

# Periodic orbit analysis of a system with continuous symmetry—A tutorial

Nazmi Burak Budanur,<sup>1,a)</sup> Daniel Borrero-Echeverry,<sup>1,2</sup> and Predrag Cvitanović<sup>1</sup>

<sup>1</sup>*School of Physics and Center for Nonlinear Science, Georgia Institute of Technology, Atlanta, Georgia 30332, USA*

<sup>2</sup>*Department of Physics, Reed College, Portland, Oregon 97202, USA*

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Dynamical systems with translational or rotational symmetry arise frequently in studies of spatially extended physical systems, such as Navier-Stokes flows on periodic domains. In these cases, it is natural to express the state of the fluid in terms of a Fourier series truncated to a finite number of modes. Here, we study a 4-dimensional model with chaotic dynamics and  $SO(2)$  symmetry similar to those that appear in fluid dynamics problems. A crucial step in the analysis of such a system is symmetry reduction. We use the model to illustrate different symmetry-reduction techniques. The system's relative equilibria are conveniently determined by rewriting the dynamics in terms of a symmetry-invariant polynomial basis. However, for the analysis of its chaotic dynamics, the "method of slices," which is applicable to very high-dimensional problems, is preferable. We show that a Poincaré section taken on the "slice" can be used to further reduce this flow to what is for all practical purposes a unimodal map. This enables us to systematically determine all relative periodic orbits and their symbolic dynamics up to any desired period. We then present cycle averaging formulas adequate for systems with continuous symmetry and use them to compute dynamical averages using relative periodic orbits. The convergence of such computations is discussed. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4923742>]

**Periodic orbit theory provides a way to compute dynamical averages for chaotic flows by means of cycle averaging formulas that relate the time averages of observables to the spectra of unstable periodic orbits. Standard cycle averaging formulas are valid under the assumption that the stability multipliers of all periodic orbits have a single marginal direction corresponding to time evolution and are hyperbolic in all other directions. However, if a dynamical system has  $N$  continuous symmetries, periodic orbits are replaced by relative periodic orbits, invariant  $(N + 1)$ -dimensional tori with marginal stability in  $(N + 1)$  directions. Such exact invariant solutions arise in studies of turbulent flows, such as pipe flow or plane Couette flow, which have continuous symmetries. In practice, the translational invariance of these flows is approximated in numerical simulations by using periodic domains so that the state of the fluid is conveniently expressed as a Fourier series, truncated to a large but finite number (from tens to thousands) of Fourier modes. This paper is a tutorial on how such problems can be analyzed using periodic orbit theory. We illustrate all the necessary steps using a simple "two-mode" model as an example.**

## I. INTRODUCTION

Recent experimental observations of traveling waves in pipe flows have confirmed the intuition from dynamical systems theory that invariant solutions of Navier-Stokes equations play an important role in shaping the state space of

turbulent flows.<sup>1</sup> When one casts fluid flow equations in a particular basis, the outcome is an infinite dimensional dynamical system that is often equivariant under transformations such as translations, reflections, and rotations. For example, when periodic boundary conditions are imposed along the streamwise direction, the equations for pipe flow retain their form under the action of streamwise translations, azimuthal rotations, and reflections about the central axis, i.e., they are equivariant under the actions of the  $SO(2) \times O(2)$  symmetry group. In this case, it is natural to express the state of the fluid in a Fourier basis. However, as the system evolves, the nonlinear terms in the equations mix the various modes, so that the state of the system evolves not only along the symmetry directions, but also along directions transverse to them. This complicates the dynamics and gives rise to high dimensional coherent solutions such as relative equilibria and relative periodic orbits. These take on the roles played by equilibria and periodic orbits in flows without symmetry.

There is an extensive literature on equivariant dynamics, which can be traced back to Poincaré's work on the 3-body problem.<sup>2</sup> Early references in the modern dynamical systems literature that we know of are the works of Smale,<sup>3</sup> Field,<sup>4</sup> and Ruelle.<sup>5</sup> Our goal here is not to provide a comprehensive review of this literature, or study its techniques in generality. For those, we refer the reader to monographs by Golubitsky and Stewart,<sup>6</sup> and by Field.<sup>7</sup> Our aim here is much more modest: We would like to provide a hands-on introduction to some of the concepts from equivariant dynamical systems theory, with an emphasis on those aspects relevant to the application of the periodic orbit theory to these systems. To this end, we undertake a step-by-step tutorial approach and illustrate each concept on a two-mode  $SO(2)$  equivariant

<sup>a)</sup>Electronic mail: budanur3@gatech.edu

normal form that has the minimal dimensionality required for chaotic dynamics. We provide visualizations of geometrical concepts, whenever possible. While the example studied here has no physical significance, such an analysis should ultimately be applicable to numerical solutions of turbulent flows on periodic domains, once sufficiently many exact invariant solutions become numerically accessible.

The rest of the paper is organized as follows: In Section II, we define basic concepts and briefly review the relevant symmetry reduction literature. In Section III, we introduce the two-mode model system, discuss several of its symmetry-reduced representations, and utilize a symmetry-reduced polynomial representation to find the only relative equilibrium of the system. In Section IV, we show how the method of slices can be used to quotient the symmetry and reduce the dynamics onto a symmetry-reduced state space or "slice." A Poincaré section taken on the slice then reduces the 4-dimensional chaotic dynamics in the full state space to an approximately one-dimensional, unimodal Poincaré return map. The return map is then used to construct a finite grammar symbolic dynamics for the flow and determine all relative periodic orbits up to a given period. In Section V, we present cycle averaging formulas adequate for systems with continuous symmetries and use the relative periodic orbits calculated in Section IV to calculate dynamically interesting observables. Finally, in Section VI, we discuss possible applications of the method of slices to various spatially extended systems.

The main text is supplemented by two Appendices. Appendix A describes the multi-shooting method used to calculate the relative periodic orbits. Appendix B discusses how periodic Schur decomposition can be used to determine their Floquet multipliers, which can differ by 100s of orders of magnitude even in a model as simple as the two-mode system.

## II. CONTINUOUS SYMMETRIES

A dynamical system  $\dot{a} = v(a)$  is said to be *equivariant* under the group  $G$  of symmetry transformations if

$$v(a) = D(g)^{-1}v(D(g)a), \tag{1}$$

for every point  $a$  in the state space  $\mathcal{M}$  and every element  $g \in G$ , where  $g$  is an abstract group element and  $D(g)$  is its  $[d \times d]$  matrix representation. Infinitesimally, the equivariance condition (1) can be expressed as a vanishing Lie derivative<sup>8</sup>

$$Tv(a) - A(a)t(a) = 0, \tag{2}$$

where  $A(a)$  is the  $[d \times d]$  stability matrix with elements  $A_{ij}(a) = \partial v_i / \partial a_j$ ,  $t(a) = Ta$  is the group tangent at  $a$ , and  $T$  is the  $[d \times d]$  generator of infinitesimal transformations, such that  $D(\theta) = \exp(\theta T)$ , where the phase  $\theta \in [0, 2\pi)$  parametrizes the group action. (We shall interchangeably use notations  $D(g)$  and  $D(\theta)$ .) In general, there is a generator associated with each continuous symmetry. For the simple model considered here, which has a single  $SO(2)$

symmetry, there is only one parameter  $\theta$ , so we only have one generator  $T$ .

If the trajectory of a point  $a_q$  coincides with its group orbit, i.e., for every  $\tau$  there is a group transformation such that

$$a(\tau) = a_q + \int_0^\tau d\tau' v(a(\tau')) = D(\theta(\tau)) a_q, \tag{3}$$

$a_q$  is a point on *relative equilibrium*  $q$ . In our case, this is a 1-torus in state space. Expanding both sides of Eq. (3) for infinitesimal time verifies that the group tangent and the velocity vector are parallel, i.e.,  $v(a_q) = \dot{\theta}(0)t(a_q)$ . By symmetry, this must hold for all  $a(\tau) \in q$ , so for relative equilibria the *phase velocity* is constant,  $\dot{\theta}(\tau) = c$ . Multiplying the equivariance condition (2) by  $c$ , we find that velocity is a marginal stability eigenvector in the reference frame co-moving with the relative equilibrium,

$$(A(a) - cT)v(a) = 0, \quad a \in \mathcal{M}_q. \tag{4}$$

A state space point  $a_p$  lies on a *relative periodic orbit* of period  $T_p$  if its trajectory first intersects its group orbit after a finite time  $T_p$ ,

$$a(T_p) = a_p + \int_0^{T_p} d\tau' v(a(\tau')) = D(\theta_p) a_p, \tag{5}$$

with a phase  $\theta_p$ . In systems with  $SO(2)$  symmetry, relative periodic orbits are topologically 2-tori, where the trajectory of  $a_p$  generically traces out the torus ergodically by repeating the same path shifted by the group action  $D(\theta_p)$  after each prime period  $T_p$ . As we will see in Section IV, these tori can be very convoluted and difficult to visualize. In special cases where  $\theta_p = 0$ , the solution is a periodic orbit, a 1-dimensional loop in state space and the 2-torus is generated by all actions of the symmetry group on this loop.

