14
Matter in motion

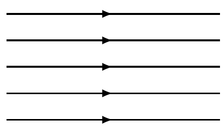
The running water in a brook, the streaming wind and the rolling sea, are all examples of fluids in motion. The richness of fluid motion is most clearly exposed by a picture of a waterfall. It is one of the wonders of nature that all this richness is a consequence of the Newtonian equations of motion for continuous matter. One must marvel at the fact that the simple partial differential equations we can write down with very little effort should contain all the variety of fluid phenomena.

But this fact also warns us that analytic solutions to these equations can only be expected in idealized and highly constrained situations. Nevertheless, when found and then often in severe approximation, such solutions still offer insight into the underlying mechanisms which experiments and computer calculations may carry to the domain of reality.

The motion of solids is generally less rich than fluid motion, and it is precisely for this reason we use solids to build structures like houses, bridges, and machines. But fluids and solids are extremes, and there are many transition materials with properties in between. It is therefore important as far as possible to analyze matter in motion without distinguishing between particular types of matter.

In this chapter the two basic mechanical equations governing the motion of continuous matter will be formulated. One concerns conservation of mass and states that the only way the mass of a volume of matter can change is through flow of material across its surface. The other is nothing but Newton's second law for continuous systems. Together with expressions for the forces at play in the material, these equations form a complete set of mechanical equations of motion for the mass density and velocity fields. In this chapter, we shall "only" apply these equations the whole universe. Later chapters will deal with more earthly aspects.

14.1 The velocity field

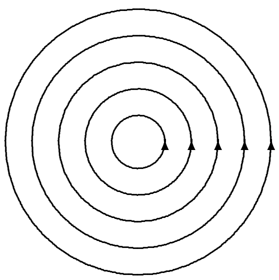


Streamlines for rigid body translation.

The mechanical state of continuous matter is not only described by a mass density field $\rho(\boldsymbol{x}, t)$, but also by a velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, defined such that the total momentum of the material in a small volume dV in the neighborhood of $\boldsymbol{x}$ is

$$\boxed{d\boldsymbol{\mathcal{P}} = \boldsymbol{v}\, dM = \boldsymbol{v}\, \rho\, dV} \ . \tag{14-1}$$

The *density of momentum* is consequently $\rho\boldsymbol{v}$. The definition $\boldsymbol{v} = d\boldsymbol{\mathcal{P}}/dM$ also shows that the velocity field in a point $\boldsymbol{x}$ should be understood as the center-of-mass velocity of the material in a small — though not too small — volume near $\boldsymbol{x}$, *i.e.* of a material particle.



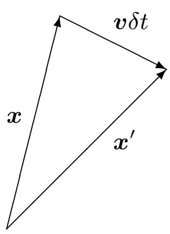
Streamlines for rigid body rotation.

Example 14.1.1: The velocity field for the material of a non-rotating rigid body moving with constant velocity $\boldsymbol{v}_0$ is $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}_0$. If instead the body is rotating around the origin of the coordinate system with constant angular velocity vector $\boldsymbol{\Omega}_0$, the velocity field is $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{\Omega}_0 \times \boldsymbol{x}$.

Flow and displacement

Although the concept of a velocity field is meaningful for all kinds of continuous matter, it is of greatest importance for fluids, where large scale material motion, called *flow*, is the dominant phenomenon. In solids,the dominant feature is *displacement* of material, and the velocity field plays a less fundamental role (see chapter 9).

Flow may on the other hand be viewed as a continual sequence of infinitesimal displacements. In a tiny time interval, δt, the center-of-mass of a material particle is displaced from the point $\boldsymbol{x}$ to



The displacement of a material particle in a small interval of time.

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}(\boldsymbol{x}, t)\, \delta t \ . \tag{14-2}$$

The actual displacement of the particle is given by the vector

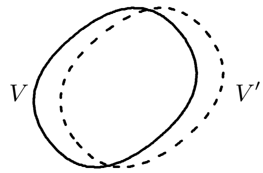
$$\delta\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}, t)\, \delta t \ . \tag{14-3}$$

Repeating such infinitesimal displacements while advancing time, it is at least in principle possible to follow the progress of each and every material particle.

The continuum hypothesis

In solids, where the progress of material particles is reasonably orderly, this makes sense and one may speak meaningfully about the displacement of a definite volume of material over a finite time. In fluids, the paths of material particles tend to become so intertwined with each other that it makes very little sense to speak of the displacement of a volume over longer times. Think, for example, of a bucket of water thrown into a waterfall. The chaotic motion in the fall mixes the original volume of water so thoroughly with the other volumes that it is physical nonsense to speculate on the whereabouts of the original material particles over a longer time interval.

Over sufficiently small time intervals, it makes better sense to speak about the motion of individual material particles in a fluid. In chapter 1 we discussed how the molecules in a small volume of gas are rapidly replaced by other molecules, but concluded that it was still physically meaningful to speak about material particles in a statistical sense. This point of view is sometimes called the *continuum hypothesis*, thus allowing for the possibility that there could be situations in which it breaks down.

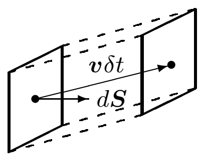


Displacement of a volume of material in a small time interval. The change in volume is given by the thin surface layer between the dashed and fully drawn curves.

Rate of change of comoving volume

In a small time interval δt, all the material particles in a volume V are simultaneously displaced according to (14-2) to fill out another volume V' in the vicinity the original volume. The smallness of the displacement permits us to calculate the change in volume from the small movement of the surface. Since a surface element $d\boldsymbol{S}$ is displaced through the vector distance $\boldsymbol{v}\delta t$, it scoops up a tiny volume $\boldsymbol{v}\delta t \cdot d\boldsymbol{S}$ (counted with sign), so that the total change in the volume V becomes

$$\delta V = V' - V = \oint_S \boldsymbol{v}\delta t \cdot d\boldsymbol{S} \ .$$

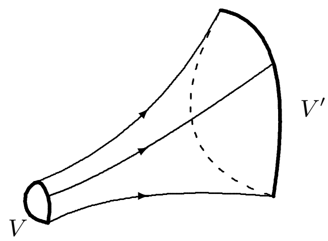


A surface element $d\boldsymbol{S}$ scoops up the volume $\boldsymbol{v}\delta t \cdot d\boldsymbol{S}$ in a small time interval δt.

