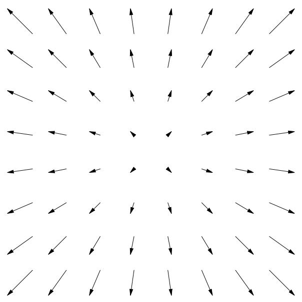
9 Strain

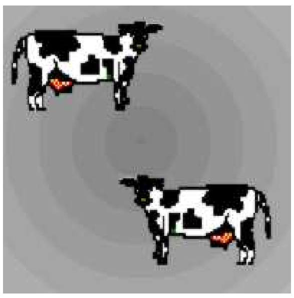
All materials deform when subjected to external forces, but different materials react in different ways. Elastic materials bounce back again to the original configuration when the forces cease to act. Others are plastic and retain their shape after deformation. Viscoelastic materials behave like elastic solids under fast deformation, but creep like viscous liquid over longer periods of time. Elasticity is itself an idealization, limited to a certain range of forces. If the external forces become excessive, most materials become plastic and undergo permanent deformation, even fracture.

When a body is deformed, its material is displaced away from its original position. Small deformations are mathematically much easier to handle than large deformations, where parts of a body become greatly and non-uniformly displaced relative to other parts, as for example when you crumble a piece of paper. A rectilinear coordinate system embedded in the original body and deformed along with the material of the body, becomes a curvilinear coordinate system after the deformation. It can therefore come as no surprise that the general theory of finite deformation is mathematically at the same level of difficulty as general curvilinear coordinate systems. Luckily, our buildings and machines are rarely subjected to such violent treatment, and in most practical cases the deformation may be assumed to be tiny.

The description of continuous deformation inevitably leads to the introduction of a new tensor quantity, the *strain tensor*, which characterizes the state of local deformation or *strain* in a material. Strained relations between neighboring material particles (as well as among people) evoke tension or stress, but in this chapter we shall focus exclusively on the formalism for strain. The discussion of the stress-strain relationship for elastic materials is postponed to the following chapter.



Uniform dilatation. The arrows indicate how material particles are displaced.



A cow studying its reflection. It is physically impossible to reflect a cow in the origin of the coordinate system.

9.1 Displacement

The prime example of deformation is a *uniform dilatation*, in which the coordinates of every material particle in a body are multiplied by a constant factor, $\kappa > 1$. The coordinates of a particle originally situated in the point $\boldsymbol{x}$ thus become

$$\boldsymbol{x}' = \kappa \boldsymbol{x} \tag{9-1}$$

after the dilatation. Contraction is also included by this expression for κ-values in the interval $0 < \kappa < 1$. Negative values of κ are not physical, because besides dilatation or contraction they contain a reflection ($\boldsymbol{x} \to \boldsymbol{x}' = -\boldsymbol{x}$) of the body in the origin of the coordinate system.

> While one might think of, probably rather unethical, ways to shrink or expand a cow, there is no way, ethical or not, to reflect it in the origin. It could of course be done moving it atom by atom, but that seems technically unfeasible at present. Furthermore, a reflected cow would presumably run into digestive problems because of the reversal of the handedness of its biomolecules.

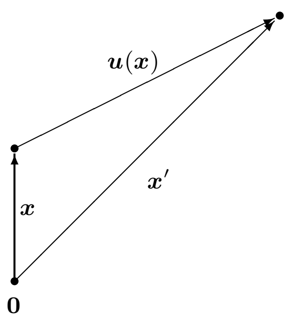
The only point which does not change place during a uniform dilatation is the origin of the coordinate system. Although it superficially looks as if the origin of the coordinate system plays a special role, this is not really the case. All relative distances between material particles scale in the same way, $|\boldsymbol{x}' - \boldsymbol{y}'| = \kappa\,|\boldsymbol{x} - \boldsymbol{y}|$, independently of the origin of the coordinate system. There is no special center for a uniform dilatation, neither geometrically nor physically. The origin of the coordinate system is simply an *anchor point* for the mathematical description of dilatation.

The displacement field

In an elastic solid the atoms retain their relations to the neighbors, so that each atom by and large is always surrounded by the same neighboring atoms. Only their mutual distances change a little with the deformation.

The simultaneous displacement of all material particles in a body may mathematically be described by a displacement vector field, $\boldsymbol{u}(\boldsymbol{x})$, such that a material particle, originally in the point $\boldsymbol{x}$, after the displacement is found in

$$\boxed{\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})}\,. \tag{9-2}$$



The displacement field.

The position $\boldsymbol{x}$ refers to a reference state of the material which we shall arbitrarily call undeformed, but this state may very well itself be highly deformed relative to another state. The value $\boldsymbol{u}(\boldsymbol{x})$ defined as the displacement suffered by a material particle *originally* situated at $\boldsymbol{x}$ in the undeformed state, whereas after the displacement it is *actually* situated at $\boldsymbol{x}'$.

There is nothing wrong in mathematically referring to a position where the material particle used to be, as long as we keep in mind that the physical position of the particle is $\boldsymbol{x}'$. After the displacement the original body no more exists, except in our imagination. It is, of course, also possible to express the displacement field in terms of the actual position of the body (see section 9.5).

