
(a) \(\gamma^\mu \gamma^\nu = \pm \gamma^\nu \gamma^\mu\) where the sign is ‘+’ for \(\mu = \nu\) and ‘−’ otherwise. Hence for any product \(\Gamma\) of the \(\gamma\) matrices, \(\gamma^\mu \Gamma = (-1)^{n_\mu} \gamma^\mu\) where \(n_\mu\) is the number of \(\gamma^\nu \neq \gamma^\mu\) factors of \(\Gamma\). For \(\Gamma = \gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3\), \(n_\mu = 3\) for any \(\mu = 0, 1, 2, 3\); thus \(\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu\).

(b) First,
\[
(\gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3)^\dagger = -i(\gamma^3)^\dagger (\gamma^2)^\dagger (\gamma^1)^\dagger (\gamma^0)^\dagger = +i\gamma^3 \gamma^2 \gamma^1 \gamma^0
\]
\[
= +i((\gamma^3 \gamma^2) \gamma^1) \gamma^0 = (-1)^3 i \gamma^0 ((\gamma^3 \gamma^2) \gamma^1)
\]
\[
= (-1)^3 \gamma^0 (\gamma^1 (\gamma^3 \gamma^2)) = (-1)^3 (\gamma^3 \gamma^2)
\]
\[
= +i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \equiv +\gamma^5.
\]

Second,
\[
(\gamma^5)^2 = \gamma^5 (\gamma^5)^\dagger = (i\gamma^0 \gamma^1 \gamma^2 \gamma^3)(i\gamma^3 \gamma^2 \gamma^1 \gamma^0) = -\gamma^0 \gamma^1 \gamma^2 (\gamma^3 \gamma^2) \gamma^2 \gamma^1 \gamma^0
\]
\[
= +\gamma^0 \gamma^1 (\gamma^2 \gamma^2) \gamma^1 \gamma^0 = -\gamma^0 (\gamma^1 \gamma^1) \gamma^0 = +\gamma^0 \gamma^0 = +1.
\]

(c) Any four distinct \(\gamma^\kappa, \gamma^\lambda, \gamma^\mu, \gamma^\nu\) are \(\gamma^0, \gamma^1, \gamma^2, \gamma^3\) in some order. They all anticommute with each other, hence \(\gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu = \epsilon^{\kappa \lambda \mu \nu} \gamma^0 \gamma^1 \gamma^2 \gamma^3 \equiv -i \epsilon^{\kappa \lambda \mu \nu} \gamma^5\). The rest is obvious.

(d) \(i \epsilon^{\kappa \lambda \mu \nu} \gamma_\kappa \gamma^5 = \gamma_\kappa \gamma^{[\kappa \gamma^\lambda \gamma^\mu \gamma^\nu]}\)
\[
= \frac{1}{4} \gamma_\kappa \left(\gamma^{[\kappa \gamma^\lambda \gamma^\mu \gamma^\nu]} - \gamma^{[\lambda \gamma^\mu \gamma^\nu]} \gamma^{\kappa} + \gamma^{[\lambda \gamma^\mu \gamma^\kappa]} \gamma^{\nu} - \gamma^{[\lambda \gamma^\mu \gamma^\nu]} \gamma^{\kappa}\right)
\]
\[
= \frac{1}{4} \left(4 \gamma^{[\lambda \gamma^\mu \gamma^\nu]} + 2 \gamma^{[\lambda \gamma^\mu \gamma^\nu]} + 4 \gamma^{[\lambda \gamma^\mu \gamma^\nu]} + 2 \gamma^{[\nu \gamma^\mu \gamma^\lambda]}\right)
\]
\[
= \frac{1}{4} \left(4 + 2 + 0 - 2\right) \gamma^{[\lambda \gamma^\mu \gamma^\nu]} = \gamma^{[\lambda \gamma^\mu \gamma^\nu]}.
\]
Proof by inspection: In the Weyl basis, the 16 matrices are