We shall introduce a special notation for the rate of change $\delta V/\delta t$ of such a *comoving volume*,

$$\boxed{\frac{\delta V}{\delta t} = \oint_S \boldsymbol{v} \cdot d\boldsymbol{S}} \ . \tag{14-4}$$

The reason for the special notation is that this is not an ordinary time derivative, and we shall soon meet a number of similar expressions for the rate of change in other quantities following the flow of matter. The operator $\delta/\delta t$ is for this reason called the *material time derivative* (see also page 227) .



A comoving volume expands in a diverging velocity field, and contracts in a converging field.

Example 14.1.2: Let $\boldsymbol{v} = \kappa\boldsymbol{x}$ be a uniformly expanding velocity field. Then the comoving rate of change of the volume of a sphere of radius r centered at the origin becomes

$$\frac{\delta V}{\delta t} = \oint_{|\boldsymbol{x}|=r} \kappa\boldsymbol{x} \cdot d\boldsymbol{S} = 4\pi\kappa r^3 \ , \tag{14-5}$$

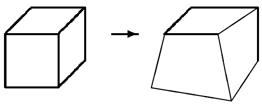
because the surface area of the sphere is $4\pi r^2$ and its normal points along $\boldsymbol{x}$.

By means of Gauss' theorem (6-3) the surface integral may be converted to a volume integral,

$$\frac{\delta V}{\delta t} = \int_V \mathsf{div}\, \boldsymbol{v}\, dV \ . \tag{14-6}$$

We may interpret the integrand as the rate of change of the volume dV of a material particle (which by definition is comoving)

$$\boxed{\frac{\delta(dV)}{\delta t} = \mathsf{div}\,\boldsymbol{v}\,dV} \;. \tag{14-7}$$



A material particle is also deformed by the flow because the velocity field varies from place to place within it.

Thus the volume of a material particle swells and shrinks according to the divergence or convergence of the velocity field. In problem 14.1 this relation is proven directly without recourse to Gauss' theorem.

Incompressible flow

Most liquids are rather incompressible under ordinary circumstances and, as we shall see later, gases may often be taken to be effectively incompressible when flow speeds are much less than the velocity of sound. One should, however, not forget that all materials may in principle be compressed. *Incompressibility* is always an approximation, and should for this reason rather be viewed as a property of the flow than of the fluid.

Incompressibility means that material particles can neither expand nor contract, and comparing with (14-7) it follows that the divergence of the velocity field must vanish,

$$\boxed{\mathsf{div}\,\boldsymbol{v} = 0} \;. \tag{14-8}$$

Equivalently, we may use the global equation (14-4) and find

$$\boxed{\oint_S \boldsymbol{v}\cdot d\boldsymbol{S} = 0} \;. \tag{14-9}$$

This equation therefore tells us that the net rate of material flux through the surface S must vanish. In other words, incompressible fluid cannot accumulate anywhere, but will flow right through any closed surface.

Example 14.1.3: The flow described by the stationary velocity field

$$\boldsymbol{v} = (\sin x \cos y, -\cos x \sin y, 0) \tag{14-10}$$

is incompressible, because

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \cos x \sin y - \cos x \sin y = 0 \tag{14-11}$$

Due to the periodicity in both x and y, the flow forms regular array of stationary whirls, one of which is shown in fig. 14.1. There is probably no practical way of generating this pattern.

Leonardo's law

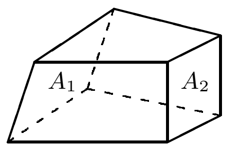
Already Leonardo da Vinci knew — and used — that the water speed decreases when a canal or river becomes wider or deeper and increases when it becomes narrower or shallower [6]. He discovered the simple law that the product of the cross-sectional area of a canal and the flow velocity in the canal is everywhere the same.

Leonardo da Vinci (1452–1519). Italian renaissance artist, architect, scientist and engineer. A universal genius that made fundamental contributions to nearly all fields. Also a highly practical man who concerned himself with the basic mechanical principles behind everyday machines, and sometimes also future machines, such as the helicopter.

Consider, for example, an aqueduct or canal and mark two fixed planar cross sections A_1 and A_2, both orthogonal to the general direction of flow. Leonardo's law then says that the (average) water velocities v_1 and v_2 through these cross sections must obey the relation

$$A_1 v_1 = A_2 v_2 \ . \tag{14-12}$$

The law expresses the rather self-evident fact that the same volume of incompressible water has to pass any cross section of the canal per unit of time. In practice we use it unconsciously all the time when dealing with water.



Aqueduct with varying cross section. The volume of water passing through the cross section A_1 is the same as the amount of water passing through A_2, or $A_1v_1 = A_2v_2$ where v_1 and v_2 are average flow velocities.

Formally, Leonardo's law follows from the global condition of incompressibility (14-9). Together with the sides of the canal, the two cross sections define a volume, to which the condition can be applied. Since no water can flow through the sides of the canal, the surface integral only receives contributions from the two cross sections, and consequently, taking both normals along the general direction of the canal, we get

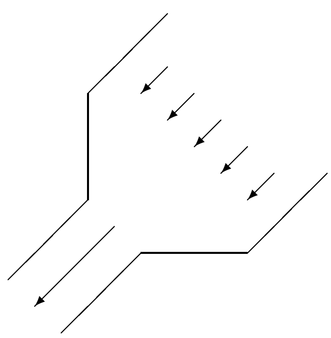
$$\oint \boldsymbol{v} \cdot d\boldsymbol{S} = \int_{A_2} \boldsymbol{v} \cdot d\boldsymbol{S} - \int_{A_1} \boldsymbol{v} \cdot d\boldsymbol{S} = \boldsymbol{0} \ . \tag{14-13}$$

The average flow velocity through a cross section A of the canal is

$$v_A = \frac{1}{A} \int_A \boldsymbol{v} \cdot d\boldsymbol{S} \ . \tag{14-14}$$

It then follows from (14-13) that the product of the area of any planar cross section and the average flow velocity through the cross section, $A\, v_A$, is the same everywhere along the canal, *i.e.* Leonardo's law.

> **Example 14.1.4:** A hypodermic syringe contains a few cc's of liquid in a small chamber about 1 cm in diameter and a few centimeters long (unless it is meant for a horse, where the dimensions are much larger). The liquid is injected through a hollow needle with an inner diameter of about 1 mm in the course of a few seconds. Since the ratio of cross sections is 100, Leonardo's law tells us that the speed of the liquid in the needle is about 100 times larger than the speed with which the piston of the syringe is pushed, *i.e.* of the order of meters per second.



