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# Fractal «Aggregates» in the Complex Plane.

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**Abstract.** – Aggregates generated by probabilistic diffusion are fractals reminiscent of the deterministically generated Julia sets: the growth measure of the aggregate is equivalent to the «electric field» around the Julia set and both fractals have a characteristic hairy structure with the fields diverging at the tips. We conjecture that their $f(\alpha)$ spectra share the same qualitative features. For Julia sets the part of the spectrum corresponding to the tips (smallest $\alpha$'s) is robust under small parameter changes. In the fjords the electric field is vanishingly small, the associated $\alpha$'s diverge and $f(\alpha)$'s are noisy, poorly convergent and highly sensitive to small variations of parameters. The fjords are screened, and this screening can manifest itself in $f(\alpha)$ as a «phase transition» at the Hausdorff dimension.

Fractal aggregates have been observed in a variety of experiments [1, 2], as well as in numerical simulations of diffusion-limited aggregation (DLA) or other clustering models [3, 4]. The aggregates grow rapidly at the tips, but very sluggishly in the ends of the fjords between the branches of the aggregates. In this note we shall argue that in general the ends of these fjords will be screened and inactive: in the $f(\alpha)$ spectrum one expects a «tip phase» and a «fjord phase» of a very different nature, possibly separated by a phase transition at the Hausdorff dimension. Our arguments rely on «hairy aggregates» developed by the theory of complex iterations, but the lessons are of physical interest: they suggest that only the small $\alpha$'s (those which measure the accumulation at tips) are experimentally measurable. The large $\alpha$'s corresponding to the fjords are noisy and experimentally inaccessible. This makes the Hausdorff dimension hard to measure, even for the simplest mathematical models of fractal aggregates. Indeed, the existing numerical and experimental $f(\alpha)$ spectra for aggregates show strong fluctuations for large $\alpha$ values and are not inconsistent with a phase transition [2].

Both the diffusion field around an aggregate and the electrostatic potential around a conductor satisfy the Laplace equation [3, 4]. One expects the electric field to diverge at the tips and vanish in the fjords, but in general the evaluation of the electric field is difficult. The

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utility of our analogy between the fractal aggregates and the Julia sets rest on the observation of Douady and Hubbard [5] that the 2-dimensional electrostatics has a particularly simple solution if the «aggregate» is a Julia set. However, it must be emphasized that unlike the physical dynamically growing fractal aggregates, our Julia sets are static, and we use them only to develop intuition about the expected qualitative features of the $f(\alpha)$ spectrum for a given static «aggregate».

For the Julia sets numerical computation of the electric field is straightforward; we refer the reader to ref. [6] for an introduction into the theory of complex iterations in general, and computation of the electric field in particular. As a matter of fact, as we are interested only in the density of electric field lines on the Julia set itself (which, in the case of a physical aggregate, is the growth measure), we do not even need to evaluate the electric field. This density is generated by the inverse iterates of the polynomial map under investigation, and the extremal $\alpha$'s are available analytically in terms of the eigenvalues of the fixed points and the cycles of the mapping.

Consider, as a prototype, map [7, 8]

$$f(z) = z^2 + c, \tag{1a}$$

which has the fixed points (here and in the following, 0 refers to using $+\sqrt{}$, 1 to using $-\sqrt{}$)

$$z_0 = \frac{1+\sqrt{1-4c}}{2}, \qquad z_1 = \frac{1-\sqrt{1-4c}}{2}. \tag{1b}$$