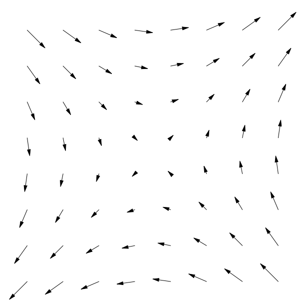
Since we have put no restrictions on the displacement field, the transformation (9-2) is equivalent to an arbitrary vector transformation, $\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$. The only reason to split it into the identity $\boldsymbol{x}$ and a displacement $\boldsymbol{u}(\boldsymbol{x})$, is that the displacement in most applications may be considered "small". But displacement has dimension of length, so we need to be more specific about the meaning of "small", for example that for all $\boldsymbol{x}$

$$|\boldsymbol{u}(\boldsymbol{x})| \ll L \ , \tag{9-3}$$

where L is a measure of the size of the body. We shall see below that for a consistent definition of "small", it is more reasonable to demand the displacement field to vary slowly over the body.

Linear displacements

The general, unrestricted displacement includes all kinds of ordinary rigid body moves, such as translations, rotations and reflection, and it would be wrong to classify such displacements as deformations. Sailing a submarine at the surface of the water will not deform it, but only displace it, whereas taking it to the bottom of the sea will. A real deformation must involve changes in geometric relationships, *i.e.* lengths and angles, in the body.



Plot of the two-dimensional linear displacement field $\boldsymbol{u} = (\alpha\, y, \alpha\, x, 0)$ for $-1 < x < 1$ and $-1 < y < 1$. The material is dilated along one diagonal and contracted along the other. These are the principal directions of strain (see problem 9.9).

Although the displacement in practical cases will always be a non-linear function of the coordinates, it is of interest to begin by analyzing linear displacements, such as the uniform dilatation (9-1). In the most general case, a linear transformation of the coordinates takes the form

$$\boldsymbol{x}' = \mathbf{A} \cdot \boldsymbol{x} + \boldsymbol{b} \ , \tag{9-4}$$

where $\mathbf{A}$ is a non-singular matrix and $\boldsymbol{b}$ is a constant vector. There is strong similarity between the class of linear displacements and the transformations of Cartesian coordinates discussed in section 2.5, but the class of linear displacements is the larger, because the matrix $\mathbf{A}$ is not restricted to be orthogonal. The conceptual difference lies in the interpretation of the displacement field,

$$\boldsymbol{u}(\boldsymbol{x}) = (\mathbf{A} - \mathbf{1}) \cdot \boldsymbol{x} + \boldsymbol{b} \ , \tag{9-5}$$

as a real shift of the material, as opposed to a change in the way coordinates are calculated.

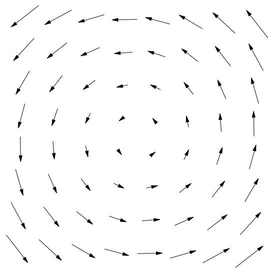
As was the case for Cartesian transformations, the general linear displacement may also be resolved into simpler types, namely translation along a coordinate axis, rotation by a fixed angle around a coordinate axis, and scaling by a fixed factor along a coordinate axis, whereas the physically impossible reflections are excluded. We shall not prove here that the general linear displacement may be resolved in this way, but instead rely on geometric intuition (see, however, problem 9.18).


Simple translation.

Simple translation: A rigid body translation of the material through a distance b along the x-axis is described by the displacement field

$$u_x = b \; , \quad u_y = 0 \; , \quad u_z = 0 \; . \tag{9-6}$$

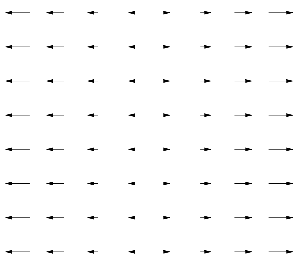
Since the geometric relationships in a body are unchanged under translation, it should not be classified as deformation.


Simple rotation.

Simple rotation: Likewise, a rigid body rotation through the angle ϕ around the z-axis is described by the displacement field

$$\begin{aligned} u_x &= -x\,(1-\cos\phi) - y\,\sin\phi \; , \\ u_y &= x\,\sin\phi - y\,(1-\cos\phi) \; , \\ u_z &= 0 \; . \end{aligned} \tag{9-7}$$

Again we do not consider a rotation to be a deformation.


Simple scaling.

Simple scaling: Finally, multiplying all distances along the x-axis by the factor κ, the displacement field becomes

$$\begin{aligned} u_x &= (\kappa - 1)\,x \; , \\ u_y &= 0 \; , \\ u_z &= 0 \; . \end{aligned} \tag{9-8}$$

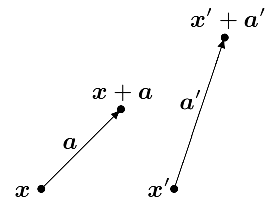
Uniform dilation (9-1) is a combination of three such scalings along the three coordinate axes by the same factor. Intuitively, simple scaling implies deformation for $\kappa \neq 1$.

9.2 Local deformation

Displacement is, as demonstrated above, not the same as deformation. All the parts of a body may be simultaneously displaced by the same amount, or bodily rotated, without straining the geometric relations between them. What is needed is a measure of the actual change of spatial relations between different parts of the material, also called *strain*.