The flow velocity in the thin needle of a syringe is much higher than in the liquid chamber.

Leondardo's law is *absolutely not* valid for compressible fluids. When you pump your bicycle by pushing a piston into a cylindric chamber filled with air, the average flow velocity will decrease towards the end of the chamber where it has to vanish, because no air can pass through there. But the cross section of the chamber is constant, so the product of cross section and average velocity is *not* constant throughout the chamber.

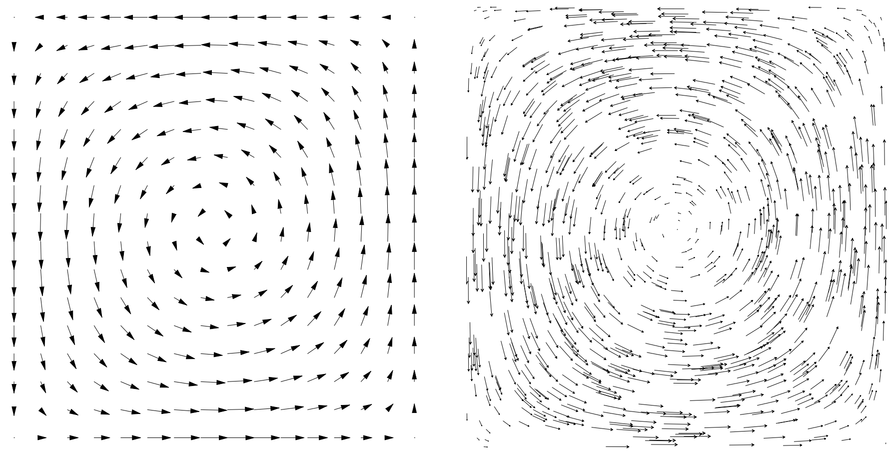


Figure 14.1: *Regular (left) and random (right) plots of little arrows for the incompressible flow* $\boldsymbol{v} = (\sin x \cos y, -\cos x \sin y, 0)$ *in the region* $0 \leq x \leq \pi$ *and* $0 \leq y \leq \pi$ *(see also example 14.1.3). The regular plot shows the velocity vector in the 256 points of a Cartesian* 16×16 *grid, whereas the random plot is created by selecting 1000 points randomly. The skewness seen in the regular plot is a graphical artifact. The whirl (and its mirror images) are repeated periodically throughout the xy-plane.*

14.2 Visualization of flow patterns

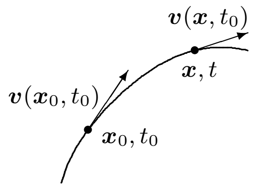
Wind and water currents are normally invisible unless polluted by foreign matter. A gentle breeze in the air can be observed from the motion of dust particles dancing in the sunshine or the undulations of smoke from a cigarette. Even a tornado first becomes visible when water vapor condenses near it center or debris is picked up and thrown around. Modern technology does on the other hand permit us indirectly to "see" velocity fields. Doppler radar is used for tracking and visualizing damaging winds in violent storms, and likewise, Doppler acoustics is used to visualize blood flow in the heart.

Little arrows: In computational fluid dynamics, the velocity field is calculated numerically on a grid of points in space, and the instantaneous velocity field, or rather the displacement in a small time interval, is often visualized by means of little arrows attached to the grid points, each of a length and direction proportional to the velocity field in the point (see Fig. 14.1). Sometimes it is more illustrative to draw the arrows from a random selection of points because the random plot permits longer arrows and a higher density which makes the picture more dramatic.

Streamlines: In section 3.2 we discussed how the gravitational field could be visualized by means of field lines, defined to be curves that everywhere had the gravitational field at a fixed time as tangent. Similar field lines, called *streamlines*, can be defined for the velocity field as curves that are everywhere tangent to the velocity field at a fixed time. Such curves are solutions to the ordinary differential equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}, t_0) \ , \tag{14-15}$$

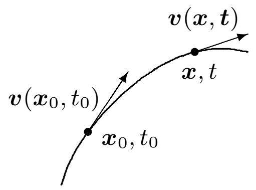
where the velocity field is calculated for a fixed value of time t_0. Starting in any point $\boldsymbol{x}_0$ at $t = t_0$ we may use this equation to determine the path $\boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{x}_0, t_0)$ of a streamline. Because the velocity field is evaluated at a fixed moment in time t_0 there will be only one tangent and thus only one streamline through every point of space. Streamlines depict the velocity field at a single instant in time and can never intersect.



A streamline is everywhere tangent to the velocity field at a given time t_0.

Particle orbits: Imagine you drop a tiny particle — a speck of dust — into a fluid, and watch how it is carried along with the fluid in its motion. The speck of dust should be so small that its inertia and mass plays no role, but on the other hand so large that it is not buffeted much around by collisions with individual molecules (*i.e.* Brownian motion). The path $\boldsymbol{x} = \boldsymbol{x}(t)$ which it follows is called a *particle orbit* and is determined by the differential equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}, t) \ . \tag{14-16}$$



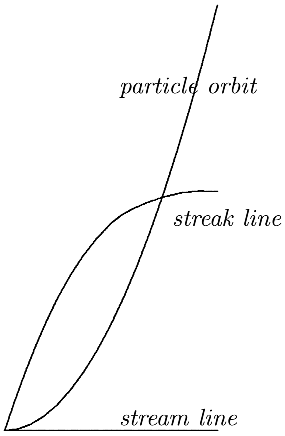
A particle orbit is everywhere parallel with the instantaneous velocity field.

Given a starting point $\boldsymbol{x}_0$ and a starting time t_0, the path $\boldsymbol{x}(t, \boldsymbol{x}_0, t_0)$ may be calculated from this equation for all times t. There will be only one particle orbit going through each point in space at a fixed instant, but different orbits may cross each other and even themselves as long as it happens at different times.

Streaklines: A standard method for visualizing fluid flow, in for example wind tunnels, is to inject smoke (or dye) into the fluid at a constant rate, as was done by every industrial smokestack before factories were required to scrub their exhaust fumes. This leads to long *streaklines* of smoke weaving through the fluid. Since smoke particles are tiny they must follow particle orbits, so that a streakline is obtained from the particle orbit $\boldsymbol{x}(t, \boldsymbol{x}_0, t_0)$ by varying the start time t_0 while keeping fixed the observation time t and the point $\boldsymbol{x}_0$ from which the smoke emerges.