The linear stability of relative periodic orbits is captured by their *Floquet multipliers*  $\Lambda_{p,j}$ , the eigenvalues of the Jacobian  $\hat{J}_p$  of the time-forward map  $a(\tau) = f^\tau(a(0))$ .  $\hat{J}_p$  is defined as

$$\hat{J}_p = D(-\theta_p)J_p^T(a_p), \quad \text{where } J_j^\tau(a(0)) = \frac{\partial a_i(\tau)}{\partial a_j(0)}. \tag{6}$$

The magnitude of  $\Lambda_{p,j}$  determines whether a small perturbation along its corresponding eigendirection (or Floquet vector) will expand or contract after one period. If the magnitude of  $\Lambda_{p,j}$  is greater than 1, the perturbation expands; if it is less than 1, the perturbation contracts. In systems with  $N$  continuous symmetries, relative periodic orbits have  $(N + 1)$  marginal directions ( $|\Lambda_{p,j}| = 1$ ), which correspond to the temporal evolution of the flow and the  $N$  symmetries. By applying symmetry reduction, the marginal Floquet multipliers corresponding to the symmetries are replaced by 0, so that periodic orbit theory, which requires that the flow have only one marginal direction, becomes applicable.

*Symmetry reduction* is a coordinate transformation that maps all the points on a group orbit  $D(\theta)a$ , which are equivalent from a dynamical perspective, to a single representative

point in a symmetry reduced space. After symmetry reduction, relative equilibria and relative periodic orbits are converted to equilibria and periodic orbits in a reduced state space without loss of dynamical information; the full state space trajectory can always be retrieved via a reconstruction equation.

One well-studied technique for symmetry reduction, which works well for low-dimensional dynamical systems, such as the Lorenz system, is to recast the dynamical equations in terms of invariant polynomials.<sup>9</sup> However, there are multiple difficulties associated with using these techniques. Computing invariants is a non-trivial problem, and even for the simple case of  $SO(2)$ , computer algebra methods for finding invariants become impractical for systems with more than a dozen dimensions.<sup>10</sup> Moreover, the projection of a linear equivariant vector field onto orbit space is not necessarily a linear operation.<sup>11</sup> This means that even when conducting basic operations such as the linearization of nonlinear vector fields, special attention has to be paid to the choice of invariants, even when it is possible to find them. In contrast, the method of slices,<sup>12–18</sup> which we study in detail here, is a symmetry reduction scheme applicable to high-dimensional flows like the Navier-Stokes equations.<sup>19</sup>

### A. Method of slices

In a system with  $N$  continuous symmetries, a *slice*  $\hat{\mathcal{M}}$  is a codimension  $N$  submanifold of  $\mathcal{M}$  that cuts every group orbit once and only once. In the *method of slices*, the solution of a  $d$ -dimensional dynamical system is represented as a symmetry-reduced trajectory  $\hat{a}(\tau)$  within the  $(d - N)$ -dimensional slice and  $N$  time dependent group parameters  $\theta(\tau)$ , which map  $\hat{a}(\tau)$  to the full state space by the group action  $D(\theta(\tau))$  that defines a *moving frame*.

The idea goes back to Cartan,<sup>20</sup> and there is a rich literature on the method of slices (in variety of guises) and its applications to problems in dynamical systems theory: notable examples include the work of Field,<sup>21</sup> Krupa,<sup>22</sup> and Ashwin and Melbourne,<sup>23</sup> who used slicing to prove rigorous results for equivariant systems. Fels and Olver<sup>24,25</sup> used the method of moving frames to compute invariant polynomials. Haller and Mezić<sup>26</sup> used the method of slices, under the name “orbit projection map,” to study three-dimensional volume-preserving flows. Our presentation closely follows that of Refs. 12 and 13, the former of which derives the “reconstruction equation” for the template fitting method of Ref. 27.

A general definition of a slice puts no restriction on its shape and offers no guidance on how to construct it. For Lie groups, it is computationally convenient to use a local, linear approximation to the slice (a *slice hyperplane*) constructed in the neighborhood of a point  $\hat{a}'$ . (For a general discussion of how a local slice is defined with the help of the tubular neighborhood theorem, the reader is referred to Refs. 7, 28, and 29). The point  $\hat{a}'$  is called the *slice template* and the slice hyperplane is then defined as the hyperplane that contains  $\hat{a}'$  and is perpendicular to its group tangent  $t' = T\hat{a}'$ . The relationship between a template, its slice hyperplane, and symmetry-reduced trajectories is illustrated in Fig. 1.

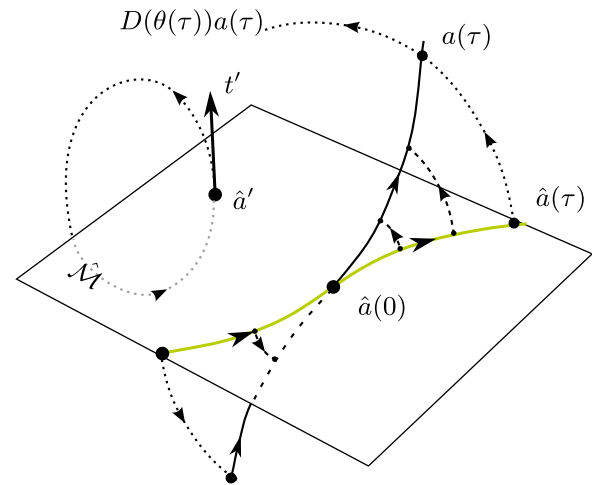


FIG. 1. The slice hyperplane  $\hat{\mathcal{M}}$  is a hyperplane that contains the template point  $\hat{a}'$  and is normal to its group tangent  $t'$ . It intersects all group orbits (dotted lines) in an open neighborhood of  $\hat{a}'$ . The full state space trajectory  $a(\tau)$  (solid black line) and the reduced state space trajectory  $\hat{a}(\tau)$  (thick green line) belong to the same group orbit  $\mathcal{M}_{a(\tau)}$  and are equivalent up to a group rotation  $D(\theta(\tau))$ . Adapted from Ref. 8.

Reduced trajectories  $\hat{a}(t)$  can be obtained in two ways: by post-processing data or by reformulating the dynamics and integrating directly in the slice hyperplane. The post-processing method (also called the *method of moving frames*<sup>24,30</sup>) can be applied to both numerical and experimental data. Here, one takes the data in the full state space and looks for the time dependent group parameter that brings the trajectory  $a(\tau)$  onto the slice. That is, one finds  $\theta(\tau)$  such that  $\hat{a}(\tau) = D(-\theta(\tau))a(\tau)$  satisfies the slice condition

$$\langle \hat{a}(\tau) - \hat{a}' | t' \rangle = 0. \quad (7)$$

In the second implementation (valid only for abelian groups), one reformulates the dynamics as

$$\hat{v}(\hat{a}) = v(\hat{a}) - \dot{\theta}(\hat{a}) t(\hat{a}), \quad (8a)$$

$$\dot{\theta}(\hat{a}) = \langle v(\hat{a}) | t' \rangle / \langle t(\hat{a}) | t' \rangle, \quad (8b)$$

which can then be directly integrated to get the symmetry-reduced trajectory  $\hat{a}(\tau)$  and the reconstruction angle  $\theta(\tau)$ . In Eq. (8),  $\hat{v}$  is the projection of the full state space velocity  $v(\hat{a})$  onto the slice hyperplane. For a derivation, see Ref. 8.

While early studies<sup>12,13,31</sup> applied the method of slices to a single solution at a time, studying the nonlinear dynamics of extended systems requires symmetry reduction of global objects, such as strange attractors and inertial manifolds. In this spirit, Ref. 14 used the method of slices to quotient the  $SO(2)$  symmetry from the chaotic dynamics of complex Lorenz flow. They showed that the singularity of the reconstruction equation that occurs when the denominator in Eq. (8b) vanishes (e.g., when the group tangents of the trajectory and the template are orthogonal) causes the reduced flow to make discontinuous jumps. The set of points  $\hat{a}^*$  where this occurs satisfies

$$\langle t(\hat{a}^*) | t' \rangle = 0, \quad (9)$$

and make up the *slice border* (studied in detail in Ref. 15).