\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \]

\[ i\gamma^i, \gamma^j = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \quad i\gamma^i[0, \gamma^j] = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & +i\sigma^i \end{pmatrix}, \] (S.4)

\[ \gamma^5\gamma^0 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \quad \gamma^5\gamma^1 = \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \]

and their linear independence is self-evident. Since there are only 16 independent \(4 \times 4\) matrices altogether, any such matrix \(\Gamma\) is a linear combination of the matrices (S.4). \(\text{Q.E.D.}\)

Algebraic Proof: Without making any assumption about the matrix form of the \(\gamma^\mu\) operators, let us consider the Clifford algebra \(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\). Because of these anticommutation relations, one may re-order any product of the \(\gamma\)'s as \(\pm \gamma^0 \cdots \gamma^0 \gamma^1 \cdots \gamma^1 \gamma^2 \cdots \gamma^2 \gamma^3 \cdots \gamma^3\) and then further simplify it to \(\pm (\gamma^0 \text{ or 1}) \times (\gamma^1 \text{ or 1}) \times (\gamma^2 \text{ or 1}) \times (\gamma^3 \text{ or 1})\). The net result is (up to a sign or \(\pm i\) factor) one of the 16 operators \(1, \gamma^\mu, i\gamma^i[\mu, \gamma^\nu], -i\gamma^i[\lambda, \gamma^\mu] = \epsilon^{\mu\nu\rho} \gamma^5 \gamma^\rho\) (cf. (d)) or \(i\gamma^i[\kappa, \gamma^\mu] = \epsilon^{\kappa \lambda \mu \nu} \gamma^5\) (cf. (c)). Consequently, any operator \(\Gamma\) algebraically constructed of the \(\gamma^\mu\)'s is a linear combination of these 16 operators.

Incidentally, the algebraic argument explains why the \(\gamma^\mu\) (and hence all their products) should be realized as \(4 \times 4\) matrices since any lesser matrix size would not accommodate 16 independent products. That is, the \(\gamma\)'s are \(4 \times 4\) matrices in four spacetime dimensions; different dimensions call for different matrix sizes. Specifically, in spacetimes of even dimensions \(d\), there are \(2^d\) independent products of the \(\gamma\) operators, so we need matrices of size \(2^{d/2} \times 2^{d/2}: 2 \times 2\) in two dimensions, \(4 \times 4\) in four, \(8 \times 8\) in six, \(16 \times 16\) in eight, \(32 \times 32\) in ten, etc., etc.

In odd dimensions, there are only \(2^{d-1}\) independent operators because \(\gamma^{d+1} \equiv (i)\gamma^0 \gamma^1 \cdots \gamma^{d-1}\) — the analogue of the \(\gamma^5\) operator in 4d — commutes rather than anticommutes with all the \(\gamma^\mu\) and hence with the whole algebra. Consequently, one has two distinct representations of the Clifford algebra — one with \(\gamma^{d+1} = +1\) and one with \(\gamma^{d+1} = -1\) — but in each representation there are only \(2^{d-1}\) independent operator products, which call for the matrix size of \(2^{(d-1)/2} \times 2^{(d-1)/2}\). For example, in three spacetime dimensions (two space, one time), can take \((\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, i\sigma_2)\) for \(\gamma^4 \equiv i\gamma^0 \gamma^1 \gamma^2 = +1\) or \((\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, -i\sigma_2)\) for \(\gamma^4 = -1\), etc., etc.
Problem 2(a):
Despite anticommutativity of the fermionic fields, the Hermitian conjugation of an operator product reverses the order of operators without any extra sign factors, thus \( (\Psi_\alpha^\dagger \Psi_\beta)^\dagger = +\Psi_\beta^\dagger \Psi_\alpha \). Consequently, for any 4 \( \times \) 4 matrix \( \Gamma \), \( (\Psi^\dagger \Gamma \Psi)^\dagger = +\Psi^\dagger \Gamma^\dagger \Psi \), and hence \( (\Psi \Gamma \Psi)^\dagger = \Psi \Gamma \Psi \) where \( \Gamma = \gamma^0 \Gamma^\dagger \gamma^0 \) is the Dirac conjugate of \( \Gamma \).