At large spatial distances, deformation can be very complex. Think of all the loops and knots that weavers make from a roll of yarn. We should for this reason not expect to find a simple formalism for global deformation. Weaving, folding, winding, writhing, wringing, and squashing may bring particles that originally were far apart into close proximity. Even the wildest weave consists, however, locally of small pieces of straight yarn that have only been translated, rotated, stretched or contracted, but not folded, spindled or mutilated. We may therefore expect to find a much simpler description of deformation for very small pieces of matter.

Displacement of a needle



Displacement of a tiny material needle. It may be translated, rotated, and shrunk or stretched, but only the latter is a true deformation.

Consider a tiny elongated piece of material, a material vector or "needle", connecting the point $\boldsymbol{x}$ with $\boldsymbol{x} + \boldsymbol{a}$. After the displacement this needle will be situated between the points $\boldsymbol{x}'$ and $\boldsymbol{x}' + \boldsymbol{a}'$, where $\boldsymbol{x}'$ is given by (9-2) and similarly $\boldsymbol{x}' + \boldsymbol{a}' = \boldsymbol{x} + \boldsymbol{a} + \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{a})$. Solving for $\boldsymbol{a}'$ we find

$$\boldsymbol{a}' = \boldsymbol{a} + \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{a}) - \boldsymbol{u}(\boldsymbol{x}) \ . \tag{9-9}$$

The needle is now assumed to be so small that we may expand the displacement field $\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{a})$ to first order in $\boldsymbol{a}$. For the x-component of the field, we find

$$\begin{aligned} u_x(\boldsymbol{x} + \boldsymbol{a}) &\approx u_x(\boldsymbol{x}) + a_x \frac{\partial u_x(\boldsymbol{x})}{\partial x} + a_y \frac{\partial u_x(\boldsymbol{x})}{\partial y} + a_z \frac{\partial u_x(\boldsymbol{x})}{\partial z} \\ &= u_x(\boldsymbol{x}) + (\boldsymbol{a} \cdot \boldsymbol{\nabla}) u_x(\boldsymbol{x}) \ . \end{aligned}$$

Since $\boldsymbol{a} \cdot \boldsymbol{\nabla}$ is a scalar operator acting in the same way on each component of a vector, we may after collecting the other components write the displacement rules for an infinitesimal needle unambiguously in the form

$$\boxed{\boldsymbol{a}' = \boldsymbol{a} + (\boldsymbol{a} \cdot \boldsymbol{\nabla})\, \boldsymbol{u}(\boldsymbol{x})} \ . \tag{9-10}$$

Not surprisingly, since it is a relation between infinitesimal quantities, this transformation is *linear* in $\boldsymbol{a}$. In index notation, it may be written more transparently as,

$$a'_i = a_i + \sum_j (\nabla_j u_i) a_j \ , \tag{9-11}$$

which shows that the coefficients of the linear transformation of $\boldsymbol{a}$ are computed from the derivatives of the displacement field, $\nabla_j u_i$, also called the *displacement gradients*.

Example 9.2.1: For a simple rotation (9-7), the matrix of displacement gradients becomes

$$\{\nabla_j u_i\} = \begin{pmatrix} -1 + \cos\phi & -\sin\phi & 0 \\ \sin\phi & -1 + \cos\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9-12}$$

where the index i enumerates the rows and j the columns. For small angle of rotation, $|\phi| \ll 1$, the displacement gradients are all small, and the matrix simplifies to

$$\{\nabla_j u_i\} = \phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9-13}$$

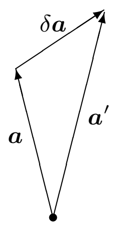
to lowest order in ϕ.

Small displacement gradients

Since both displacements and coordinates have dimension of length, the displacement gradients are all *dimensionless* quantities, *i.e.* pure numbers, and this makes it meaningful to speak of small displacement gradients in an absolute way. We shall say that a displacement field is slowly varying, if

$$|\nabla_j u_i(\boldsymbol{x})| \ll 1 \tag{9-14}$$

for all i, j and all $\boldsymbol{x}$. This does not mean that the displacement itself is small, because it could include a rigid body translation to any place in the universe. But if we require that there is a single point in the body, an *anchor point*, which is not displaced, then this possibility is excluded, and a slowly varying field must also be small in the sense of (9-3).



The change in a needle vector is small when the displacement gradients are small.

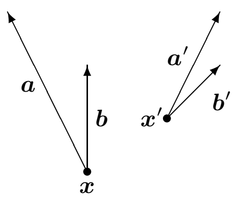
By and large, the opposite is also true. A small displacement field satisfying (9-3) everywhere, will also be smoothly varying, though there are notable exceptions. If you, for example, make a crease in your shirt when you iron it, the displacement gradients will be large in the crease although none of the shirt's material is greatly displaced relative to the size of the shirt.

For a slowly varying displacement, the vector change in a needle

$$\boxed{\delta\boldsymbol{a} \equiv \boldsymbol{a}' - \boldsymbol{a} = (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{u}} \tag{9-15}$$

is always small compared to the length of $\boldsymbol{a}$, *i.e.* $|\delta\boldsymbol{a}| \ll |\boldsymbol{a}|$. Except in section 9.5, where a few aspects of finite deformations are studied, we shall from now on assume that the displacement field is small and smoothly varying.