Relating the various flow lines

For a time-dependent velocity field the relationship between the three types of lines can be hard to imagine. Think for example of the smoke trail from a chimney or the contrails of a jet when the winds are changing (problem 14.3). For *steady flow*, where the velocity field is independent of time, $\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{x})$, the particle orbits evidently coincide with the streamlines, and since the streamlines in this case can only depend on the time difference $t - t_0$, the streaklines will also coincide with them. One should always remember that streamlines are quite misleading for unsteady flow. Even if the streamlines look nice as for example in an ocean surface wave, the actual particle orbits may be quite chaotic.



The three types of flow lines from example 14.2.1.

Example 14.2.1: To expose the difference between the three types of field lines consider a spatially uniform two-dimensional velocity field of the form $\boldsymbol{v}(\boldsymbol{x}, t) = (a, bt, 0)$ where the y-component everywhere grows linearly with t. The stream lines are in this case determined by

$$\frac{d\boldsymbol{x}}{dt} = (a, bt_0, 0) , \tag{14-17}$$

and are straight lines

$$x = x_0 + a(t - t_0) , \tag{14-18}$$

$$y = y_0 + bt_0(t - t_0) . \tag{14-19}$$

The particle orbits are determined by

$$\frac{d\boldsymbol{x}}{dt} = (a, bt, 0) , \tag{14-20}$$

and are parabolas

$$x = x_0 + a(t - t_0) \tag{14-21}$$

$$y = y_0 + \frac{1}{2}b(t^2 - t_0^2) \tag{14-22}$$

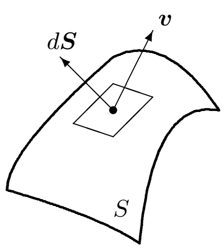
Varying t_0 in the interval $-\infty < t_0 < t$ the streak lines are seen also to be parabolas curving the opposite way of the particle orbits.

Taking $x_0 = y_0 = 0$ and $a = b = 1$, the streamline at $t_0 = 0$ runs along the x-axis, $y = 0$, and the particle orbit becomes $y = x^2/2$. At a given moment of time t, the corresponding streakline, obtained by varying the start time in the interval $0 < t_0 < t$, is described by $y = x(t - x/2)$ with maximum at $x = t$.

14.3 Mass conservation

In Newtonian mechanics, mass is *conserved*. The mass of a collection of point particles ("molecules") can only change by addition or removal of particles. Since all matter is made from molecules, this must mean that the only way the mass in a given volume of continuous matter can change, is by mass flowing in or out of the volume through its surface. This almost trivial remark leads to the first of the two central equations of continuum dynamics, the *equation of continuity.*

It is perhaps in place to remark that mass conservation is only valid in classical, non-relativistic physics. In relativity, mass is equivalent to energy through Einstein's famous formula, $E = mc^2$, and only the total energy of a closed system is conserved.



Matter moves with velocity $\boldsymbol{v}(\boldsymbol{x}, t)$ through every surface element $d\boldsymbol{S}$ near $\boldsymbol{x}$.

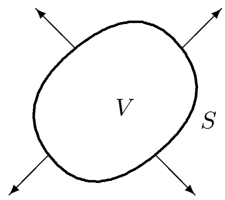
Flux of mass

Consider a piece of a fixed open or closed surface, S. The (signed) amount of mass passing through a tiny surface element $d\boldsymbol{S}$ in the time interval dt will be $\rho\boldsymbol{v}dt\cdot d\boldsymbol{S}$. Integrating over the whole surface and dividing by dt, we find the total amount of mass transported through S per unit of time,

$$Q = \int_S \rho \, \boldsymbol{v} \cdot d\boldsymbol{S} \; , \tag{14-23}$$

also called the *flux of mass* through the surface S. According to this formula, the density of momentum, $\rho\boldsymbol{v}$, also specifies how much mass that flows through a unit area in a unit of time. It may therefore alternatively be viewed as the *current density of mass*.

Example 14.3.1: Water is discharged from a fixed hose at a rate of $Q = 1$ kg/s. The hose has cross section $A = 1$ cm^2, so that the average current density at the exit of the tube is $\langle \rho v \rangle = Q/A = 10,000$ kg/m^2/s. Since the water density is constant, $\rho = 1000$ kg/m^2, the average flow speed is $\langle v \rangle = 10$ m/s. Directing the water hose vertically upwards, the water will reach a height of $h \approx \langle v \rangle^2/2g_0 = 5$ m.



The only way the amount of mass can diminish in a fixed volume V is by a net outflow through its surface S, and conversely.

Global mass conservation

Since mass can neither be created nor destroyed, the rate of gain of mass in a volume, V, must be balanced by the rate of loss through its surface S, *i.e.*

$$\boxed{\frac{d}{dt} \int_V \rho \, dV + \oint_S \rho \, \boldsymbol{v} \cdot d\boldsymbol{S} = 0} \; . \tag{14-24}$$

This is the *global equation of mass conservation* for an arbitrary fixed control volume. It will be generalized to moving control volumes in chapter 16.

If mass was not conserved, the right hand side would be non-vanishing and contain *sources* accounting for the rate of mass creation and destruction (see problem 14.9). In chapter 16, we shall meet other equations of the same general form with non-vanishing sources, namely the transport equations for momentum, energy, and angular momentum.

The equation of continuity

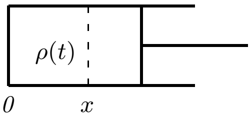
Since the control volume is fixed, we may pass the time derivative in through the volume integral and use Gauss' theorem (6-3) on the resulting surface integral, to get

$$\frac{d}{dt} \int_V \rho \, dV + \oint_S \rho \, \boldsymbol{v} \cdot d\boldsymbol{S} = \int_V \left(\frac{\partial \rho}{\partial t} + \mathsf{div} \, (\rho \boldsymbol{v}) \right) dV = 0 \; .$$

Since this has to be true for any volume V, we conclude that mass conservation requires

$$\boxed{\frac{\partial \rho}{\partial t} + \mathsf{div} \, (\rho \boldsymbol{v}) = 0} \; , \tag{14-25}$$

all over space and time. This is the *equation of continuity*, first obtained by Euler (1757). Although derived from global mass conservation by specializing to a fixed volume, it is itself a local relation completely without reference to macroscopic volumes.