Now consider the 16 matrices which appear in the bilinears (1). Obviously \( S^\dagger = +S \). We saw in class that \( \gamma^\mu = +\gamma^\mu \), and this gives us \( (V^\mu)^\dagger = +V^\mu \). We also saw that \( i\gamma^\mu \gamma^\nu = -i\gamma^\nu \gamma^\mu = +i\gamma^{\mu\nu} \), and this gives us \( (T^\mu{}^\nu)^\dagger = +T^{\mu\nu} \). As to the \( \gamma_5 \) matrix, it is Hermitian (cf. 1.(b)) and anticommutes with \( \gamma^0 \), hence \( \gamma_5^\dagger = \gamma^0 (\gamma_5)^\dagger \gamma^0 = +\gamma^0 \gamma_5 \gamma^0 = -\gamma_5 \) and therefore \( i\gamma_5 = +i\gamma_5 \), which gives us \( P^\dagger = +P \). Finally, \( \gamma_5 \gamma^\mu = -\gamma^\mu \gamma_5 = +\gamma^\mu \gamma_5 \), which gives us \( (A^\mu)^\dagger = +A^\mu \). Thus, by inspection, all the bilinears (1) are Hermitian. \( \text{Q.E.D.} \)

Problem 2(b):
Under a continuous Lorentz symmetry \( x \mapsto x' = Lx \), the Dirac spinor field and its conjugate transform according to

\[
\Psi'(x') = M(L)\Psi(x = L^{-1}x'), \quad \overline{\Psi}(x') = \overline{\Psi}(x = L^{-1}x')M^{-1}(L), \quad (S.5)
\]

hence any bilinear \( \overline{\Psi} \Gamma \Psi \) transforms according to

\[
\overline{\Psi}(x')\Gamma\Psi(x') = \overline{\Psi}(x)\Gamma'\Psi(x) \quad (S.6)
\]

where

\[
\Gamma' = M^{-1}(L)\Gamma M(L). \quad (S.7)
\]

Obviously, for \( \Gamma = 1 \), \( \Gamma' = M^{-1}M = 1 \). According to homework set #5 (problem 3(d)), for \( \Gamma = \gamma^\mu \), \( \Gamma' = M^{-1}\gamma^\mu M = L^\mu{}^\nu \gamma^\nu \). Similarly, \( M^{-1}\gamma^\mu \gamma^\nu M = (M^{-1}\gamma^\mu M)(M^{-1}\gamma^\nu M) = L^\mu{}^\nu \gamma^\kappa \times \)
and use (2) to get

\[
\sum_{m=1}^{\infty} \frac{f(k)}{(m-1)!} \int \tilde{d}k_1 \ldots \tilde{d}k_m f_1 \ldots f_m a_1^\dagger \ldots a_m^\dagger |0\rangle
\]

so

\[
\frac{\langle \psi | a(k) | \psi \rangle}{\langle \psi | \psi \rangle} = f(k) \quad \frac{\langle \psi | a^\dagger(k) | \psi \rangle}{\langle \psi | \psi \rangle} = f^*(k)
\]

where I took the Hermitian conjugate to find the eigenvalue of \(a^\dagger(k)\).

c) Taking the mode expansion

\[
\phi(x, t) = \int \tilde{d}k \left[ a(k)e^{ikx} + a^\dagger(k)e^{-ikx} \right]
\]

(4)

gives the expectation value

\[
\langle \phi(x,t) \rangle = \int \tilde{d}k \left[ f(k)e^{ikx} + f^*(k)e^{-ikx} \right]
\]

(5)

d) The second derivative with respect to time is

\[
\left\langle \dot{\phi}(x, t) \right\rangle = -\int \tilde{d}k \omega^2 \left[ f(k)e^{ikx} + f^*(k)e^{-ikx} \right]
\]

and the Laplacian is

\[
\langle \nabla^2 \phi(x,t) \rangle = -\int \tilde{d}k \textbf{k} \cdot \textbf{k} \left[ f(k)e^{ikx} + f^*(k)e^{-ikx} \right]
\]

so

\[
\partial_\mu \partial^\mu \langle \phi(x, t) \rangle = \int \tilde{d}k \left( \omega^2 - \textbf{k} \cdot \textbf{k} \right) \left[ f(k)e^{ikx} + f^*(k)e^{-ikx} \right] = m^2 \langle \phi(x,t) \rangle
\]

indicating that \(\langle \phi \rangle\) satisfies the Klein-Gordon equation.