Two strategies have been proposed in order to handle this problem: The first attempts to try to identify a template such that slice singularities are not visited by the dynamics.<sup>14</sup> The second uses multiple "charts" of connected slice hyperplanes,<sup>12,15</sup> switching between charts when the dynamics approach the border of a particular chart. The latter approach was applied to complex Lorenz flow by Cvitanović *et al.*<sup>16</sup> and to pipe flow by Willis, Cvitanović, and Avila.<sup>17</sup> However, neither approach is straightforward to apply, particularly in high-dimensional systems.

**B. First Fourier mode slice**

A third strategy has recently been proposed by Budanur *et al.*,<sup>18</sup> who considered Fourier space discretizations of partial differential equations (PDEs) with SO(2) symmetry. They showed that in these cases a simple choice of slice template, associated with the first Fourier mode, results in a slice in which it is highly unlikely that generic dynamics visit the neighborhood of the singularity. If the dynamics do occasionally come near the singularity, these close passages can be regularized by means of a time rescaling. Here, we shall illustrate this approach, which we call the "first Fourier mode slice," and apply it to a model system with two modes that will be described in Section III.

In the discussion so far, we have not specified any constraints on the symmetry group to be quotiented beyond the requirement that it be abelian as required for Eq. (8) to be valid. Since we are interested in spatially extended systems with translational symmetry, and in order to keep the notation compact, we restrict our discussion to one dimensional PDEs describing the evolution of a field  $u(x, t)$  in a periodic domain. By expressing the solutions in terms of a Fourier series

$$u(x, \tau) = \sum_{k=-\infty}^{\infty} u_k(\tau) e^{ikx}, \quad u_k = x_k + iy_k, \quad (10)$$

the translationally invariant PDE can be replaced by a system of coupled nonlinear ordinary differential equations (ODEs) for the Fourier coefficients equivariant under the 1-parameter compact group of SO(2) rotations.

Truncating the expansion to  $m$  modes, we write the real and imaginary parts of the Fourier coefficients with  $k \geq 1$  as the state vector  $a = (x_1, y_1, x_2, y_2, \dots, x_m, y_m)$ . The action of the SO(2) group on this vector can then be expressed as a block diagonal matrix

$$D(\theta) = \begin{pmatrix} R(\theta) & 0 & \cdots & 0 \\ 0 & R(2\theta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R(m\theta) \end{pmatrix}, \quad (11)$$

where

$$R(n\theta) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \quad (12)$$

is the rotation matrix for  $n$ th Fourier mode. The Lie algebra element for  $D(\theta)$  is given by

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -2 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -m \\ 0 & 0 & 0 & 0 & \cdots & m & 0 \end{pmatrix}. \quad (13)$$

In order to construct a slice hyperplane for such a system, we choose the following slice template:

$$\hat{a}' = (1, 0, \dots, 0). \quad (14)$$

The slice condition (7) then constrains points on the reduced trajectory to the hyperplane given by

$$\hat{a} = (\hat{x}_1, 0, \hat{x}_2, \hat{y}_2, \dots, \hat{x}_m, \hat{y}_m). \quad (15)$$

As discussed earlier, group orbits should cross the slice once and only once, which we achieve by restricting the slice hyperplane to the half-space where  $\hat{x}_1 > 0$ . In general, a slice hyperplane can be constructed by following a similar procedure for any choice of template. However, the power of choosing template (14) becomes apparent by computing the border (9) of its slice hyperplane. The points on Eq. (15) lie on the slice border only if  $\hat{x}_1 = 0$ . This means that as long the dynamics are such that the magnitude of the first mode never vanishes, every group orbit is guaranteed to have a unique representative point on the slice hyperplane. By symmetry, any template of the form  $\hat{a}' = (\hat{x}'_1, \hat{y}'_1, 0, \dots, 0)$  would work just as well. The slice template (14) was chosen for notational and computational convenience.

More insight can be gained by writing the symmetry-reduced evolution equations (8) explicitly for template (14)

$$\hat{v}(\hat{a}) = v(\hat{a}) - \frac{\dot{y}_1(\hat{a})}{\hat{x}_1} t(\hat{a}), \quad (16a)$$

$$\dot{\theta}(\hat{a}) = \frac{\dot{y}_1(\hat{a})}{\hat{x}_1}. \quad (16b)$$

Since the argument  $\phi_1$  of a point  $(x_1, y_1)$  in the first Fourier mode plane is given by  $\phi_1 = \tan^{-1} \frac{y_1}{x_1}$ , its velocity is

$$\dot{\phi}_1 = \frac{x_1}{r_1^2} \dot{y}_1 - \frac{y_1}{r_1^2} \dot{x}_1, \quad (17)$$

where  $r_1^2 = x_1^2 + y_1^2$ . Therefore, on the slice hyperplane (15), where  $\hat{y}_1 = 0$ ,

$$\dot{\theta}(\hat{a}) = \dot{\phi}_1(\hat{a}). \quad (18)$$

That is, for our choice of template (14), the reconstruction phase coincides with the phase of the first Fourier mode. This makes this choice of template more natural from a group-theoretic point of view than the physically motivated templates used in Refs. 12–17.

In general, additional care must be taken when the dynamics approach the slice border  $\hat{x}_1 = 0$ . Whenever this happens, the near-divergence of  $\hat{v}$  can be regularized by introducing a rescaled time coordinate<sup>18</sup> such that  $d\hat{\tau} = d\tau/\hat{x}_1$ . However, in our analysis of the two-mode system that we introduce below, we omit this step since points with a vanishing first mode are in an invariant subspace of the flow and, hence, are never visited by the dynamics.

### C. Geometric interpretation of the first Fourier mode slice

Before moving on to our analysis of the two-mode model, we first discuss the geometrical interpretation of the first Fourier mode slice. The slice defined by Eq. (14), along with the directional constraint  $\hat{x}_1 > 0$ , fixes the phase of the first complex Fourier mode to 0. This can also be seen from Eq. (18), which shows that if the first Fourier mode slice (14) is used as a template, the reconstruction phase is the same as the phase of the first Fourier mode (18).

In complex representation, we can express the relationship between Fourier modes ( $z_n = x_n + iy_n$ ) and their representative points ( $\hat{z} = \hat{x}_n + i\hat{y}_n$ ) on the slice hyperplane by the U(1) action

$$\hat{z}_n = e^{-in\phi_1} z_n. \quad (19)$$

This relation provides another interpretation for the slice border: For template (14), the slice border condition (9) defines the slice border as those points where  $|\hat{z}_1| = |z_1| = 0$ . At these points, the phase of the first Fourier mode is not well-defined and hence neither is the transformation (19).

This is illustrated in Fig. 2, where the first Fourier mode slice hyperplane is shown along with the group orbits of points with decreasing  $|z_1|$ . When the magnitude of the first

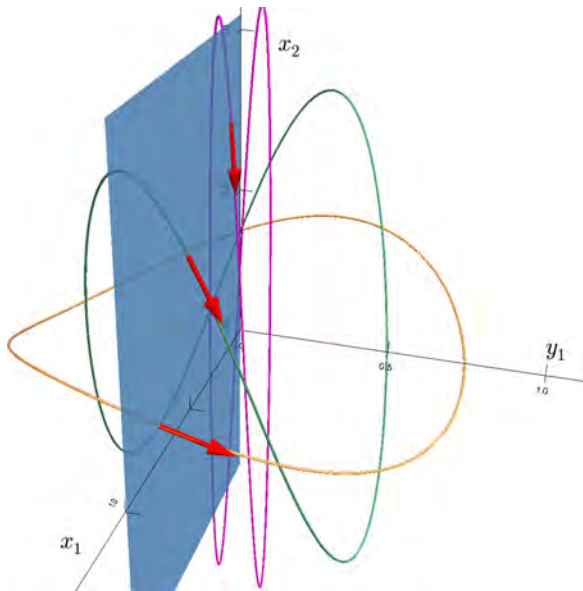


FIG. 2. SO(2) group orbits of state space points (0.75, 0, 0.1, 0.1) (orange), (0.5, 0, 0.5, 0.5) (green), (0.1, 0, 0.75, 0.75) (pink), and the first mode (15) slice hyperplane (blue). The group tangents at the intersections with the slice hyperplane are shown as red arrows. As the magnitude of the first Fourier mode decreases relative to the magnitude of the second one, so does the group tangent angle to the slice hyperplane.

mode is small relative to that of the second (pink curve), the group tangent at the representative point for the group orbit (i.e., where the group orbit and the slice hyperplane intersect) has a larger component parallel to the slice hyperplane. If the magnitude of the first mode was exactly 0, the group tangent would lie entirely on the slice hyperplane, satisfying the slice border condition.