Bicycle pump. The velocity field varies linearly with x.

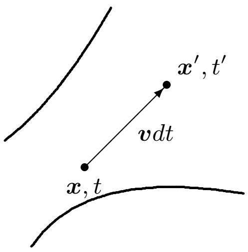
Example 14.3.2: A piston is pushed *slowly* into a gas-filled cylindric chamber (a bicycle pump), such that the density at time t is $\rho(t)$ everywhere in the chamber. It is reasonable to assume that only the x-component of the velocity field is non-vanishing, and from the equation of continuity we find

$$\frac{\partial v_x}{\partial x} = -\frac{\dot{\rho}(t)}{\rho(t)} \ .$$

The solution which vanishes for $x = 0$ is $v_x = -x\dot{\rho}(t)/\rho(t)$, and varies linearly with the distance x along the chamber as you might have guessed.

Using that $\mathsf{div}\,(\rho \boldsymbol{v}) = \rho \mathsf{div}\,\boldsymbol{v} + \boldsymbol{v} \cdot \nabla\rho$ the equation of continuity may be written

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho = -\rho\, \mathsf{div}\, \boldsymbol{v} \ . \tag{14-26}$$



A particle may be swept along with the flow into a region of different density and velocity.

If the density is constant everywhere and at all times, *i.e.* $\rho(\boldsymbol{x}, t) = \rho_0$, then the left hand side vanishes, and it follows that the material is incompressible, $\mathsf{div}\, \boldsymbol{v} = 0$, as one would expect. But the opposite is not necessarily the case. If the material is incompressible so that the divergence vanishes, the expression on left hand side must also vanish, but that does apparently not mean that the density has to be constant. We shall now see, what it really means.

Moving along with the flow

How does the world look from the point of view of a small object riding along with the motion of the material? A speck of dust being sucked into a vacuum cleaner will find itself in a region with higher air velocity and lower pressure and density than outside, even if the flow of air is completely steady with air velocity, pressure and density being constant in time everywhere (because you have stopped moving the cleaner). The ambient flow of matter may thus contribute to changes in physical quantities in the neighborhood of *comoving* particles.

A particle near the point $\boldsymbol{x}$ at time t riding along with the flow will at time $t' = t + \delta t$ have been displaced to the point $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}(\boldsymbol{x}, t)\delta t$. Expanding to first order in δt we find the change in mass density

$$\begin{aligned}
\delta\rho &= \rho(\boldsymbol{x} + \boldsymbol{v}\delta t, t + \delta t) - \rho(\boldsymbol{x}, t) \\
&= v_x \delta t \frac{\partial \rho(\boldsymbol{x}, t)}{\partial x} + v_y \delta t \frac{\partial \rho(\boldsymbol{x}, t)}{\partial y} + v_z \delta t \frac{\partial \rho(\boldsymbol{x}, t)}{\partial z} + \delta t \frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} \\
&= \left(\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho \right) \delta t \ .
\end{aligned}$$

This shows that the *comoving* rate of change of density is

$$\boxed{\frac{\delta \rho}{\delta t} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho} \ . \tag{14-27}$$

Returning to the question of incompressibility, we see that setting $\mathsf{div}\,\boldsymbol{v} = 0$ in (14-26) forces the comoving rate of change to vanish. In other words, the density must be constant near a particle that moves along with the incompressible flow.

Example 14.3.3: A rigid body, for example a boulder rolling down a mountainside, is by definition incompressible. Although its mass density may vary according to the mineral composition, it stays — of course — the same in the neighborhood of any particular mineral grain, independently of how the boulder rolls.

Constant mass of material particle

A material particle has mass $dM = \rho\, dV$. We know from the discussion in chapter 1 that the molecules of a material particle are rapidly replaced by other molecules, but the mass ought nevertheless to be constant (within the chosen precision) in the continuum description. The rate of change of the mass can be calculated from the comoving rate of change (14-27) of the density and the comoving volume rate of change (14-7), and we find as expected,

$$\begin{aligned}\frac{\delta(dM)}{\delta t} = \frac{\delta(\rho\, dV)}{\delta t} &= \frac{\delta\rho}{\delta t}\, dV + \rho\,\frac{\delta(dV)}{\delta t}\\ &= \left(\frac{\partial\rho}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\rho\right) dV + \rho\,\mathsf{div}\,\boldsymbol{v}\, dV\\ &= 0\ .\end{aligned}$$

In the last step we used the equation of continuity (14-26).

Material time derivative

The differential operator

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} \tag{14-28}$$

is called the *material time derivative* (also often denoted D/Dt). It is a mixed time and space differential operator, which may be applied to any field, for example also the trivial vector field $\boldsymbol{x}$, for which we get,

$$\frac{\delta\boldsymbol{x}}{\delta t} = (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x},t)\ .$$

The material derivative of the velocity itself becomes the *comoving acceleration*,

$$\boldsymbol{w} = \frac{\delta\boldsymbol{v}}{\delta t} = \frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\ , \tag{14-29}$$

The non-linear term $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is often called the *convective* term or the *inertia* term, because it stems from the material environment that sweeps the particle along. For *steady flow* where $\partial\boldsymbol{v}/\partial t = \boldsymbol{0}$, this is the only cause of acceleration. It is most important for fluids where large-scale flow with strong spatial variation in velocities may occur, whereas for small amplitude vibrations, *i.e.* sound waves in solids or fluids, it is normally negligible (see section 15.2).

14.4 Continuum dynamics

Newton's second law for an arbitrary material particle with mass $dM = \rho\, dV$ becomes

$$dM\boldsymbol{w} = d\boldsymbol{\mathcal{F}} \;, \tag{14-30}$$

where $\boldsymbol{w}$ is the comoving acceleration (14-29) and $d\boldsymbol{\mathcal{F}}$ is the force acting on the particle. It was shown in section 8.4, that in mechanical equilibrium the force on a material particle can be written as

$$d\boldsymbol{\mathcal{F}} = \boldsymbol{f}^* \, dV \tag{14-31}$$

where $\boldsymbol{f}^*$ is the *effective force density*, defined in (8-20). Dividing (14-30) by dV, we get

$$\boxed{\rho \frac{\delta \boldsymbol{v}}{\delta t} = \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \right) = \boldsymbol{f}^*} \;, \tag{14-32}$$

also called *Cauchy's equation of motion* (1827). Together with the continuity equation (14-25), this equation governs the dynamics of all continuous systems.