**Problem 2:**

a) The field operator \(\phi(x, t)\) must satisfy antiperiodic boundary conditions.

\[
\phi(x + L, t) = -\phi(x, t) = e^{i\pi} \phi(x, t)
\]

suggesting that all modes must have wavenumbers of the form

\[
k_n = \frac{\pi}{L}(2n + 1)
\]

**Problem 4**
for some integer $n$. Thus, we have a mode expansion
\[
\phi(x, t) = \sum_n [e^{ik_n(x-t)}a(k_n) + e^{-ik_n(x-t)}a^+(k_n)]
\]

b) The Green's function is given by
\[
G_F(x, x') = \int \frac{dk \, d\omega}{(2\pi)^2} \frac{e^{-i\omega \Delta t + i k \Delta x}}{k^2 - \omega^2} = \int \int d\omega \, d\alpha \, \frac{e^{-i\omega \Delta t + i k \Delta x}}{k^2 + \alpha^2} e^{-\alpha \omega^2 + i\omega \Delta \tau}
\]
\[
= \int \int d\omega \, d\alpha \, \frac{e^{-i\omega \Delta t + i k \Delta x}}{4\pi\alpha} e^{-\frac{1}{4\pi}((\Delta \tau)^2 + (\Delta x)^2)}
\]
\[
= -i \int \int d\omega \, d\alpha \, \frac{e^{-u((\Delta \tau)^2 + (\Delta x)^2)}}{4\pi u}
\]

I got the second line by Wick rotating (Peskin and Shroeder, p. 193 or Srednicki, p. 216) and the line after that by using the identity
\[
\frac{1}{B} = \int_0^\infty d\alpha \, e^{-\alpha B}
\]

The integral on the last line is formally divergent, but note that
\[
-\frac{\partial}{\partial B} \int_0^\infty d\alpha \, e^{-\alpha B} = \int_0^\infty d\alpha \, e^{-\alpha B}
\]
in order to recover
\[
G_F(x, x') = -\frac{i}{4\pi} \ln [(\Delta \tau)^2 + (\Delta x)^2]
\]
\[
= -\frac{1}{2\pi} \ln |x - x'|
\]
after Wick rotating things back.

c) Start with
\[
G_F(x, x') = \sum_{k_n} \int \frac{dk \, d\omega}{(2\pi)^2} \frac{e^{i(k_n x - \omega t)}}{k_n^2 - \omega^2}
\]
\[
= \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikx}}{k^2} \sum_n (2\pi) \delta \left( k - (2n + 1)\frac{\pi}{L} \right)
\]
\[
= \sum_m \int \frac{dk \, d\omega}{(2\pi)^2} \frac{e^{imL}}{k^2} e^{i\pi m e^{imkL}}
\]
\[
= \sum_m (-1)^m G_F(x + mL, x')
\]
using the Poisson sum formula on the third line.

**d)** The Lagrangian for the theory is

\[ \mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \]

with canonical momentum \( \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \) so the Hamiltonian is

\[ \mathcal{H} = \Pi \dot{\phi} - \mathcal{L} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi_x^2 \]

**e)** The point-splitting Hamiltonian is

\[ \mathcal{H}_\epsilon \equiv \frac{1}{2} \dot{\phi}(x,t) \dot{\phi}(x+\epsilon,t) + \frac{1}{2} \phi_{x,x}(x,t) \phi_{x,x}(x+\epsilon,t) \]

so

\[ \langle 0| \mathcal{H}_\epsilon(x,t) |0 \rangle = -\frac{1}{2} (\partial_0^2 - \partial_x^2) G_F(x,x) \]

In flat space, this is simply

\[ -\frac{1}{2\pi \epsilon^2} \]

and in the box it is

\[ -\frac{1}{2\pi \epsilon^2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(nL)^2} \]

As expected, both Hamiltonians diverge in the \( \epsilon \to 0 \) limit.