The inverse map has two branches, $f_\varepsilon^{-1}$, where $\varepsilon$ is 0 or 1: here $f_0^{-1}(y) = \sqrt{y-c}$ and $f_1^{-1}(y) = -\sqrt{y-c}$. Depending on the value of the parameter $c$, one distinguishes three situations. 1) If the critical point $z_c$ (defined by $f'(z_c) = 0$) converges to a stable periodic orbit, the basin of attraction has finite measure. Its border is the Julia set, the union of all unstable periodic orbits. 2) If $z_c$ is preperiodic to an unstable periodic orbit, *i.e.* $f^n(z_c) = f^{n-k}(z_c)$, the basin of attraction shrinks to zero, but the Julia set is still connected. The corresponding parameter values are called a Misiurewicz points and are denoted [6] $(n, k)$. We shall refer to such Julia sets as «hairy». 3) If $z_c$ iterates to infinity, the Julia set disintegrates into «Cantor dust» of dimension smaller than one. Here we concentrate on the hairy Julia sets, such as the set plotted in fig. 1, motivated by their similarity to the experimentally and numerically observed fractal aggregates. (Sets with finite but thin basins of attraction might also be acceptable as models of fractal aggregates.)

The potential $U(z)$ is calculated by conformally mapping [5, 6] the Julia set onto the unit circle

$$U(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f^n(z)|. \tag{2}$$

Notice that $U = 0$ on the Julia set, *i.e.* the set is «grounded». Around a point $z_i$ on the Julia set the potential scales as

$$U(z_i + \varepsilon) \sim \varepsilon^{\alpha_i}. \tag{3}$$

For example, for the fixed point $z_0$ it follows from (1) that in one iteration the potential doubles, whereas the length scale $\varepsilon$ is expanded by the eigenvalue of $f$ at $z_0$:

$$\alpha_{\min} = \frac{\log 2}{\log |f'(z_0)|}. \tag{4}$$

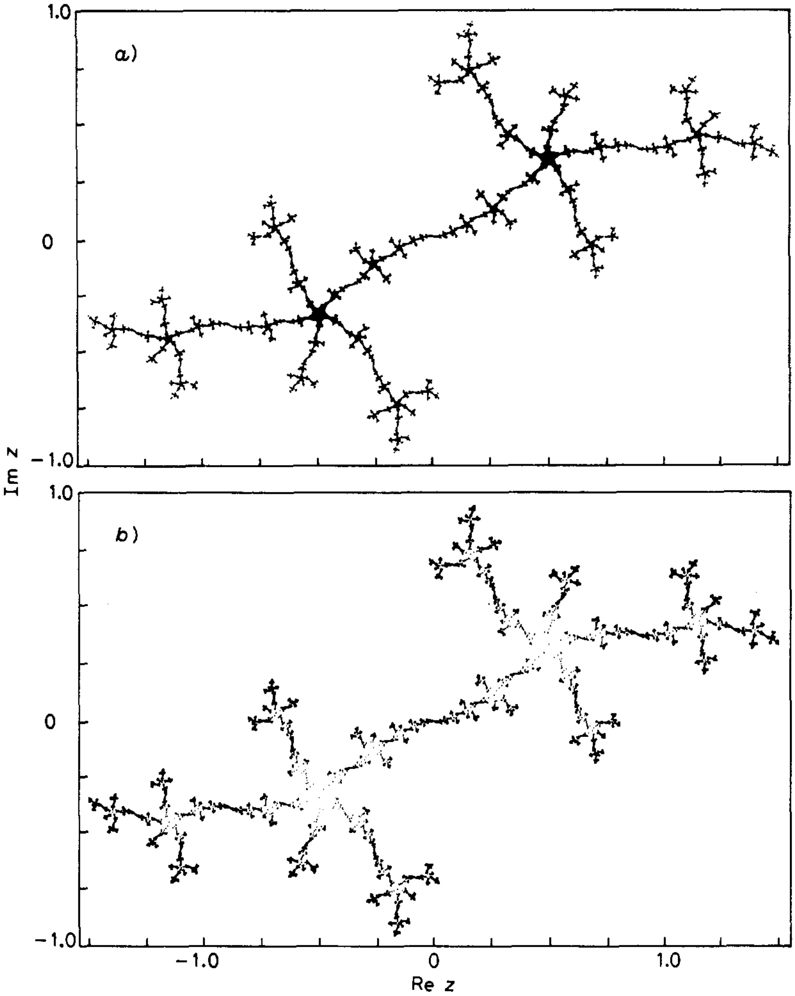


Fig. 1. – *a*) The Julia set for $c = -0.636\,754 - 0.685\,031i$. At this parameter value $z_c$ lands on the fixed-point $z_0$ in five iterations. This particular Julia set is chosen here for its visual resemblance to DLA aggregates (from the point of view of the theory of complex iterations, five-fold branchings play no special role). The set is approximated by scanning a grid of size $1000 \times 1000$ and retaining those initial $z$'s whose first 25 iterates have not diverged outside the frame of the plot. *b*) The same Julia set as in *a*). Plotted are the first $2^{16}$ tips and fjords obtained by inverse iteration from the fixed points $z_0$ and $z_1$. The density of points is proportional to the electric field, which is equivalent to the harmonic measure of the «aggregate».

What is special about the fixed point $z_0$? It is located at the tip with the strongest divergence of the field, and thus the minimal value of $\alpha$. Starting from this tip, inverse iterations of (1) generate all other tips on the Julia set (fig. 1*b*)). $\alpha_{\min} < 1$, so the electric field ($\boldsymbol{E} = \nabla U$) diverges as $E(z_0 + \varepsilon) \sim \varepsilon^{\alpha - 1}$.

The role of the other fixed point, $z_1$, is more delicate. The associated $\alpha_1 = \log 2/\log|f'(z_1)|$ is larger than one, so the electric field vanishes at $z_1$. For the particular hairy Julia set depicted in fig. 1*a*), $z_1$ is the location of the «primary» five-fold branching; its backward iterates generate all fjords in fig. 1*b*), and $\alpha_1$ is the maximal $\alpha$. However, this Julia set is extremely atypical; as we shall see, in general one expects existence of long unstable cycles of marginal stability, with corresponding $\alpha$'s arbitrarily large. For the example of fig. 1$a$) all tips and fjords are preimages of the two fixed points, so the corresponding $\alpha$'s can be evaluated from inverse iterates of the map. Going one step backwards, the potential is halved and the length scale is multiplied by the derivative of the map so with the terminology $z_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_n}=f_{\varepsilon_n}^{-1}\circ\dots\circ f_{\varepsilon_1}^{-1}(z_{\varepsilon_0})$ we obtain the $2^{n+1}$ $\alpha$-values on the $n$-th step as

$$\begin{cases} \text{Tips: } \alpha_{0,\varepsilon_1,\dots,\varepsilon_n}=\dfrac{\log(2^{n+1})}{\log|f^{n\prime}(z_{0,\varepsilon_1,\dots,\varepsilon_n})|}=\dfrac{\log(2^{n+1})}{\log|2^{n+1}z_{0,\varepsilon_1,\dots,\varepsilon_n}\dots z_0|}, \\ \text{Fjords: } \alpha_{1,\varepsilon_1,\dots,\varepsilon_n}=\dfrac{\log(2^{n+1})}{\log|f^{n\prime}(z_{1,\varepsilon_1,\dots,\varepsilon_n})|}=\dfrac{\log(2^{n+1})}{\log|2^{n+1}z_{1,\varepsilon_1,\dots,\varepsilon_n}\dots z_1|}, \end{cases} \tag{5}$$

where $\varepsilon_i=0,1$. If the critical point is traversed, we use $\log((1/2)\sqrt{|z_{\varepsilon_0,\varepsilon_1,\dots,\varepsilon_n}\dots z_{\varepsilon_0}|})$ as denominator, because the inverse map of (1$a$) behaves like $\frac{1}{2}\sqrt{z}$ around the critical point.