In Ref. 32, a polar coordinate representation of a two Fourier mode normal form is obtained by defining the  $G$ -invariant phase:  $\Phi = \phi_2 - 2\phi_1$  and three symmetry invariant coordinates  $\{r_1, r_2 \cos \Phi, r_2 \sin \Phi\}$ . One can see by direct comparison with Eq. (19), which yields  $\hat{z}_1 = r_1$  and  $\hat{z}_2 = r_2 e^{i\Phi}$ , that this representation is a special case ( $m=2$ ), of the slice defined by Eq. (14). Corresponding ODEs for the polar representation for this two-mode system were obtained in Ref. 32 by chain rule and substitution. Note that the method of slices provides a general form (16a) for symmetry reduced time evolution.

### III. TWO-MODE SO(2)-EQUIVARIANT FLOW

Dangelmayr,<sup>33</sup> Armbruster, Guckenheimer and Holmes,<sup>34</sup> Jones and Proctor,<sup>35</sup> and Porter and Knobloch<sup>32</sup> (for more details, see Sect. XX.1 in Golubitsky *et al.*<sup>36</sup>) have previously investigated bifurcations in 1:2 resonance ODE normal form models to third order in the amplitudes. Here, we use this model as a starting point from which we derive what may be one of the simplest chaotic systems with continuous symmetry. We refer to this as the two-mode system

$$\begin{aligned} \dot{z}_1 &= (\mu_1 - i e_1) z_1 + a_1 z_1 |z_1|^2 + b_1 z_1 |z_2|^2 + c_1 \bar{z}_1 z_2 \\ \dot{z}_2 &= (\mu_2 - i e_2) z_2 + a_2 z_2 |z_1|^2 + b_2 z_2 |z_2|^2 + c_2 z_1^2, \end{aligned} \quad (20)$$

where  $z_1$  and  $z_2$  are complex and all parameters real-valued. The parameters  $\{e_1, e_2\}$  break the reflectional symmetry of the O(2)-equivariant normal form studied by Dangelmayr<sup>33</sup> leading to an SO(2)-equivariant system. This complex two mode system can be expressed as a 4-dimensional system of real-valued first order ODEs by substituting  $z_1 = x_1 + iy_1$ ,  $z_2 = x_2 + iy_2$ , so that

$$\begin{aligned} \dot{x}_1 &= (\mu_1 + a_1 r_1^2 + b_1 r_2^2 + c_1 x_2) x_1 + c_1 y_1 y_2 + e_1 y_1, \\ \dot{y}_1 &= (\mu_1 + a_1 r_1^2 + b_1 r_2^2 - c_1 x_2) y_1 + c_1 x_1 y_2 - e_1 x_1, \\ \dot{x}_2 &= (\mu_2 + a_2 r_1^2 + b_2 r_2^2) x_2 + c_2 (x_1^2 - y_1^2) + e_2 y_2, \\ \dot{y}_2 &= (\mu_2 + a_2 r_1^2 + b_2 r_2^2) y_2 + 2c_2 x_1 y_1 - e_2 x_2, \end{aligned} \quad (21)$$

where  $r_1^2 = x_1^2 + y_1^2$ ,  $r_2^2 = x_2^2 + y_2^2$ .

The large number of parameters ( $\mu_1, \mu_2, a_1, a_2, b_1, b_2, c_1, c_2, e_1, e_2$ ) in this system makes full exploration of the parameter space impractical. Following in the tradition of Lorenz,<sup>37</sup> Hénon,<sup>38</sup> and Rössler,<sup>39</sup> we have tried various choices of parameters until settling on the following set of values, which we will use in all numerical calculations presented here:

$$\begin{array}{cccccccccc} \mu_1 & \mu_2 & e_1 & e_2 & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \\ -2.8 & 1 & 0 & 1 & -1 & -2.66 & 1 & -0 & -7.75 & 1 \end{array}. \quad (22)$$

This choice of parameters is far from the bifurcation values studied by previous authors,<sup>32–35</sup> so that the model has no physical interpretation. However, these parameters yield chaotic dynamics, making the two-mode system a convenient minimal model for the study of chaos in the presence of a continuous symmetry: It is a 4-dimensional SO(2)-equivariant model, whose symmetry-reduced dynamics are chaotic and take place on a three-dimensional manifold. For another example of parameter values that result in chaotic dynamics, see Ref. 32.

It can be checked by inspection that Eq. (20) is equivariant under the U(1) transformation

$$(z_1, z_2) \rightarrow (e^{i\theta} z_1, e^{i2\theta} z_2). \tag{23}$$

In the real representation (21), the SO(2) group action (23) on a state space point  $a$  is given  $\exp(\theta T)a$ , where  $a^\top = (x_1, y_1, x_2, y_2)$  and  $T$  is the Lie algebra element

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}. \tag{24}$$

One can easily check that the real two-mode system (21) satisfies the equivariance condition (2).

From Eq. (20), it is obvious that the equilibrium point  $(z_1, z_2) = (0, 0)$  is an invariant subspace and that  $z_1 = 0, z_2 \neq 0$  is a 2-dimensional flow-invariant subspace

$$\dot{z}_1 = 0, \quad \dot{z}_2 = (\mu_2 - i e_2 + b_2 |z_2|^2) z_2, \tag{25}$$

with a single circular relative equilibrium of radius  $r_2 = \|z_2\| = \sqrt{-\mu_2/b_2}$  with phase velocity  $c = -e_2/2$ . At the origin, the stability matrix  $A$  commutes with  $T$ , and so, can be block-diagonalized into two  $[2 \times 2]$  matrices. The eigenvalues of  $A$  at  $(0, 0, 0, 0)$  are  $\lambda_1 = \mu_1$  with multiplicity 2 and  $\lambda_2 = \mu_2 \pm i e_2$ . In the  $(x_1, y_1, x_2, y_2)$  coordinates, the eigenvectors for  $\lambda_1$  are  $(1, 0, 0, 0)$  and  $(0, 1, 0, 0)$  and the eigenvectors for  $\lambda_2$  are  $(0, 0, 1, 0)$  and  $(0, 0, 0, 1)$ .

In contrast,  $z_2 = 0$  is not, in general, a flow-invariant subspace since the dynamics

$$\dot{z}_1 = (\mu_1 - i e_1) z_1 + a_1 z_1 |z_1|^2, \quad \dot{z}_2 = c_2 z_1^2$$

take the flow out of the  $z_2 = 0$  plane.

### A. Invariant polynomial bases

Before continuing our tutorial on the use of the method of slices using the first Fourier mode slice, we briefly discuss the symmetry reduction of the two-mode system using invariant polynomials. While representations of our model in terms of invariant polynomials and polar coordinates are useful for cross-checking our calculations in the full state space  $a^\top = (x_1, x_2, y_1, y_2)$ , their construction requires a bit of algebra even for this simple 4-dimensional flow. For very high-dimensional flows, such as Kuramoto-Sivashinsky and Navier-Stokes flows, we do not know how to carry out such

constructions. As discussed in Refs. 32–34, for the two-mode system, it is easy to construct a set of four real-valued SO(2) invariant polynomials

$$\begin{aligned} u &= z_1 \bar{z}_1, & v &= z_2 \bar{z}_2 \\ w &= z_1^2 \bar{z}_2 + \bar{z}_1^2 z_2, & q &= (z_1^2 \bar{z}_2 - \bar{z}_1^2 z_2)/i. \end{aligned} \tag{26}$$

The polynomials  $[u, v, w, q]$  are linearly independent, but related through one syzygy,

$$w^2 + q^2 - 4u^2v = 0 \tag{27}$$

that confines the dynamics to a 3-dimensional manifold  $\hat{\mathcal{M}} = \mathcal{M}/\text{SO}(2)$ , which is a symmetry-invariant representation of the 4-dimensional SO(2) equivariant dynamics. We call this the reduced state space. By construction,  $u \geq 0, v \geq 0$ , but  $w$  and  $q$  can be of either sign. That is explicit if we express  $z_1$  and  $z_2$  in polar coordinates ( $z_1 = |u|^{1/2} e^{i\phi_1}, z_2 = |v|^{1/2} e^{i\phi_2}$ ), so that  $w$  and  $q$  take the form

$$\begin{aligned} w &= 2 \text{Re}(z_1^2 \bar{z}_2) = 2u|v|^{1/2} \cos \psi, \\ q &= 2 \text{Im}(z_1^2 \bar{z}_2) = 2u|v|^{1/2} \sin \psi, \end{aligned} \tag{28}$$

where  $\psi = 2\phi_1 - \phi_2$ .