**e)** Subtracting the flat Hamiltonian from the box Hamiltonian eliminates the \( \epsilon \) dependence, leaving a difference of

\[ -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(nL)^2} = \frac{1}{12L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi}{12L^2} \]

**Problem 3:**

The problem is simplified a lot if we break the field operator \( \phi(x) \) up into parts

\[ \phi(x) = \phi_+(x) + \phi_-(x) \]

where \( \phi_+(x) \) (vice \( \phi_-(x) \)) depends only on creation (vice annihilation) operators.

\[ \phi_+(x) = \int \tilde{a} a^\dagger(k) e^{-ikx} \quad \phi_-(x) = \int \tilde{a} a(k) e^{ikx} \]
Solution

a) We have $g_{\mu\nu}\sigma_a^\mu \sigma_b^\nu$, which must be proportional to $\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}$, since no other Lorenz invariant tensor has the right combination of indices.

Now, if

$$g_{\mu\nu}\sigma_a^\mu \sigma_b^\nu = c \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}$$

we can find $c$ by evaluating

$$g_{\mu\nu}\sigma_a^\mu \sigma_b^\nu = -g_{00}1_{11}1_{22} + g_{33}3_{11}3_{22} = -2\epsilon_{12}\epsilon_{\dot{1}\dot{2}}$$

so $c = 2$.

b) This is easiest working backwards.

$$-\frac{1}{2}(\xi\sigma^\mu\chi^1)(\psi\sigma_\mu)_{\dot{a}} = -\frac{1}{2}g_{\mu\nu}(\xi^b\sigma^\mu\chi^\dot{b})\psi^a\sigma_{a\dot{a}} = \xi^b\chi^\dot{b}\psi^a\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} = (\xi\psi)\chi_{\dot{a}}$$

Problem 3:

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(i\bar{\psi}\gamma^\mu\partial_\mu \psi - m\bar{\psi}\psi - im'\bar{\psi}\gamma_5\psi)$$

a) We perform a chiral transformation

$$\psi \rightarrow e^{i\alpha\gamma_5}\psi$$

$$\bar{\psi} \rightarrow \bar{\psi} e^{-i\alpha\gamma_5}$$

where we can anti-commute every power of $\gamma_5$ in the exponential past the $\gamma_0$ in the definition of $\psi$. Then the derivative term in the Lagrangian becomes

$$\bar{\psi}e^{i\alpha\gamma_5}\gamma^\mu \partial_\mu e^{i\alpha\gamma_5}\psi = \bar{\psi}\gamma^\mu \partial_\mu e^{-i\alpha\gamma_5} e^{i\alpha\gamma_5}\psi = \bar{\psi}\gamma^\mu \partial_\mu \psi$$

b) Transforming the mass terms gives

$$m\bar{\psi}e^{2i\alpha\gamma_5}\psi + im'\bar{\psi}e^{2i\alpha\gamma_5}\gamma_5\psi$$

Fortunately, we can use $(\gamma_5)^2 = 1$ to simplify the exponential.

$$e^{2i\alpha\gamma_5} = \sum_{n=0}^{\infty} \frac{(2i\alpha)^2 n}{(2n)!} + \gamma_5 \frac{(2i\alpha)^{2n+1}}{(2n+1)!} = \cos 2\alpha + i\gamma_5 \sin 2\alpha$$

so the sum of the mass terms is

$$\bar{\psi} \left[(m \cos 2\alpha - m' \sin 2\alpha) + i\gamma_5 (m' \cos 2\alpha + m \sin 2\alpha)\right] \psi$$
Rotating the chiral mass term away requires finding a value of $\alpha$ such that

$$m' \cos 2\alpha + m \sin 2\alpha = 0$$

which is satisfied by

$$\frac{m'}{m} = - \tan 2\alpha$$

so the new mass is given by

$$\sqrt{m^2 + m'^2}$$