The natural measure of interest for aggregates is the harmonic measure [9], *i.e.* the probability of growth is proportional to the strength of the electric field, or, as explained above, the density of inverse iterates (fig. 1$b$)). From the $\alpha$'s of the tips and fjords calculated above and the harmonic measure we form the partition sum [10, 11]

$$\Gamma_n(q,\tau)=\sum_i\frac{p_i^q}{l_i^\tau}.$$

Since the set can be conjugated to a circle, each tip (or fjord) has equal probability, $p_i=2^{-(n+1)}$, while the covering interval is given by $l_i=\mathrm{d}z_{\varepsilon_0,\dots,\varepsilon_n}/\mathrm{d}z_c=2^{-n+1/\alpha_{\varepsilon_0,\dots,\varepsilon_n}}$, the distance between neighbouring points [12] in fig. 1$b$). The Legendre transform $f(\alpha)=$ $=-\tau(q)+q\alpha$, $\alpha=\mathrm{d}\tau(q)/\mathrm{d}q$ yields the spectrum of fig. 2.

As expected for aggregates with harmonic measure [13], this spectrum passes through the point (1, 1): the information dimension $D_1$ equals on. In the present context this can be

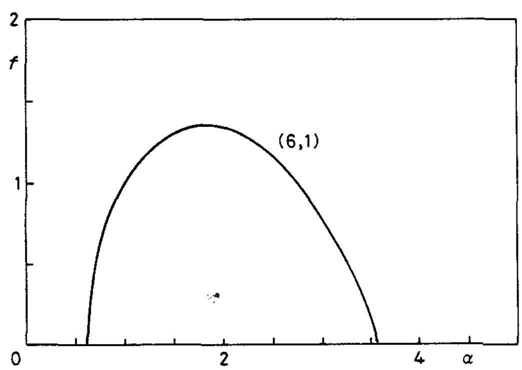


Fig. 2.

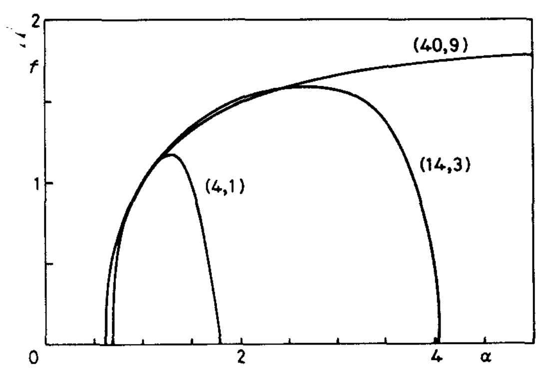


Fig. 3.

Fig. 2. – The $f(\alpha)$ curve for the Julia set in fig. 1. The values of $\alpha_{\min}$, $\alpha_{\max}$ are calculated from the eigenvalues of the fixed points $z_0$, $z_1$, respectively.

Fig. 3. – The $f(\alpha)$ curves for the first three steps in the inverse period tripling sequence: $z_c$ is iterated into a fixed point, into a 3-cycle, and into a 9-cycle. The corresponding values (see (1)) of $c$ are: $-0.228115-1.115142i$, $-0.00148215-0.836948i$, $-0.017893-0.784231i$. Note that $\alpha_{\max}\to\infty$, indicating that the ends of fjords are screened by a phase transition.

verified by a simple calculation. The information dimension is an average over $\alpha_i$, which, at the $n$-th level of the resolution, is given by

$$\frac{1}{D_1} = \frac{1}{2^{n+1}} \sum_{i=1}^{2^{n+1}} \alpha_i^{-1}. \tag{6}$$

When calculating the $\alpha$-values (5) we should, in principle, be able to use almost any point $z$ as starting point. For the hairy sets studied here the critical point, $z_c$, is on the Julia set and evaluating (5) for backwards iterates of $z_c$ we find

$$\frac{1}{D_1} = 1 + \lim_{n\to\infty} \frac{1}{2^n \log 2} \log|f^{(n)}(z_c) - z|. \tag{7}$$

As mentioned above, if the Julia set is connected, iterates of the critical point $z_c$ are bounded, so the information dimension tends to 1. If iterates of $z_c$ escape to infinity, the Julia set falls apart into Cantor dust, and the information dimension is less than one.