Cauchy's strain tensor



Displacement of a pair of infinitesimal material needles may affect their lengths as well as the angle between them.

In order to study the changes in geometry due to displacement, we consider the scalar product, $\boldsymbol{a} \cdot \boldsymbol{b}$, between two needles. Since the scalar product is unchanged by translation and rotation of the neighborhood of $\boldsymbol{x}$, it ought to be a useful measure for change in geometry. Using (9-15), we find the change in the scalar product $\delta(\boldsymbol{a} \cdot \boldsymbol{b}) \equiv \boldsymbol{a}' \cdot \boldsymbol{b}' - \boldsymbol{a} \cdot \boldsymbol{b}$ to first order in the small displacement gradients

$$\begin{aligned}
\delta(\boldsymbol{a} \cdot \boldsymbol{b}) &= \delta\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \delta\boldsymbol{b} \\
&= (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{u} \cdot \boldsymbol{b} + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{u} \cdot \boldsymbol{a} \\
&= \sum_{ij} (\nabla_i u_j + \nabla_j u_i)\, a_i b_j
\end{aligned}$$

In the last line we have cast the rather ugly vector expression in the much more elegant index notation, replacing all dot-products by explicit sums, such that we may write

$$\delta(\boldsymbol{a} \cdot \boldsymbol{b}) = 2 \sum_{ij} u_{ij} a_i b_j \tag{9-16}$$

where the symmetric combination of displacement gradients in the parenthesis is given a special symbol (with a conventional factor 1/2)

$$\boxed{u_{ij} = \frac{1}{2}\big(\nabla_i u_j + \nabla_j u_i\big)} \,, \tag{9-17}$$

called *Cauchy's strain tensor* (or just the *strain tensor*). It pays to write out all its components explicitly, once and for all. On the diagonal, they are

$$u_{xx} = \nabla_x u_x \;, \quad u_{yy} = \nabla_y u_y \;, \quad u_{zz} = \nabla_z u_z \;, \tag{9-18}$$

whereas off the diagonal one has

$$\begin{aligned} u_{xy} = u_{yx} &= \frac{1}{2}(\nabla_x u_y + \nabla_y u_x) \\ u_{yz} = u_{zy} &= \frac{1}{2}(\nabla_y u_z + \nabla_z u_y) \\ u_{zx} = u_{xz} &= \frac{1}{2}(\nabla_z u_x + \nabla_x u_z) \end{aligned} \tag{9-19}$$

Had we not assumed that the displacement was slowly varying, there would also have been terms quadratic in the displacement gradients, and the strain tensor might take large values (see section 9.5). But with our assumption of small displacement gradients (9-14), the strain tensor is likewise small.

Example 9.2.2: For a uniform dilatation $\boldsymbol{u} = \alpha\,\boldsymbol{x}$, the strain gradients become $\nabla_j u_i = \alpha\delta_{ij}$ and are small for $|\alpha| \ll 1$. Cauchy's strain tensor becomes $u_{ij} = \alpha\,\delta_{ij}$.

Example 9.2.3: The displacement field $\boldsymbol{u} = \phi(-y, x, 0)$ describes a rotation through a small angle $|\phi| \ll 1$ around the z-axis. From the antisymmetry of the matrix of strain gradients (9-13), it follows that the strain tensor vanishes, as expected.

Example 9.2.4: The linear displacement $\boldsymbol{u} = \alpha(2y, x, 0)$ with $|\alpha| \ll 1$ has a matrix of displacement gradients

$$\{\nabla_j u_i\} = \alpha \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \tag{9-20}$$

and Cauchy's strain tensor becomes

$$\{u_{ij}\} = \frac{3}{2}\alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \tag{9-21}$$

which is symmetric as it should be.

9.3 Geometrical meaning of the strain tensor

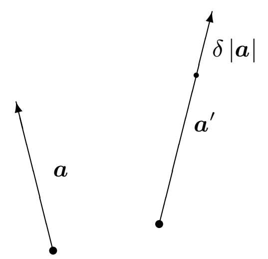
The strain tensor contains all the relevant information about changes in geometric relationships, such as lengths of material needles and the angles between them. Other geometric quantities, for example area and volume, are also changed under a deformation.

It is useful for the following discussion to define the *projection* u_{ab} of a tensor u_{ij} on the directions of two arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$u_{ab} = \frac{\sum_{ij} u_{ij} a_i b_j}{|\boldsymbol{a}|\,|\boldsymbol{b}|}\ . \tag{9-22}$$

so that we may write (9-16)

$$\delta(\boldsymbol{a}\cdot\boldsymbol{b}) = 2\,|\boldsymbol{a}|\,|\boldsymbol{b}|\,u_{ab} \tag{9-23}$$



The change in length $\delta\,|\boldsymbol{a}|$ is determined by the diagonal projection of the strain tensor.

