3.16 Line distribution $\rho dv = \lambda dz$ along the z-axis. Put $r = \sqrt{x^2 + y^2}$.

a) Substitute $z' = z - r\sinh\psi$

$$\Phi = -G\int_{-a}^{a} \frac{\lambda dz'}{\sqrt{r^2 + (z - z')^2}} = -G\lambda(\sinh^{-1}\frac{z + a}{r} - \sinh^{-1}\frac{z - a}{r})$$

where $\sinh^{-1} u = \log(u + \sqrt{u^2 + 1})$ is the inverse hyperbolic sine.

$$g_z = -\frac{\partial\Phi}{\partial z} = G\lambda\left(\frac{1}{\sqrt{(r^2 + (z + a)^2}} - \frac{1}{\sqrt{(r^2 + (z - a)^2}}\right)$$

$$g_r = -\frac{\partial\Phi}{\partial r} = -G\lambda\frac{1}{r}\left(\frac{z + a}{\sqrt{(r^2 + (z + a)^2}} - \frac{z - a}{\sqrt{(r^2 + (z - a)^2}}\right)$$

b) For $r, |z| \to \infty$: $\Phi \to -G\dfrac{2a\lambda}{r}$, $g_z \to -G2a\lambda\dfrac{z}{r^2}$, $g_r \to -G\dfrac{2a\lambda}{r^2}$.

c) For $a \to \infty$: $\Phi \to -2G\lambda\log\dfrac{a}{r}$, $g_z \to -2G\lambda\dfrac{z}{a^2}$, $g_r \to -\dfrac{2G\lambda}{r}$.

3.17 Use cylindrical coordinates (r, ϕ, z).

a)

$$\Phi = -G\sigma\int_0^a sds\int_0^{2\pi} d\phi\frac{1}{\sqrt{z^2 + r^2 + s^2 - 2rs\cos\phi}}$$

b) $\Phi = -2\pi G\sigma(\sqrt{z^2 + a^2} - |z|)$.

c) $\Phi \to -G\dfrac{\sigma\pi a^2}{|z|}$

d) $\Phi \to -2\pi G(a - |z|)$ for $a \to \infty$.

4 Fluids at rest

4.1 Put $h_1 = 9$ m, $h_2 = 6$ m and $w = 12$ m. Atmospheric pressure is p_0. a) $p_1 = p_0 + \rho_0 g_0(h_1 - z)$. $p_2 = p_0 + \rho_0 g_0(h_2 - z)$. b) $F = \frac{1}{2}(h_1^2 - h_2^2)w\rho_0 g_0 = 2.7 \times 10^6$ N. c) $M = \frac{1}{6}(h_1^3 - h_2^3)w\rho_0 g_0 = 10^7$ Nm. d) $M/F = 3.7$ m.

4.3 Put $h = 3$ m and $d = 2a = 30$ cm. The horizontal pressure force on the hemisphere is equal to the pressure force on the vertical plane through the center of the sphere. The linear rise of pressure with depth makes the pressure act with its average value at the center. So the force becomes $\rho_0 g_0 h\pi d^2/4 = 2100$ N. If you do not like this argument, it is also possible with some effort to integrate the pressure force directly in spherical coordinates

$$\begin{aligned}\boldsymbol{\mathcal{F}} &= -\int_{\text{half-sphere}} (p - p_0)\, d\boldsymbol{S} = -\int_{\text{half-sphere}} (p - p_0)\boldsymbol{e}_r\, dS \\ &= -a^2\int_0^{\pi} d\theta\int_{-\pi/2}^{\pi/2} d\phi\,\sin\theta\,(p - p_0)\,\boldsymbol{e}_r\end{aligned}$$

Using that

$$p = p_0 + \rho_0 g_0 (h - a\cos\theta)$$

where h is the depth of the center of the lamp, we obtain for the x-component

$$\begin{aligned}
\mathcal{F}_x &= -a^2 \int_0^{\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi \, \sin\theta \, \rho_0 g_0 (h - a\cos\theta) \sin\theta\cos\phi \\
&= -2a^2 \rho_0 g_0 \int_0^{\pi} d\theta \sin^2\theta (h - a\cos\theta) \\
&= -\rho_0 g_0 h \pi a^2
\end{aligned}$$

4.5 a) $K = n(p+B)$. b) Put $h = \dfrac{n}{n-1}\dfrac{p_0+B}{\rho_0 g_0} = 35\ \mathsf{km}$. Then $\rho = \rho_0(1-z/h)^{1/(n-1)}$, $p + B = (p_0 + B)(1 - z/h)^{n/(n-1)}$. c) $(\rho - \rho_0)/\rho_0 = 4.3\ \%$ at $z = -10\ \mathsf{km}$.

4.6 The pressure must be continuous across the boundary and thus of the form

$$p = p_b \begin{cases} (\rho/\rho_1)^\gamma & \text{in the core} \\ (\rho/\rho_2)^\gamma & \text{in the mantle} \end{cases} \tag{4-A1}$$

where ρ_1 is the density in the core and ρ_2 the density at the mantle at the boundary. The common pressure on both sides of the boundary is p_b.

4.7 a) $4000\ \mathsf{kg/m^3}$. b) $10\ \mathsf{m}$.

4.8 a) The surface inside the tube will be at the same level $h_1 + h_2$ as in the jar. b) The heavy liquid in the tube must initially rise to the same level h_1 as in the jar. When the light liquid is poured in, the surface of the heavy liquid in the tube must rise further to balance the weight of the light and rise to a height $h_1 + h_2\rho_2/\rho_1 < h_1 + h_2$.

4.9

a) $p(z) = \rho_0 g_0 z$.

b) $0 \le \dfrac{1}{A}\displaystyle\int_A (z - I_1)^2 \, dS = I_2 - I_1^2$.

c) $\mathcal{F} = \displaystyle\int_A p(z)\, dS = \rho_0 g_0 I_1$.

d) $\mathcal{M} = \displaystyle\int_A p(z) z \, dS = \rho_0 g_0 I_2$.

e) $z_P = \dfrac{\mathcal{M}}{\mathcal{F}} = \dfrac{I_2}{I_1} \ge I_1 = z_M$.

f) Since the width of the triangle is bz/h at depth z, the area of the triangle is $A = \displaystyle\int_0^h b\,\frac{z}{h}\, dz = \frac{1}{2}hb$ (which is of course well-known) . It then follows that $I_1 = \frac{2}{3}h$ and $I_2 = \frac{1}{2}h^2$, and thus $z_M = \frac{2}{3}h$ and $z_P = \frac{3}{4}h$. Clearly $z_P > z_M$.