The dynamical equations for  $[u, v, w, q]$  follow from the chain rule, which yields

$$\begin{aligned} \dot{u} &= \bar{z}_1 \dot{z}_1 + z_1 \dot{\bar{z}}_1, & \dot{v} &= \bar{z}_2 \dot{z}_2 + z_2 \dot{\bar{z}}_2, \\ \dot{w} &= 2\bar{z}_2 z_1 \dot{z}_1 + 2z_2 \bar{z}_1 \dot{\bar{z}}_1 + z_1^2 \dot{\bar{z}}_2 + \bar{z}_1^2 \dot{z}_2, \\ \dot{q} &= (2\bar{z}_2 z_1 \dot{z}_1 - 2z_2 \bar{z}_1 \dot{\bar{z}}_1 + z_1^2 \dot{\bar{z}}_2 - \bar{z}_1^2 \dot{z}_2)/i. \end{aligned} \tag{29}$$

Substituting Eq. (20) into Eq. (29), we obtain a set of four SO(2)-invariant equations,

$$\begin{aligned} \dot{u} &= 2\mu_1 u + 2a_1 u^2 + 2b_1 uv + c_1 w, \\ \dot{v} &= 2\mu_2 v + 2a_2 uv + 2b_2 v^2 + c_2 w, \\ \dot{w} &= (2\mu_1 + \mu_2)w + (2a_1 + a_2)uw + (2b_1 + b_2)v w \\ &\quad + 4c_1 uv + 2c_2 u^2 + (2e_1 - e_2)q, \\ \dot{q} &= (2\mu_1 + \mu_2)q + (2a_1 + a_2)uq \\ &\quad + (2b_1 + b_2)vq - (2e_1 - e_2)w. \end{aligned} \tag{30}$$

Note that the O(2)-symmetry breaking parameters  $\{e_1, e_2\}$  of the Dangelmayr normal form system<sup>33</sup> appear only in the relative phase combination  $(2e_1 - e_2)$ , so one of the two can be set to zero without loss of generality. This consideration motivated our choice of  $e_1 = 0$  in Eq. (22). Using the syzygy (27), we can eliminate  $q$  from Eq. (30) to get

$$\begin{aligned} \dot{u} &= 2\mu_1 u + 2a_1 u^2 + 2b_1 uv + c_1 w, \\ \dot{v} &= 2\mu_2 v + 2a_2 uv + 2b_2 v^2 + c_2 w, \\ \dot{w} &= (2\mu_1 + \mu_2)w + (2a_1 + a_2)uw + (2b_1 + b_2)v w \\ &\quad + 4c_1 uv + 2c_2 u^2 + (2e_1 - e_2)(4u^2v - w^2)^{1/2}. \end{aligned} \tag{31}$$

This invariant basis can be used either to investigate the dynamics directly or to visualize solutions<sup>9</sup> computed in the full equivariant basis (20).

### B. Equilibria of the symmetry-reduced dynamics

The first step in elucidating the geometry of attracting sets is the determination of their equilibria. We shall now show that the problem of determining the equilibria of the symmetry-reduced two-mode (30) system  $[u^*, v^*, w^*, q^*]$  can be reduced to finding the real roots of a multinomial expression. First, we define

$$A_1 = \mu_1 + a_1 u + b_1 v, \quad A_2 = \mu_2 + a_2 u + b_2 v, \quad (32)$$

and rewrite Eq. (30) as

$$\begin{aligned} 0 &= 2A_1 u + c_1 w, & 0 &= 2A_2 v + c_2 w, \\ 0 &= (2A_1 + A_2) w + 2(c_2 u + 2c_1 v) u \\ &\quad + (2e_1 - e_2) q, \\ 0 &= (2A_1 + A_2) q - (2e_1 - e_2) w. \end{aligned} \quad (33)$$

We already know that  $[0, 0, 0, 0]$  and  $[0, -\mu_2/b_2, 0, 0]$  are the only roots in the  $u=0$  and  $v=0$  subspaces, so we are looking only for the  $u > 0, v > 0, w, q \in \mathbb{R}$  solutions; there could be non-generic roots with either  $w=0$  or  $q=0$ , but not both simultaneously, since the syzygy (27) precludes that. Either  $w$  or  $q$  can be eliminated by obtaining the following relations from Eq. (33):

$$\begin{aligned} w &= -\frac{2u}{c_1} A_1 = -\frac{2v}{c_2} A_2, \\ q &= \frac{2(-2e_1 + e_2)uv}{c_2 u + 2c_1 v}. \end{aligned} \quad (34)$$

Substituting Eq. (34) into Eq. (33), we get two bivariate polynomials whose roots are the equilibria of system (30)

$$\begin{aligned} f(u, v) &= c_2 u A_1 - c_1 v A_2 = 0, \\ g(u, v) &= (4A_1^2 u^2 - 4c_1^2 u^2 v)(c_2 u + 2c_1 v)^2 \\ &\quad + 4c_1^2 (-2e_1 + e_2)^2 u^2 v^2 = 0. \end{aligned} \quad (35)$$

We divide the common multiplier  $u^2$  from the second equation and by doing so, eliminate one of the two roots at the origin, as well as the  $[0, -\mu_2/b_2, 0, 0]$  root within the invariant subspace (25). Furthermore, we scale the parameters and variables as  $\tilde{u} = c_2 u, \tilde{v} = c_1 v, \tilde{a}_1 = a_1/c_2, \tilde{b}_1 = b_1/c_1, \tilde{a}_2 = a_2/c_2, \tilde{b}_2 = b_2/c_1$  to get

$$\tilde{f}(\tilde{u}, \tilde{v}) = \tilde{u} \tilde{A}_1 - \tilde{v} \tilde{A}_2 = 0, \quad (36)$$

$$\tilde{g}(\tilde{u}, \tilde{v}) = (\tilde{A}_1^2 - c_1 \tilde{v})(\tilde{u} + 2\tilde{v})^2 + e_2^2 \tilde{v}^2 = 0, \quad (37)$$

where  $\tilde{A}_1 = \mu_1 + \tilde{a}_1 \tilde{u} + \tilde{b}_1 \tilde{v}$  and  $\tilde{A}_2 = \mu_2 + \tilde{a}_2 \tilde{u} + \tilde{b}_2 \tilde{v}$ .

Solving coupled bivariate polynomials such as Eqs. (36) and (37) is not, in general, a trivial task. However, for the choice of parameters given by Eq. (22), Eq. (36) yields  $\tilde{v} = (\mu_1 + \tilde{a}_1 \tilde{u})/(\mu_2 + \tilde{a}_2 \tilde{u})$ . Substituting this into Eq. (37) makes it a fourth order polynomial in  $u$ , which we can solve. Only the non-negative, real roots of this polynomial correspond to relative equilibria in the two-mode state space since  $u$  and  $v$  are the squares of first and second mode amplitudes, respectively. Two roots satisfy this condition, the equilibrium at the origin

$$p_{EQ} = [0, 0, 0, 0], \quad (38)$$

and the relative equilibrium

$$p_{TW} = [0.193569, 0.154131, -0.149539, -0.027178]. \quad (39)$$

Note that by setting  $b_2=0$ , we send the relative equilibrium at  $[0, -\mu_2/b_2, 0, 0]$  to infinity. Thus, Eq. (39) is the only relative equilibrium of the two-mode system for our choice of parameters. While this is an equilibrium in the invariant polynomial basis, in the  $SO(2)$ -equivariant, real-valued state space this is a 1-dimensional relative equilibrium group orbit. The point on this orbit that lies in first Fourier mode slice is (see Fig. 4(c))

$$(x_1, y_1, x_2, y_2) = (0.439966, 0, -0.386267, 0.070204). \quad (40)$$

We computed the linear stability eigenvalues and eigenvectors of this relative equilibrium by analyzing the stability matrix within the first Fourier mode slice  $\hat{A}_{ij}(\hat{a}) = \partial \hat{v}_i / \partial \hat{a}_j|_{\hat{a}}$ , resulting in linear stability eigenvalues

$$\lambda_{1,2} = 0.05073 \pm i 2.4527, \quad \lambda_3 = -5.5055, \quad \lambda_4 = 0. \quad (41)$$

The 0 eigenvalue corresponds to the direction transverse to the slice. We expect this since the reduced trajectory evolution equation (8a) keeps the solution within the slice. The imaginary part of the expanding complex pair sets the “winding time” in the neighborhood of the equilibrium to  $T_w = 2\pi/\text{Im}(\lambda_1) = 2.5617$ . The large magnitude of the contracting eigenvalue  $\lambda_3$  yields a very thin attractor in the reduced state space, so that in a planar Poincaré section, the two-mode flow is almost one dimensional, as shown in Figs. 5(a) and 5(b).