Different types of materials, solids, fluids, and all kinds of transition forms, are characterized by different expressions for the effective force density, in particular the part due to contact forces. The last two and a half centuries of continuum physics have essentially "only" been an exploration of the rich ramifications of this dynamical equation.

Field equations of motion

The equation of continuity (14-25) and the above dynamic equation may be written in the form of *equations of motion* for the four field components, ρ and $\boldsymbol{v}$,

$$\frac{\partial \rho}{\partial t} = -\mathsf{div}\,(\rho \boldsymbol{v}) \;, \tag{14-33a}$$

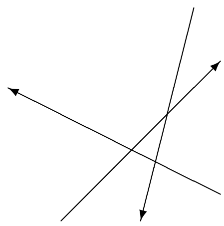
$$\frac{\partial \boldsymbol{v}}{\partial t} = -(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \frac{1}{\rho}\boldsymbol{f}^* \;. \tag{14-33b}$$

Knowing the density and velocity fields at a given time together with the effective body force density (which usually also depends on these fields), the above equations allow us to calculate the rate of change of the fields. If the forces depend on non-mechanical fields, for example temperature, special equations of motion are also needed for those fields to make the system complete.

In computational continuum mechanics, continuous fields are implemented on a finite grid of points covering the system of interest, and then the equations of motion are used to step the fields forward (or backward) in time by a small amount. In practice, computational continuum mechanics is quite demanding on computers as well as on human ingenuity [38].

14.5 Little bangs and Big Bang

A cloud of non-interacting particles, grains or fragments, is perhaps a poor model for continuous matter, but it is nevertheless of interest to study the equation of motion for the velocity field in this most simple case where all volume and contact forces are absent. It may even be used as a primitive model for the expanding universe with galaxies playing the role of grains. But the lack of interaction violates the continuum conditions discussed in chapter 1, and no dynamic smoothing of the fields will occur. A grain given a certain initial velocity will like a ghost continue unhindered with the same velocity through the cloud for all time, and thus have infinite mean free path. The model should definitely be taken with a grain of salt, if not a whole cloud.



In a cloud of free particles all particles move on straight lines with constant velocity for ever and ever.

Explosion field

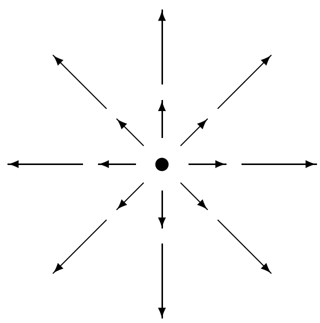
Suppose that the cloud is created in an explosion where the fragments have stopped interacting immediately after and now move freely away from each other. Since there are neither contact forces nor body forces, $\boldsymbol{f}^* = \mathbf{0}$, and every material particle must be unaccelerated. Consequently, the comoving acceleration (14-29) must vanish everywhere,

$$\boxed{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = 0} \,. \tag{14-34}$$

In spite of this being the simplest possible dynamical equation, it looks complicated enough, and if presented with it without any other explanation, we would have some difficulty solving it because of its non-linearity.

Underneath, we know, however, that it only implements the law of inertia, with all particles moving at constant velocity along straight lines. If the explosion happened in the point $\boldsymbol{x} = \mathbf{0}$ at time $t = 0$, the fragments were almost instantly given random velocities in all directions. After the explosion, the fragments will be separated according to their velocities with the fastest fragments being farthest away. At time t, the velocity of any fragment found at $\boldsymbol{x}$ must be

$$\boxed{\boldsymbol{v}(\boldsymbol{x}, t) = \frac{\boldsymbol{x}}{t}} \,, \tag{14-35}$$



Explosion fragments get separated according to initial velocity, because those fragments that accidentally get the largest initial velocity will ever after be farthest away from the center.

independently of how the explosion started out. To see that this indeed satisfies (14-34), we calculate the x-component

$$\begin{aligned}\frac{\partial \boldsymbol{v}_x}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}_x &= \frac{\partial (x/t)}{\partial t} + \frac{x}{t}\frac{\partial (x/t)}{\partial x} + \frac{y}{t}\frac{\partial (x/t)}{\partial y} + \frac{z}{t}\frac{\partial (x/t)}{\partial z} \\ &= -\frac{x}{t^2} + \frac{x}{t^2} \\ &= 0 \,,\end{aligned}$$

and similarly for the y and z components.

Hubble's law

The explosion field (14-35) is of the same form as Hubble's law, which states that all galaxies move away from us (and each other) with speeds that are proportional to their distances, usually written as $\boldsymbol{v} = H\boldsymbol{x}$. In general relativity, this is understood as a consequence of a uniform expansion of space itself since the initial Big Bang. Comparing with (14-35), the constant of proportionality H, called Hubble's constant, is seen to be a measure of the (inverse) age of the Universe $H = 1/t$. Although first determined in 1924, it has been very difficult for astronomy to settle on a reliable experimental value for the Hubble constant. The best current value [**?**] seems to be around $H \approx 55$ km/s/Mpc $= 1.78 \times 10^{-18}\ \mathsf{s}^{-1}$ where one megaparsec (Mpc) is a little more than three million light years. The inverse comes to $1/H \approx 18$ billion years.

This is, however, an overestimate for the age of the universe. A more correct calculation of the expansion of the universe since the Big Bang must take into account the slowing-down due to gravitation (see section 14.6). The galactic recession velocities that are observed today are in fact lower than earlier in the history of the universe, and the actual age of the universe is smaller than $1/H$. If gravitation is barely able to turn the expansion into a contraction again, the age of the universe becomes $2/3H \approx 12$ billion years. Recent observations indicate that the age is at least 12.5 billion years [31].

∗ The mass density

The mass density just before the explosion at $t = 0$ does not matter much for what happens later. The explosion is a cataclysmic event where large unknown forces distribute essentially random velocities to all the fragments of the body that existed before the explosion.