Change of length: The change in length of a needle is found by setting $\boldsymbol{b} = \boldsymbol{a}$, and using that $\delta(\boldsymbol{a}^2) = 2\,|\boldsymbol{a}|\,\delta\,|\boldsymbol{a}|$, we find

$$\boxed{\frac{\delta\,|\boldsymbol{a}|}{|\boldsymbol{a}|} \equiv \frac{|\boldsymbol{a}'| - |\boldsymbol{a}|}{|\boldsymbol{a}|} = u_{aa}}\ . \tag{9-24}$$

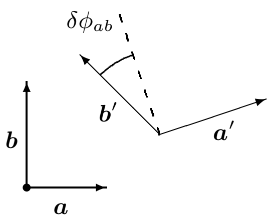
The diagonal strain projection u_{aa} is thus the *fractional change of lengths* in the direction of $\boldsymbol{a}$.

Change of angle: In the same way, we may from $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\phi_{ab}$ calculate the change in angle, $\delta\phi_{ab}$, between two needles, $\boldsymbol{a}$ and $\boldsymbol{b}$, under the deformation. Since

$$\delta(|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\phi_{ab}) = \delta\,|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\phi_{ab} + |\boldsymbol{a}|\,\delta\,|\boldsymbol{b}|\cos\phi_{ab} - |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\phi_{ab}\delta\phi_{ab}$$

we obtain by means of (9-24)

$$\delta\phi_{ab} \equiv \phi'_{ab} - \phi_{ab} = -\frac{2u_{ab}}{\sin\phi_{ab}} + (u_{aa} + u_{bb})\cot\phi_{ab} \tag{9-25}$$



The off-diagonal projections of the strain tensor determine the change in angle for originally orthogonal needles.

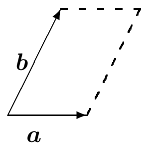
For orthogonal vectors, *i.e.* for $\phi_{ab} = \pi/2$, the change in angle simplifies to

$$\boxed{\delta\phi_{ab} = -2u_{ab}}\ . \tag{9-26}$$

The off-diagonal projections of the strain tensor thus determine the change in angle between originally orthogonal needles.

Change of area: The infinitesimal change in the area of the parallelogram $\boldsymbol{S} = \boldsymbol{a} \times \boldsymbol{b}$ spanned by two needles is calculated using (9-15). We find, keeping all the time $\boldsymbol{\nabla}$ to the left of $\boldsymbol{u}$, and using the "double-cross" rule (2-72) twice

$$\begin{aligned}\delta(\boldsymbol{a} \times \boldsymbol{b}) &= \delta\boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \delta\boldsymbol{b} \\ &= (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{u} \times \boldsymbol{b} + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{a} \times \boldsymbol{u} \\ &= ((\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{b}) \times \boldsymbol{u} \\ &= -((\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{\nabla}) \times \boldsymbol{u} \\ &= (\boldsymbol{a} \times \boldsymbol{b})\boldsymbol{\nabla} \cdot \boldsymbol{u} - \boldsymbol{\nabla}((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{u}) \ ,\end{aligned}$$



The parallelogram spanned by two infinitesimal vectors defines and elementary surface element $\boldsymbol{S} = \boldsymbol{a} \times \boldsymbol{b}$.

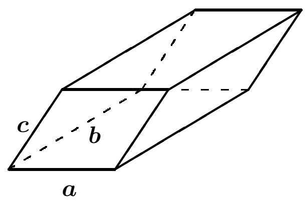
or

$$\boxed{\delta\boldsymbol{S} \equiv \boldsymbol{S}' - \boldsymbol{S} = \boldsymbol{S}\boldsymbol{\nabla} \cdot \boldsymbol{u} - \boldsymbol{\nabla}(\boldsymbol{S} \cdot \boldsymbol{u})} \ . \qquad \text{(9-27)}$$

Since it is a vector relation, it cannot be expressed entirely in terms of the strain tensor, but involves the displacement gradients in the second term.

Change of volume: The change in the volume $V = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$ spanned by three infinitesimal needles is calculated by means of the preceding result

$$\begin{aligned}\delta((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}) &= \delta(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} + (\boldsymbol{a} \times \boldsymbol{b}) \cdot \delta\boldsymbol{c} \\ &= (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}\boldsymbol{\nabla} \cdot \boldsymbol{u} - (\boldsymbol{c} \cdot \boldsymbol{\nabla})((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{u}) + (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \cdot \boldsymbol{\nabla})\boldsymbol{u} \\ &= ((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c})\boldsymbol{\nabla} \cdot \boldsymbol{u} \ .\end{aligned}$$



The parallelepiped spanned by three vectors defines an elementary volume element $V = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$.

The fractional volume change becomes

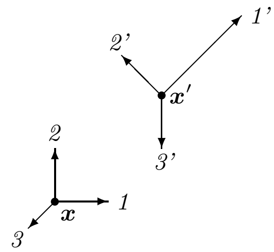
$$\boxed{\frac{\delta V}{V} \equiv \frac{V' - V}{V} = \boldsymbol{\nabla} \cdot \boldsymbol{u} = \mathsf{div}\, \boldsymbol{u} = \sum_i u_{ii}} \ , \qquad \text{(9-28)}$$

and is simply equal to the divergence of the displacement field or, equivalently, the trace of the strain tensor.