### C. No chaos when the reflection symmetry is restored

Before finishing our discussion of invariant polynomials, we make an important observation regarding the case when both of the reflection symmetry breaking parameters,  $e_1$  and  $e_2$  are set to 0. In this case,  $z_{1,2} \rightarrow \bar{z}_{1,2}$  symmetry is restored and the evolution equations for  $u, v$ , and  $w$  in Eq. (30) become independent of  $q$ . Furthermore, the time evolution equation for  $q$  becomes linear in  $q$  itself, so that it can be expressed as

$$\dot{q} = \zeta(u, v)q. \quad (42)$$

Hence, the time evolution of  $q$  can be written as

$$q(\tau) = e^{\int_0^\tau dt' \zeta(u(t'), v(t'))} q(0). \quad (43)$$

If we assume that the flow is bounded, then we can also assume that a long time average of  $\zeta$  exists. The sign of this average determines the long term behavior of  $q(\tau)$ ; it will either diverge or vanish depending on the sign of  $\langle \zeta \rangle$  being positive or negative, respectively. The former case leads to a contradiction: If  $q(\tau)$  diverges, the symmetry-invariant flow cannot be bounded since the syzygy (27) must be satisfied at all times. If  $q(t)$  vanishes, there are three invariant polynomials left, which are still related to each other by the syzygy. Thus, the flow is confined to a two dimensional manifold and

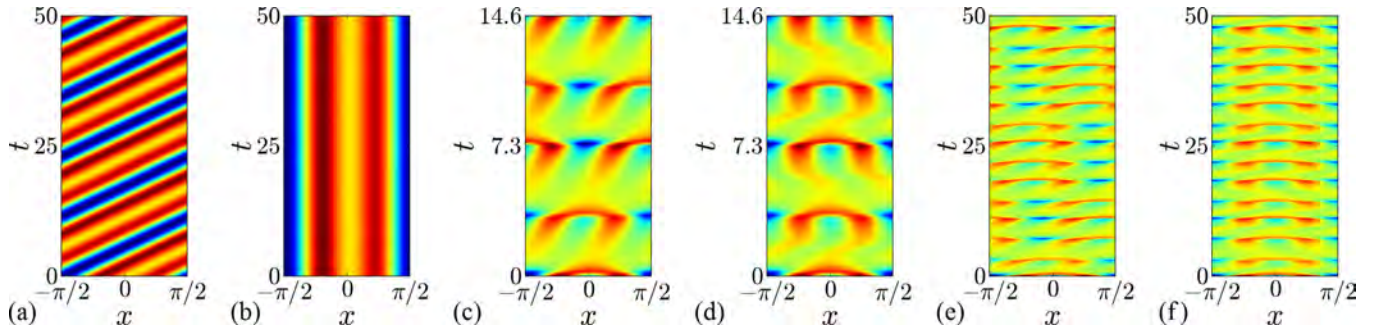


FIG. 3. The relative equilibrium TW in (a) the system's configuration space becomes an equilibrium in (b) the symmetry-reduced configuration space. Two cycles of the relative periodic orbit  $\overline{01}$  in (c) symmetry-equivariant configuration space become a periodic orbit in (d) the symmetry-reduced configuration space. (e) A typical ergodic trajectory of the two-mode system in the system's configuration space, (f) in the symmetry-reduced configuration space. The color scale used in each figure is different to enhance contrast.

cannot exhibit chaos. We must stress that this is a special result that holds for the two-mode normal form with terms up to third order.

#### D. Visualizing two-mode dynamics

We now present visualizations of the dynamics of the two-mode system in four different representations: as 3D projections of the four-dimensional real-valued state space, as 3D projections in the invariant polynomial basis, as dynamics in the 3D slice hyperplane, and as two-dimensional spacetime diagrams of the color-coded field  $u(x, \tau)$ , which is defined as follows:

$$u(x, \tau) = \sum_{k=-2}^2 z_k(\tau) e^{ikx},$$

where  $z_{-k} = \bar{z}_k$ ,  $z_0 = 0$ , and  $x \in [-\pi, \pi]$ . We can also define the symmetry reduced configuration space representation as the inverse Fourier transform of the symmetry reduced Fourier modes

$$\hat{u}(x, \tau) = \sum_{k=-2}^2 \hat{z}_k(\tau) e^{ikx},$$

where  $\hat{z}_{-k} = \bar{\hat{z}}_k$ ,  $\hat{z}_0 = 0$  and  $x \in [-\pi, \pi]$ . Figures 3(a) and 3(b) show the sole relative equilibrium TW of the two-mode system in the symmetry-equivariant and symmetry-reduced

configuration spaces, respectively. After symmetry reduction, the relative equilibrium becomes an equilibrium. Figures 3(c) and 3(d) show the relative periodic orbit  $\overline{01}$  again, respectively, in the symmetry-equivariant and symmetry-reduced configuration space representations. Similar to the relative equilibrium, the relative periodic orbit becomes a periodic orbit after symmetry reduction. Finally, Figs. 3(e) and 3(f) show a typical ergodic trajectory of the two-mode system in symmetry-equivariant and symmetry-reduced configuration space representations. Note that in each case, symmetry reduction cancels the "drifts" along the symmetry ( $x$ ) direction.

As can be seen clearly in Fig. 4(a), these drifts show up in the Fourier mode representation as  $SO(2)$  rotations. The relative equilibrium TW traces its  $SO(2)$  group orbit (green curve in Fig. 4(a)) as it drifts in the configuration space. The relative periodic orbit  $\overline{01}$  (red) and the ergodic trajectory (blue) rotate in the same fashion as they evolve. Figures 4(b) and 4(c) show a three dimensional projection onto the invariant polynomial basis and the 3-dimensional trajectory on the slice hyperplane for the same orbits. In both figures, the relative equilibrium is reduced to an equilibrium and the relative periodic orbit is reduced to a periodic orbit.

#### IV. PERIODIC ORBITS

The simple structure of the symmetry-reduced dynamics allows us to determine the relative periodic orbits of the two-

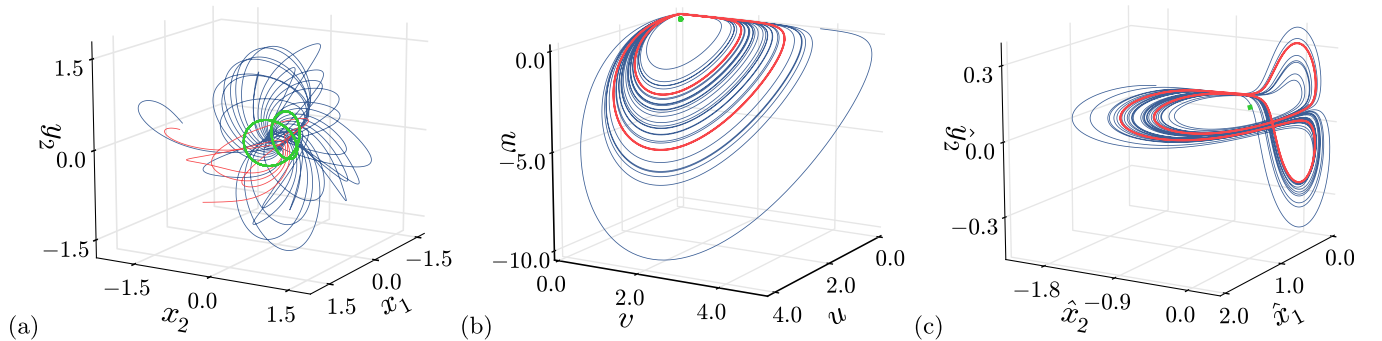


FIG. 4. The trajectories as in Figs. 3(a), 3(c), and 3(e) are colored green, red, and blue, respectively, (a) in a 3D projection of the 4-dimensional state space, (b) in terms of 3 invariant polynomials, and (c) in the 3-dimensional first Fourier mode slice hyperplane. Note that in the symmetry reduced representations (b) and (c), the relative equilibrium TW is reduced to an equilibrium, the green point; and the periodic orbit  $\overline{01}$  (red) closes onto itself after one repeat. In contrast to the invariant polynomial representation (b), in the first Fourier mode slice hyperplane (c), the qualitative difference between shifts by  $\approx \pi$  and  $\approx -\pi$  in near passages to the slice border is very clear, and it leads to the unimodal Poincaré return map of Fig. 5.



