos(\chi)\cos(m\omega t), \tag{12}$$

we have

$$[f\zeta]_n = \frac{a}{2}\cos(\chi)(\zeta_{n-m} + \zeta_{n+m}), \tag{13}$$

with appropriate modifications to ensure that ζ is real and is represented by a truncated Fourier series in (8). Equation (11) can therefore be written as

$$\mathcal{A}\hat{\zeta} = a\mathcal{B}\hat{\zeta}, \tag{14}$$

where $\hat{\zeta}$ is the vector of temporal Fourier coefficients ζ_n, $\mathcal{A}$ is the diagonal matrix of A_n , and $\mathcal{B}$ is a banded matrix containing two super- and subdiagonals.

With $\mathcal{A}$ or $\mathcal{B}$ invertible, Eq. (14) constitutes an eigenvalue problem for the forcing amplitude a and can be solved by calling a standard subroutine, in this case RG from EISPACK. At most 20 temporal Fourier coefficients suffice to resolve $\hat{\zeta}$ for typical calculations such as $(m, l) = (4, 5)$ and $a < 30g$. With fixed physical parameters, the solution of (14) yields a set of amplitudes a that depend on μ, α and k. To obtain marginal stability curves, we fix k and α, set the growth rate μ to 0, and select the smallest real positive α obtained. We find that setting the imaginary part a of the Floquet exponent to any value other than 0 and $\omega/2$ yields only physically unrealizable complex values of α, indicating that the time dependence of the marginal modes is always either harmonic ($\alpha = 0$) or subharmonic ($\alpha = \omega/2$).

References

[1] WS Edwards T Besson and LS Tuckerman. Two-frequency parametric excitation of surface waves. *Phys. Rev.* **E**, 1996.