Symmetry of the strain tensor

The strain tensor is by its definition (9-17) *symmetric* in the indices

$$\boxed{u_{ij} = u_{ji}} \ , \qquad \text{(9-29)}$$



Principal basis of deformation in a point $\boldsymbol{x}$. Apart from translation and rotation, this basis is only subject to scale changes along the principal axes.

and differs in this respect from the stress tensor, for which symmetry required further assumptions (see page 132). The symmetry implies that the strain tensor may be *diagonalized* in every point. The eigenvectors of the strain tensor in a given point are called the *principal axes* of strain, and form an orthonormal basis in every point. Whereas the angles between the principal axes are unchanged under the displacement, the signs and magnitudes of the eigenvalues determine how much the material is being stretched or contracted along the principal axes. Notice, however, that the principal basis varies from point to point (problem 9.6).

* 9.4 Work and energy

Deforming a body takes work, and in the ideal limit of infinitely slow, also called quasistatic, deformation, this work is normally saved as elastic energy in the body and may be recovered, when the deformation is released again. If, however, the deformation is done in a finite time, some energy is always lost to sound waves that are radiated away and eventually degenerate to heat. A hard steel ball may jump many times on a hard floor, but eventually it loses all its energy and comes to rest, partly due to air resistance, and partly due to radiative elastic losses in the ball and, perhaps more importantly, in the floor.

Let us assume that we displace the material in a volume V by an infinitesimal amount $\delta\boldsymbol{u}$. The total work performed by the external forces is, to first order in the displacement, the sum of the work performed by the the volume forces and by the contact forces,

$$\delta W_{\text{ext}} = \int_V \sum_i \delta u_i f_i dV + \oint_S \sum_{ij} \delta u_i \sigma_{ij} dS_j \ . \tag{9-30}$$

Here we should not worry about the stress being expressed in terms of the displaced body coordinates, whereas the displacement is a function of the coordinates in the undisplaced body. The corrections that would follow from this worry are of higher order in $\delta\boldsymbol{u}$ and can be completely disregarded. Using Gauss' theorem (5-3) on the surface integral in the first term we get

$$\begin{aligned}\delta W_{\text{ext}} &= \int_V \sum_{ij} \nabla_j (\delta u_i \sigma_{ij}) dV + \int_V \sum_i \delta u_i f_i dV \\ &= \int_V \sum_{ij} (\nabla_j \delta u_i) \sigma_{ij} dV + \int_V \sum_i \delta u_i \left(\sum_j \nabla_j \sigma_{ij} + f_i \right) dV\end{aligned}$$

In mechanical equilibrium (8-21) the second term vanishes, and the first,

$$\boxed{\delta W_{\text{int}} = \int_V \sum_{ij} \sigma_{ij} \nabla_j \delta u_i \, dV} \ , \tag{9-31}$$

must be interpreted as the work done *against* the *internal* contact forces. The work done *by* the internal forces is accordingly $-\delta W_{\text{int}}$. Although internal contact forces cancel each other in the total force (because of Newton's third law), they do not cancel in the total work, because the displacement varies from place to place. If the stresses are only due to pressure, $\sigma_{ij} = -p\delta_{ij}$, this becomes

$$\delta W = -\int_V p \, \mathsf{div} \, \delta\boldsymbol{u} \, dV \ . \tag{9-32}$$

Comparing with (9-28) we recognize that the integrand is the thermodynamic work $-p \, \delta(dV)$ performed on a material particle under the displacement.

∗ 9.5 Finite local deformation

When the condition (9-14) for slowly varying displacement is not fulfilled, we can no more use the simple Cauchy strain tensor (9-17). The local description of finite deformation (see for example [11]) is essentially equivalent to the formalism of general curvilinear coordinate systems, but because space is assumed Euclidean, not as complicated as that of truly non-Euclidean spaces.

Although many aspects of finite deformation theory were developed in the 19'th century, the subject was not fully established until the mid 20'th century through Rivlin's work on non-linear materials. Here we shall only touch briefly on the most general aspects of finite deformation theory.

Ronald Samuel Rivlin (1915–). British born engineer. Contributed to the understanding of non-linear materials during the 1940's and 1950's.

The non-linear strain tensor

For a finite deformation

$$\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}) \ , \tag{9-33}$$

there is no reason to split off a special displacement field $\boldsymbol{u}(\boldsymbol{x})$, although we shall do so in order to keep contact with the previous analysis. Under a general transformation, an infinitesimal vector, a "needle" $\boldsymbol{a}$, in the neighborhood of $\boldsymbol{x}$ is transformed into

$$a'_i = \sum_j F_{ij}(\boldsymbol{x})\, a_j \tag{9-34}$$

where the tensor field

$$F_{ij} = \frac{\partial x'_i}{\partial x_j} = \nabla_j f_i = \delta_{ij} + \nabla_j u_i \tag{9-35}$$

is called the *deformation gradient.*

The scalar product of two infinitesimal vectors becomes

$$\boldsymbol{a}' \cdot \boldsymbol{b}' = \sum_k a'_k b'_k = \sum_{ij} G_{ij}(\boldsymbol{x})\, a_i b_j \tag{9-36}$$

where

$$G_{ij} = \sum_k F_{ki} F_{kj} = \delta_{ij} + \nabla_i u_j + \nabla_j u_i + \sum_k \nabla_i u_k \nabla_j u_k \tag{9-37}$$

is called the *deformation tensor* field, and was introduced by George Green in 1841. Writing

$$G_{ij} = \delta_{ij} + 2u_{ij} \ , \tag{9-38}$$

the generalization of Cauchy's strain tensor (9-17) becomes

George Green (1793–1841). Self-taught English mathematician and mathematical physicist.