mode system by means of a Poincaré section and a return map. We illustrate this procedure in Fig. 5. Starting with an initial point close to the TW, we compute a long, symmetry-reduced ergodic trajectory by integrating Eq. (16a) and record where it crosses the Poincaré section, which we define as the plane that contains TW and is spanned the imaginary part of its unstable stability eigenvector and  $\hat{y}_2$ . We then project these points onto a basis  $(v_1, v_2)$ , which spans the Poincaré section and fit cubic splines to the data as shown in Fig. 5(b). This allows us to construct a return map along this curve, which can be expressed in terms of the distance  $s$  from TW as measured by the arc length along the cubic

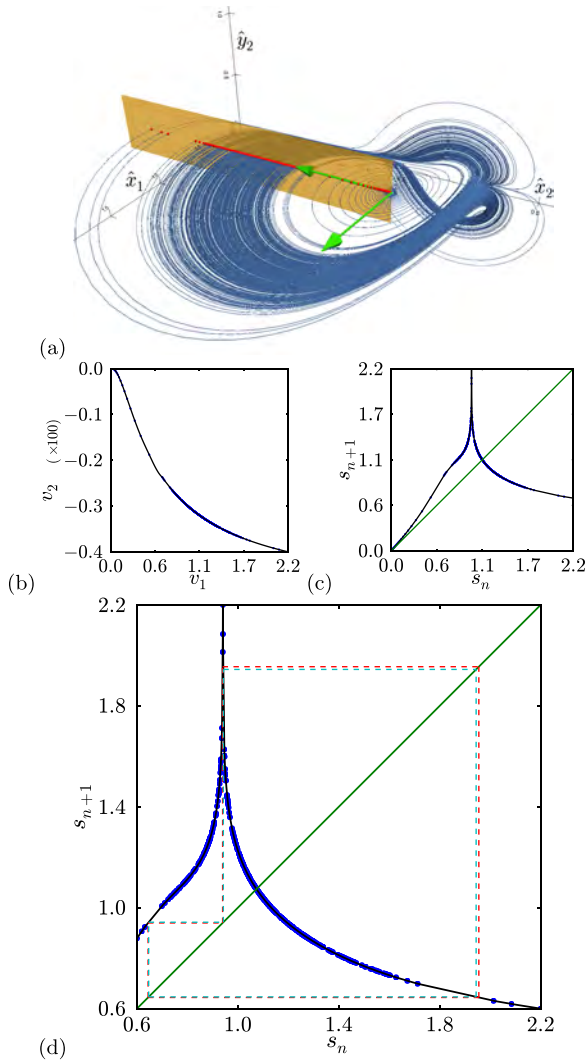


FIG. 5. (a) A symmetry-reduced ergodic trajectory within the slice hyperplane (blue). Green arrows indicate the real and imaginary parts of the complex eigenvectors  $v_u$  that span the unstable manifold of TW. The Poincaré section, which contains TW and is spanned by  $\text{Im}[v_u]$  and  $\hat{y}_2$ , is visualized as a transparent plane. Points where the flow crosses the section are marked in red. (b) A closer look at the Poincaré section shows that the attractor is very thin. Note that the vertical axis, which corresponds to the direction parallel to  $\hat{y}_2$ , is magnified by 100. All (blue) points are located relative to the TW, which is at the origin. The black curve is a cubic spline interpolation of the data. (c) By measuring arc-lengths  $s$  along the interpolation curve, a return map of the Poincaré section can be constructed. Note that once the flow exits the neighborhood of the TW ( $s < 0.6$ ) it stays on the attractor and never comes back. Thus, the data up to this point is transient. (d) The return map without the transient points framed by orbit of the critical point. Dashed lines show the 3-cycles 001 (red) and 011 (cyan).

spline fit. The resulting map, which is shown in Fig. 5(c), is unimodal with a sharp cusp located at its critical point. Note that the region  $s \in (0, 0.6)$  corresponds to the neighborhood of the relative equilibrium and is only visited transiently. Once the dynamics fall onto the chaotic attractor, this region is never visited again. Removing this region from the return map, we obtain the return map shown in Fig. 5(d), which we can then use to determine the accessible relative periodic orbits with their respective binary symbol sequences.

The unimodal return map of Fig. 5 diverges around  $s \approx 0.98$  and this neighborhood is visited very rarely by the flow. We took the pre-image of the furthest point visited by the ergodic flow,  $s_C = 0.98102264$  as the critical point of this map and coded points to the left and right hand sides of this point as “0” and “1,” respectively, and constructed a binary symbolic dynamics. Accessible periodic orbits are then those with the topological coordinates less than that of this critical point. We skip the technical details regarding symbolic dynamics and kneading theory in this tutorial since there is a rich literature on these topics and we do not employ any novel symbolic dynamics technique here. For a pedagogical introduction to the subject, we refer the reader to Refs. 8 and 40.

We are now going to summarize the procedure of locating relative periodic orbits in the state space: Suppose the binary itinerary  $\overline{I_0 I_1 \dots I_{n-1}}$ , where  $I_j = 0, 1$  corresponds to an admissible “ $n$ -cycle,” i.e., a relative periodic orbit that intersects our Poincaré section  $n$ -times. We first find arc-lengths  $\{s_0, s_1, \dots, s_n\}$  that constitute this cycle on the return map Fig. 5(d). We then find corresponding reduced state space points  $\{\hat{a}_0, \hat{a}_1, \dots, \hat{a}_{n-1}\}$ . Finally, we integrate the reduced flow and the phase (8) starting from each point  $\hat{a}_j$  until it returns to the Poincaré section, and divide this trajectory into  $N$  small pieces. As a result, we obtain  $n \times N$  state space points, durations, and phase shifts  $\{a_i^{(0)}, \tau_i^{(0)}, \theta_i^{(0)}\}$ , where  $i = 1, 2, \dots, n \times N$ , which we feed into the multiple shooting Newton solver (see Appendix A) to precisely determine the relative periodic orbit, its period, and the associated phase shift. After finding  $n \times N$  state space points  $(a_i)$ , flight times  $(\tau_i)$ , and phase shifts  $(\theta_i)$  associated with the  $n$  cycle, we can compute the stability of the orbit. We do this by computing the flow Jacobian associated with each segment of the orbit  $J^{\tau_i}(a_i)$ , so that the Jacobian associated with the relative periodic orbit is then

$$\hat{J} = D(-\theta_{n \times N}) J^{\tau_{n \times N}}(a_{n \times N}) \dots D(-\theta_2) J^{\tau_2}(a_2) D(-\theta_1) J^{\tau_1}(a_1). \quad (44)$$

This construction (44) of the Jacobian is equivalent to our definition in Eq. (6), since the group action  $g$  and the flow Jacobian  $J$  are both multiplicative and commute with each other as a consequence of  $g$ -equivariance of the flow. The form (44) is essential in determining the eigenvalues of the Jacobian (Floquet multipliers) precisely, since it allows us to use periodic Schur decomposition, as described in Appendix B.

We found the admissible cycles of the two-mode system up to the topological length 12. The binary itineraries of the seven shortest relative periodic orbits (with topological lengths up to 5), along with their periods, phase shifts,

Floquet multipliers, and Floquet exponents are listed in Table I. In Fig. 6, we show the shortest four relative periodic orbits of the two-mode system within the first Fourier mode slice hyperplane. As seen from Fig. 6, trajectories of  $\overline{001}$  (red) and  $\overline{011}$  (cyan) almost overlap in a large region of the state space. This behavior is also manifested in the return map of Fig. 5(d), where we have shown cycles  $\overline{001}$  and  $\overline{011}$  with red and cyan, respectively. This is a general property of the two-mode cycles with odd topological lengths: They come in pairs with almost equal leading (largest) Floquet exponents, see Fig. 7. Floquet exponents ( $\lambda_j$ ) characterize the rate of expansion/contraction of nearby perturbations to the relative periodic orbits and are related to Floquet multipliers ( $\Lambda_j$ ) by

$$\lambda_{p,j} = \frac{1}{T_p} \ln |\Lambda_{p,j}|, \quad j = 1, 2, \dots, d, \quad (45)$$

where the subscript  $p$  associates  $\lambda_{p,j}$  and  $\Lambda_{p,j}$  with the "prime relative periodic orbit"  $p$  and its period  $T_p$ . Having computed periods, phase shifts, and Floquet multipliers of relative periodic orbits, we are now ready to calculate dynamical averages and other statistical moments of observables using cycle averaging formulas.

## V. CYCLE AVERAGES

So far, we have explained how to find the relative periodic orbits of the two-mode system and compute their stability. However, we have not yet said anything about what to do with these numbers. We begin this section with an overview of the main results of the periodic orbit theory. Our review starts by recapitulating the presentation of Ref. 8, but then, in Section VB, explains how the theory is modified in the presence of continuous symmetries.<sup>41</sup> In Section VC, we present cycle expansions and explain how to approximate the Poincaré section in Fig. 5(d), in order to obtain a better convergence of the spectral determinants. The numerical results are discussed in Section VD.