After the explosion these fragments get separated according to velocities, as described by the explosion field (14-35). Let us assume that the probability that a fragment gets a velocity $\boldsymbol{v}$ in a small neighborhood $d^3\boldsymbol{v} = dv_x dv_y dv_x$ is $f(\boldsymbol{v})\, d^3\boldsymbol{v}$. This probability is of course normalized,

$$\int f(\boldsymbol{v})\, d^3\boldsymbol{v} = 1 \ . \tag{14-36}$$

Since the fragments with velocity $\boldsymbol{v}$ will be found at $\boldsymbol{x} = \boldsymbol{v}\, t$ at time t, we have $dV = d^3\boldsymbol{x} = t^3 d^3\boldsymbol{v}$, and the mass density becomes

$$\rho(\boldsymbol{x}, t) = \frac{M_0}{t^3} f\left(\frac{\boldsymbol{x}}{t}\right) \ , \tag{14-37}$$

where M_0 is the mass of the body that originally exploded. It is not hard to show that this density satisfies the equation of continuity (14-25) (problem 14.5).

14.6 Newtonian cosmology

Continuing our investigation of the expanding universe , we now wish to include the gravitational field in the dynamical equation (14-32), but still no contact forces, so that $\boldsymbol{f}^* = \rho \boldsymbol{g}$. The equation of motion becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} = \boldsymbol{g} \ , \tag{14-38}$$

instead of (14-34). In spite of being non-relativistic, this model captures essential elements of cosmology, although a proper understanding of course requires general relativity [13].

Cosmic democracy

In days of old, Earth was thought to be at the center of the universe. Since Copernicus, this thinking has been increasingly replaced by the more "democratic" view that the Earth, the Sun, the Galaxy are but common members of the universe of no particular distinction (except that we live here!). The end of this line of thought is the extreme Copernican view that for cosmological considerations every place in the universe is as good as any other. As we shall see, this "Cosmological Principle" or "Principle of Cosmic Democracy" is quite useful.

Nicolaus Copernicus (1473–1543), Polish astronomer who (literally) revolutionized the understanding of the planetary system in the book De revolutionibus orbium coelestium.

Mass density: It immediately follows from this principle that at a particular instant of time t, the (average) mass density cannot depend on where you are, and must thus be the same everywhere,

$$\boxed{\rho(\boldsymbol{x}, t) = \rho(t)} \ . \tag{14-39}$$

Velocity field: The Hubble expansion of the universe,

$$\boxed{\boldsymbol{v}(\boldsymbol{x}, t) = H(t)\, \boldsymbol{x}} \ , \tag{14-40}$$

with a time-dependent Hubble "constant", $H(t)$, does not look "democratic" because it seems to single out the center of the coordinate system. It is in fact completely democratic, because

$$\boldsymbol{v}(\boldsymbol{x}, t) - \boldsymbol{v}(\boldsymbol{y}, t) = H(t)(\boldsymbol{x} - \boldsymbol{y}) \ . \tag{14-41}$$

This means that an observer in a galaxy at the point $\boldsymbol{y}$ will also see the other galaxies recede from him according to the same Hubble law as ours.

Field of gravity: Can gravity be democratic? In Newtonian cosmology it is not possible to view the universe as a homogeneous whole when it comes to gravity. What is gravity in an infinite universe? Symmetry would seem to argue that it should vanish, because there is as much matter pulling from one side as from the opposite, but is that right?

To overcome this problem, let us for a while think of the universe as a huge sphere with vacuum outside and centered somewhere, perhaps right here, and let us put the origin of the coordinate system at the center of this sphere. In that case, we have seen in section 6.2 that the strength of gravity at a given point $\boldsymbol{x}$, depends only on the amount of mass, $M(r) = \frac{4}{3}\pi r^3 \rho$, inside the sphere with radius $r = |\boldsymbol{x}|$, whereas one may forget the mass outside this radius. In other words, the field of gravity is

$$\boxed{\boldsymbol{g}(\boldsymbol{x}, t) = -\frac{4}{3}\pi G \rho(t)\, \boldsymbol{x}} \ . \tag{14-42}$$

Interestingly, by the same argument as for the velocity, an observer in another galaxy at $\boldsymbol{y}$ will see a similar gravitational acceleration field around himself

$$\boldsymbol{g}(\boldsymbol{x}, t) - \boldsymbol{g}(\boldsymbol{y}, t) = -\frac{4}{3}\pi G \rho(t)(\boldsymbol{x} - \boldsymbol{y}) \ , \tag{14-43}$$

as if he/she/it were also at the center of the universe. This observer may, however, not think of his universe as a huge sphere centered on himself, but must concede that our galaxy is special, at least as long as he subscribes to Newtonian physics. In general relativity, this problem happily goes away.

Cosmological equations

Using that $\mathsf{div}\, \boldsymbol{x} = 3$, we obtain from (14-40) and the equation of continuity (14-25)

$$\boxed{\dot{\rho} = -3H\rho} \ , \tag{14-44}$$

where as before a dot denotes differentiation after time. Similarly, inserting (14-40) into (14-38) and using that $(\boldsymbol{x}\cdot\boldsymbol{\nabla})\boldsymbol{x} = \boldsymbol{x}$, this equation becomes after removal of a common factor $\boldsymbol{x}$

$$\boxed{\dot{H} + H^2 = -\frac{4\pi}{3} G \rho} \ . \tag{14-45}$$

Newtonian cosmology thus reduces to just two coupled ordinary differential equations for the mass density and the Hubble “constant”. Notice that the reference to the center of the universe has disappeared completely, and we may from now on again think of a truly infinite universe with equal rights for all observers, independent of position, occupation, creed, race, and number of tentacles.

The cosmic scale factor

The simplest way to solve these equations is by introducing a new quantity with the dimension of length, $a(t)$, called the *cosmic scale factor*, satisfying

$$\dot{a} = Ha \ . \tag{14-46}$$

From the equation of continuity (14-44) we get

$$\frac{d}{dt}(\rho a^3) = -3H\rho\, a^3 + \rho\, 3a^2 H a = 0 \;, \tag{14-47}$$

and this shows that the mass, $M = \frac{4}{3}\pi\rho(t)a(t)^3$, in an expanding sphere of radius $a(t)$ is constant in time. Eliminating H from (14-45), we obtain the following differential equation for cosmic scale factor,

$$\ddot{a} = -G\frac{M}{a^2} \;, \tag{14-48}$$

which is identical to the equation of motion for a particle moving radially in the gravitational field of a point mass M.