$$\boxed{u_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i + \sum_k \nabla_i u_k \nabla_j u_k\right)} \tag{9-39}$$

also called *Green's strain tensor*. The condition that the displacement field should change slowly across the material, $|\nabla_i u_j| \ll 1$, is of course sufficient to guarantee the smallness of the strain tensor and the linear form (9-17). The opposite is not in general true. A displacement with a large displacement gradient does not necessarily lead to a large strain tensor. The prime counterexample is a rigid body rotation through a finite angle, in which all derivatives are of order 1, whereas we know from the orthogonality of F_{ij} that the strain tensor must vanish (problem 9.14).

Example 9.5.1: For a uniform dilatation (9-1) we have $\nabla_j u_i = (\kappa - 1)\delta_{ij}$ and the strain tensor becomes,

$$u_{ij} = \frac{1}{2}\left((\kappa-1)\delta_{ij} + (\kappa-1)\delta_{ij} + (\kappa-1)^2\delta_{ij}\right) = (\kappa^2 - 1)\delta_{ij} \;, \tag{9-40}$$

for any value of κ. It vanishes for $\kappa = \pm 1$, *i.e.* for no displacement and a pure reflection in the origin. The scalar product of two needles becomes,

$$\boldsymbol{a}' \cdot \boldsymbol{b}' = \kappa^2 \boldsymbol{a} \cdot \boldsymbol{b}, \tag{9-41}$$

just reflecting that all lengths are scaled by the same amount, whereas angles are unchanged under a dilatation.

Euler versus Lagrange

It is sometimes convenient, and in a sense more physically correct, to refer displacements to the actual positions of the material particles instead of their original positions. This is called the *Eulerian* description of deformation as opposed to the *Lagrangian* description used until now.

Let us define the Eulerian displacement field $\boldsymbol{u}'(\boldsymbol{x}')$ as a function of $\boldsymbol{x}'$ with exactly the same (vector) value at the actual position as the Lagrangian field $\boldsymbol{u}(\boldsymbol{x})$ at the original position

$$\boldsymbol{u}'(\boldsymbol{x}') = \boldsymbol{u}(\boldsymbol{x}) \;. \tag{9-42}$$

Even if the two displacement fields take the same values at corresponding positions, the relation between them is non-trivial. To see this, we use (9-33) to calculate the original position $\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}' - \boldsymbol{u}'(\boldsymbol{x}')$, and inserting this into the right hand side of (9-42), we obtain a functional equation

$$\boldsymbol{u}'(\boldsymbol{x}') = \boldsymbol{u}(\boldsymbol{x}' - \boldsymbol{u}'(\boldsymbol{x}')) \;. \tag{9-43}$$

Given $\boldsymbol{u}(\boldsymbol{x})$, this equation must be solved for $\boldsymbol{u}'(\boldsymbol{x}')$, but that is in general impossible.

Example 9.5.2: For the case of a linear displacement, the equation may be solved. In the Lagrangian description we have

$$\boldsymbol{u}(\boldsymbol{x}) = (\mathbf{A} - \mathbf{1}) \cdot \boldsymbol{x} + \boldsymbol{b} \;, \tag{9-44}$$

from which we derive the displacement field in the Eulerian description

$$\boldsymbol{u}'(\boldsymbol{x}') = (\mathbf{1} - \mathbf{A}^{-1}) \cdot \boldsymbol{x}' + \mathbf{A}^{-1} \cdot \boldsymbol{b} \ . \tag{9-45}$$

Evidently, the Eulerian and Lagrangian descriptions are quite different.

Writing the inverse needle transformation in the form

$$a_i = \sum_j F'_{ij} a'_j \tag{9-46}$$

we find the Eulerian deformation gradient

$$F'_{ij}(\boldsymbol{x}') = \frac{\partial x_i}{\partial x'_j} = \delta_{ij} - \nabla'_j u'_i(\boldsymbol{x}') \ . \tag{9-47}$$

Since

$$\sum_k \mathcal{F}'_{ik}(\boldsymbol{x}') F_{kj}(\boldsymbol{x}) = \sum_k \frac{\partial x_i}{\partial x'_k} \frac{\partial x'_k}{\partial x_j} = \delta_{ij} \ , \tag{9-48}$$

the two deformation gradients are each other's inverses. This relation also connects the two displacement gradients, $\nabla_j u_i(\boldsymbol{x})$ and $\nabla'_j u'_i(\boldsymbol{x}')$. Defining

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{ij} G'_{ij} a'_i b'_j \ , \tag{9-49}$$

we obtain,

$$G'_{ij} \equiv \delta_{ij} - 2u'_{ij} = \sum_k F'_{ki} F'_{kj} \ , \tag{9-50}$$

where u'_{ij} is the non-linear strain tensor in the Eulerian description

$$u'_{ij} = \frac{1}{2} \left(\nabla'_i u'_j + \nabla'_j u'_i - \sum_k \nabla'_i u'_k \nabla'_j u'_k \right) \ . \tag{9-51}$$

This tensor was introduced by Almansi in 1911 and Hamel in 1912. The relation between the Green and Almansi strain tensors becomes

$$u'_{ij} = \sum_{kl} F'_{ki} F'_{lj} u_{kl} \ , \tag{9-52}$$

where all quantities are calculated at corresponding positions. For small displacement gradients (9-14), all differences between the two descriptions disappear.