What features of the above analysis are insensitive to particulars of the model, and are likely to be shared with physical fractal aggregates? We expect only the part of the $f(\alpha)$ spectrum corresponding to the tips to be experimentally robust. The tip with $\alpha_{\min}$ is always given by the fixed point $z_0$. As is clear from the explicit expression (1*b*) for $z_0$, $\alpha_{\min}$ is insensitive to small changes in the parameter $c$. On the other hand, the value of $\alpha_{\max}$, given by the most screened fjord, is very sensitive to minute changes of parameter, because any high order unstable cycle may become close to marginal stability, with the corresponding $\alpha$ growing without bound. $\alpha_{\max} \to \infty$ makes $q(\tau)$ or $f(\alpha)$ nonanalytic. This can be interpreted as a «phase-transition» [14-17]. In particular (as explained in ref. [14, 18]), $1/\alpha_{\max} = \mathrm{d}q/\mathrm{d}\tau = 0$ implies that $q(\tau)$ undergoes a phase transition at $-\tau = D_H$ (the Hausdorff dimension) with $q(\tau) = 0$ for $\tau \leqslant -D_H$, and $q(\tau)$ a smooth function of $\tau$ for $\tau > -D_H$. Physically, $\alpha \to \infty$ means that the corresponding fjords are screened and completely inactive [19].

To stress this point, we exhibit an example of such phase transition by following the series of reverse period triplings ([1]), with $z_c$ falling onto the fixed point after 3 iterations, onto the unstable 3-cycle, onto the unstable $3^2$-cycle, and so on (denoted $(n, k) = (4, 1)$, $(14, 3)$, $(40, 9)$, with the corresponding parameter values listed in fig. 3). The longer the cycle, the more marginal it is: for this particular sequence we can, using the universal period $n$-tupling theory of ref. [8, 20], estimate the stability of the $3^k$ cycle to be of order of $1 + \delta_{1/3}^{-k}$, $|\delta_{1/3}| \sim 10$. The corresponding $\alpha$ is proportional to $\delta^k$ and grows without bound. The results are shown in fig. 3. The tip-phase part of the $f(\alpha)$ spectrum from $(\alpha_{\min}, 0)$ up to $(1, 1)$ is basically the same for the three members of the sequence, as well as for a differently branched set of fig. 2. However, in the fjord phase $f(\alpha)$ varies wildy, $\alpha_{\max}$ diverges, and the slow convergence characteristic of phase transitions [14, 18] makes an accurate determination of the Hausdorff dimension difficult. From a practical point of view it is not important whether $\alpha_{\max}$ is infinite, or just very large: we shall refer to either situation loosely as a «phase transition». We expect the unboundedness of $\alpha$ to be a generic feature of hairy Julia sets and, by analogy, of some fractal aggregates. In other words, probabilistic diffusion is likely to leave behind inactive fjords, completely screened from the diffusive field.

In conclusion, we have utilized the Julia sets to develop some intuition about the $f(\alpha)$

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([1]) This is a generalization of the reverse 1*d* chaotic band doublings in the same sense that period $n$-tuplings of ref. [20, 8] are complex generalizations of the Feigenbaum $\delta$ and approaches the same universal limit as the corresponding sequence of period $n$-tuplings [21].

spectra of fractal aggregates. The $f(\alpha)$ spectra of Julia sets suggest that only the small $\alpha$'s (those which measure the accumulation at tips) are experimentally measurable, while the large $\alpha$'s (those corresponding to the fjords) are noisy and experimentally inaccessible. This screening of fjords can manifest itself as a phase transition at the Hausdorff dimension. In practice the $f(\alpha)$ spectrum should be reliable and easiest to measure for $\alpha$'s up to $\alpha = 1$.

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