Critical density

The above "equation of motion" implies that the "energy"

$$E = \frac{1}{2}\dot{a}^2 - \frac{GM}{a} \;, \tag{14-49}$$

must be conserved in the time evolution of the scale factor, *i.e.* $\dot{E} = 0$. Eliminating the mass, and using (14-46) to eliminate $\dot{a}$, it may be written in the form

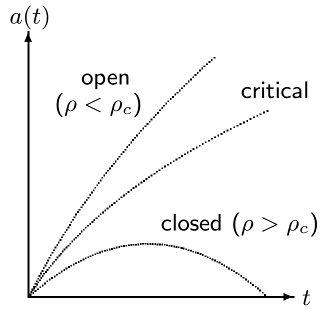
$$E = \frac{4}{3}\pi G a^2(\rho_c - \rho) \;, \tag{14-50}$$

where the *critical density*,

$$\boxed{\rho_c = \frac{3H^2}{8\pi G}} \;, \tag{14-51}$$

can be calculated from present-day observation of the Hubble "constant". Today, it is $\rho_c \approx 5.6 \times 10^{-27}$ kg/m^3, corresponding to about three protons per cubic meter.

From the particle analogy, we know that the scale factor will "escape" to infinity for $E \geq 0$, but turn around and "fall back" for $E < 0$. This means that the expansion will continue forever for $\rho \leq \rho_c$, but eventually turn around and become a contraction if $\rho > \rho_c$. The two types of universes are called *open* or *closed*, respectively. The average density of matter in the present universe is hard to estimate, but it appears to be considerably smaller than the critical value. So the prediction is that the universe is open and will expand forever.



Time evolution of the cosmic scale factor depending on the actual average mass density compared to the critical density.

Age of the universe

The energy equation (14-49) may be solved for $\dot{a}$ with the result,

$$\dot{a} = \sqrt{2\left(E + \frac{GM}{a}\right)} \;.$$

Demanding that $a = 0$ for $t = 0$, the solution is given implicitly by

$$t = \int_0^a \frac{dr}{\sqrt{2\left(E + \dfrac{GM}{r}\right)}} . \tag{14-52}$$

which must be the time elapsed since the scale factor was zero, *i.e.* since Big Bang. For the critical case, $E = 0$, the integral is easy to evaluate and we find

$$t = \frac{2}{3}\frac{a^{3/2}}{\sqrt{2GM}} \tag{14-53}$$

from which it follows that $H = \dot{a}/a = 2/3t$, or $t = 2/3H \approx 12$ billion years.

The cosmological constant

If the universe besides ordinary matter were filled with a ghostly material with a positive mass density, ρ_0, constant in both space and time, then the "energy" (14-50) should be replaced by

$$E = \frac{4}{3}\pi G a^2(\rho_c - \rho - \rho_0) , \tag{14-54}$$

where ρ as before is the density of ordinary matter still obeying (14-44).

Whatever the present-day density ρ, it it is evidently possible (by juggling ρ_0) to convert an otherwise open universe ($E > 0$) into a closed universe ($E < 0$). The new dynamic equation is found by differentiating the constant energy E after time (remembering that ρ_c is *not* constant), and becomes

$$\dot{H} + H^2 = -\frac{4\pi}{3}G(\rho - 2\rho_0) . \tag{14-55}$$

Evidently, for $\rho < 2\rho_0$, the gravitational attraction turns into a gravitational *repulsion*. The real understanding of how a positive mass density ρ_0 can give rise to an effective gravitational repulsion can only be obtained from relativistic theory [13, p. 614], which correctly takes into account the huge negative pressure accompanying ρ_0.

Einstein introduced already in 1917 the so-called *cosmological constant* (which is proportional to ρ_0) for the explicit purpose of permitting static solutions to the cosmological equations of general relativity. Clearly, the new equations admit of a non-expanding, static solution having $\rho = 2\rho_0$ and $H = 0$. Since that time we have learned that our universe is not static, but instead expands with a finite Hubble constant. In an open universe, the following peculiar scenario is now possible when $3\rho_0 < \rho_c$. For $2\rho_0 < \rho < \rho_c - \rho_0$, the effective gravity is attractive and the expansion will decelerate with time (*i.e.* $\ddot{a} = a(\dot{H} + H^2) < 0$). But sooner or later, the expansion will make the density fall below the critical value $\rho = 2\rho_0$, and the effective gravity becomes repulsive. The expansion will begin (and continue) to accelerate (*i.e.* $\ddot{a} > 0$). Recent observations indicate that the cosmic expansion is in fact accelerating [**?**].

Problems

∗ **14.1** Prove (14-7) directly from the change of the infinitesimal volume of a material particle (hint: use the Jacobi determinant of the infinitesimal displacement).

14.2 Draw stream lines, particle orbits and streak lines for a rotating velocity field $\boldsymbol{v} = a(\cos\omega t, \sin\omega t)$.

14.3 The wind suddenly turns from south to west. Draw stream lines, particle orbits and streak lines before and after the event.

14.4 A water pipe with diameter 1 inch branches into two pipes with diameters $\frac{3}{4}$ inch and $\frac{1}{2}$ inch. Water is tapped from the largest branch at the double rate as from the other. What is the ratio of velocities in the three pipes?

14.5 Show that the explosion density (14-37) satisfies the equation of continuity.

14.6 Calculate the time-derivative of $\rho(\boldsymbol{x}(t), t)$ where $\boldsymbol{x}(t)$ is a particle orbit and show that it is identical to the comoving derivative.

14.7 Consider an incompressible steady flow in a stream with constant depth, $z = d$, bounded on one side by a straight line, $y = 0$, and on the other side by a slowly varying curve, $y = h(x)$. a) Calculate the average flow velocity in the x-direction. b) Estimate the comoving acceleration in the flow. c) What should be the shape of the curve be in order for the estimate of the comoving acceleration to be constant?

14.8 Consider an incompressible steady flow in a circular tube along the x-axis with a slowly changing radius $r = a(x)$. a) Calculate the average flow velocity in the x-direction. b) Estimate the comoving acceleration in the tube, and c) determine what shape of the tube will lead to constant comoving acceleration?

14.9 Consider a universe in which matter is created everywhere at a constant rate, J, per unit of volume and time (Bondi and Gold (1948), Hoyle (1949)). Show that this allows for a steady-state cosmological solution with constant mass density and Hubble constant, and determine the rate of mass creation.