Problems

9.1 Prove that

$$2\,\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}+\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 \ . \tag{9-53}$$

and use this to show that the change in a scalar product under a deformation is derivable from changes in length.

9.2 Calculate displacement and strain tensor for the deformation

$$\begin{aligned} x' &= 5x - y + 3z \\ y' &= x + 8y \\ z' &= -3x + 4y + 5z \end{aligned}$$

9.3 Show that the general displacement rule for a an infinitesimal needle (9-10) may be written

$$\boldsymbol{a}' = \boldsymbol{a} + \boldsymbol{\phi}\times\boldsymbol{a} + \mathbf{U}\cdot\boldsymbol{a} \tag{9-54}$$

where $\phi = \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{u}$ and $\mathbf{U} = \{u_{ij}\}$ is Cauchy's strain tensor (9-17). What does the second term mean?

9.4 Show that the most general solution, for which Cauchy's strain tensor (9-17) vanishes, is

$$\begin{aligned} u_x &= A + Dy + Ez \\ u_y &= B - Dx + Fz \\ u_z &= C - Ex - Fy \end{aligned}$$

where A, B, C are arbitrary constants and D, E, F are small.

9.5 Calculate the displacement gradients and the strain tensor for the displacement field $\boldsymbol{u} = \alpha(y^2, xy, 0)$ with $|\alpha| \ll 1/L$, where L is the size of the body.

9.6 Calculate the principal directions of strain and the dilatation factors for problem 9.5.

9.7 A deformable material undergoes two successive displacements, $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{x}'' = \boldsymbol{x}' + \boldsymbol{u}'(\boldsymbol{x}')$, both having small strain. Calculate the final strain tensor for the total deformation $\widetilde{u}_{ij}$ relative to the original reference state.

9.8 Show that the infinitesimal change in the area $S = |\boldsymbol{a}\times\boldsymbol{b}|$ is

$$\delta S = |\boldsymbol{a}|\,|\boldsymbol{b}|\,\frac{u_{aa} + u_{bb} - 2u_{ab}\cos\phi_{ab}}{\sin\phi_{ab}} \tag{9-55}$$

9.9 Calculate the strain tensor for $\boldsymbol{u} = (y, x, 0)$. Determine the principal directions of strain and the change in length scales along these.

9.10 Show that the matrix $\mathbf{1} + 2\mathbf{u}$, where $\mathbf{u}$ is the strain tensor, is positive definite.

9.11 Calculate the relative volume change in a uniform expansion.

9.12 A deformable material undergoes two successive finite displacements, $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{x}'' = \boldsymbol{x}' + \boldsymbol{u}'(\boldsymbol{x}')$. Calculate the final strain tensor for the total deformation $\widetilde{u}_{ij}$ relative to the original reference state.

∗ **9.13** Calculate the non-linear strain tensor for the displacement field $\boldsymbol{u} = (Ax + Cy, Cx - By, D)$ where A, B, C, D are constants.

∗ **9.14** Show that rigid body translation and rotation are the only displacement fields with vanishing strain tensor.

∗ **9.15** Show that the infinitesimal strain tensor satisfies the relation

$$\nabla_i \nabla_j u_{kl} + \nabla_k \nabla_l u_{ij} = \nabla_i \nabla_l u_{kj} + \nabla_k \nabla_j u_{il} \ . \tag{9-56}$$

Conversely, if this relation is fulfilled for a symmetric tensor field u_{ij} then there is a displacement field such that the strain tensor is given by (9-17).

9.16 Show that the characteristic polynomial for the strain tensor may be written in the form

$$\| u_{ij} - \lambda \delta_{ij} \| = -\lambda^3 + u_{ii} \lambda^2 - \frac{1}{2}(u_{ii} u_{jj} - u_{ij} u_{ij}) \lambda + \det u_{ij} \tag{9-57}$$

∗ **9.17** Show that

$$\delta_{ij} + 2u_{ij} = \sum_k (\delta_{ik} + \nabla_i u_k)(\delta_{jk} + \nabla_j u_k) \ , \tag{9-58}$$

and use this to prove that the matrix $\{\delta_{ij} + 2u_{ij}\}$ is positive definite. Show that

$$\det \{\delta_{ij} + \nabla_i u_j\} = \sqrt{\det \{\delta_{ij} + 2u_{ij}\}} \ . \tag{9-59}$$

Use this to demonstrate that the infinitesimal volume change (9-28) is valid for any deformation with small strain tensor without assuming a slowly varying displacement field.

∗ **9.18** Show that a non-singular matrix A may be written in the form $A = U * DV$ where U and V are orthogonal matrices and D is diagonal.