### A. Classical trace formula

Consider the evolution operator  $\mathcal{L}^t$ , the action of which evolves a weighted density  $\rho(a, t)$  in the state space,

$$\begin{aligned} \rho(a', t) &= [\mathcal{L}^t \rho](a') = \int da \mathcal{L}^t(a', a) \rho(a, 0) \\ \mathcal{L}^t(a', a) &= \delta(a' - f^t(a)) e^{\beta \Omega^t(a)}, \end{aligned} \quad (46)$$

TABLE I. Itinerary, period ( $T$ ), phase shift ( $\theta$ ), Floquet multiplier ( $\Lambda$ ), and Floquet exponent ( $\lambda$ ) of the found two-mode relative periodic orbits with topological lengths up to  $n = 5$ , more (up to  $n = 12$ ) available upon request.

Itinerary	$T$	$\theta$	$\Lambda$	$\lambda$
1	3.64151221	0.08096967	-1.48372354	0.10834917
01	7.34594158	-2.94647181	-2.00054831	0.09439516
001	11.07967801	-5.64504385	-55.77844510	0.36295166
011	11.07958924	-2.50675871	54.16250810	0.36030117
0111	14.67951823	-2.74691247	-4.55966852	0.10335829
01011	18.39155417	-5.61529803	-30.00633820	0.18494406
01111	18.38741006	-2.48213868	28.41893870	0.18202976

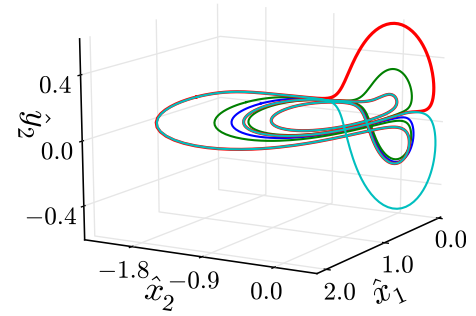


FIG. 6. Shortest four relative periodic orbits of the two-mode system:  $\overline{1}$  (dark blue),  $\overline{01}$  (green),  $\overline{001}$  (red), and  $\overline{011}$  (cyan). Note that relative periodic orbits  $\overline{001}$  and  $\overline{011}$  almost overlap everywhere except  $\hat{x}_1 \approx 0$ .

where  $\beta$  is an auxiliary variable and  $\Omega^t(a)$  is the integrated value of an observable  $\omega(a)$  along the trajectory  $a(t) = f^t(a)$ ,

$$\Omega^t(a) = \int_0^t dt' \omega(f^{t'}(a)). \quad (47)$$

When  $\beta = 0$ , the evolution operator (46) evolves the initial density of state space points to its new form after time  $t$ ; this form of the evolution operator is known as the Perron-Frobenius operator. The multiplicative weight  $\exp(\beta \Omega^t(a))$  will enable us to compute the value of the observable  $\omega$  averaged over the natural measure.

As the integrated observable  $\Omega^t(a)$ , additive along the trajectory, is exponentiated in Eq. (46), the evolution operator is multiplicative along the trajectory

$$\mathcal{L}^{t_1+t_2}(a', a) = \int da'' \mathcal{L}^{t_2}(a', a'') \mathcal{L}^{t_1}(a'', a). \quad (48)$$

This semigroup property allows us to define the evolution operator as the formal exponential of its infinitesimal generator  $\mathcal{A}$

$$\mathcal{L}^t = e^{\mathcal{A}t}. \quad (49)$$

Let  $\rho_\beta(a)$  be the eigenfunction of Eq. (46) corresponding to the leading eigenvalue of  $\mathcal{A}$  (i.e., the one with the largest real part) for a given  $\beta$ ,

$$[\mathcal{L}^t \rho_\beta](a) = e^{t s(\beta)} \rho_\beta(a). \quad (50)$$

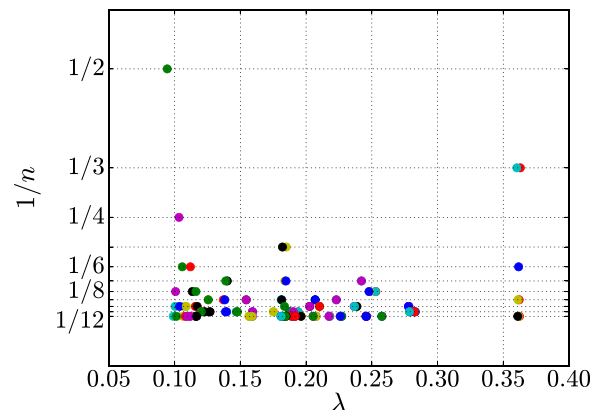


FIG. 7. Distribution of the expanding Floquet exponents of all two-mode cycles with topological lengths  $n$  from 2 to 12.

If the system under study is ergodic, then an invariant "natural measure"  $\rho_0(a)$  with eigenvalue  $s(0)=0$  exists, and the long time average of an observable is then its state space average over the natural measure

$$\langle \omega \rangle = \int da \omega(a) \rho_0(a). \quad (51)$$

By evaluating the action of the evolution operator (46) for infinitesimal times, one finds that the long-time averages of observables, as well as of their higher moments, are given by derivatives of  $s(\beta)$

$$\begin{aligned} \langle \omega \rangle &= \lim_{t \rightarrow \infty} \frac{1}{t} \langle \Omega^t \rangle = \left. \frac{\partial s(\beta)}{\partial \beta} \right|_{\beta=0}, \\ \Delta &= \lim_{t \rightarrow \infty} \frac{1}{t} \langle (\Omega^t)^2 - \langle \Omega^t \rangle^2 \rangle = \left. \frac{\partial^2 s(\beta)}{\partial \beta^2} \right|_{\beta=0}. \\ &\vdots \end{aligned} \quad (52)$$

For example, if the observable  $\omega$  is a velocity  $\dot{a} = v(a)$ , the integrated observable  $\Omega^t$  is the displacement  $a(t)$  in  $d$  dimensions, and  $\Delta/2d$  is Einstein's diffusion coefficient.

In order to obtain  $s(\beta)$ , we construct the resolvent of  $\mathcal{A}$ , by taking the Laplace transform of Eq. (49)

$$\int_0^\infty dt e^{-st} \mathcal{L}^t = (s - \mathcal{A})^{-1}, \quad (53)$$

the trace of which peaks at the eigenvalues of  $\mathcal{A}$ . By taking the Laplace transform of  $\mathcal{L}^t$  and computing its trace by  $\text{tr} \mathcal{L}^t = \int da \mathcal{L}^t(a, a)$ , one obtains the classical trace formula<sup>42</sup>

$$\sum_{\alpha=0}^\infty \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^\infty \frac{e^{r(\beta \Omega_p - s T_p)}}{|\det(1 - M_p^r)|} \quad (54)$$

that relates the spectrum of the evolution operator to the spectrum of periodic orbits. Here,  $s$  is the auxiliary variable of the Laplace transform and  $s_\alpha$  are the eigenvalues of  $\mathcal{A}$ . The outer sum on the right hand side runs over the "prime cycles"  $p$  of the system, i.e., the shortest periodic orbits of period  $T_p$ .  $\Omega_p$  is the value of the observable integrated along the prime cycle and  $M_p$  is the transverse monodromy matrix, the eigenvalues  $\Omega^t(a)$  of which are the Floquet multipliers of  $p$  with the marginal ones excluded. In the derivation of Eq. (54), one assumes that the flow has a single marginal direction, namely, the  $v(a)$  tangent to the periodic orbit, and evaluates the contribution of each periodic orbit to the trace integral by transforming to a local coordinate system where one of the coordinates is parallel to the flow, while the rest are transverse. The integral along the parallel direction contributes the factors of  $T_p$  in Eq. (54). The transverse integral over the delta function (46) contributes the factor of  $1/|\det(1 - M_p^r)|$ .

## B. Decomposition of the trace formula over irreducible representations

The classical trace formula (54) accounts for contributions from periodic orbits to long time dynamical averages.

However, relative periodic orbits of equivariant systems are almost never periodic in the full state space. In order to compute the contributions of relative periodic orbits to the trace of the evolution operator, one has to factorize the evolution operator into the irreducible subspaces of the symmetry group. For discrete symmetries, this procedure is studied in Ref. 43. For the quantum systems with continuous symmetries (abelian and 3D rotations), the factorization of the semiclassical Green's operator is carried out in Ref. 44. Reference 41 addresses the continuous factorization of the evolution operator and its trace; we provide a sketch of this treatment here.

We start by stating, without proof, that a square-integrable field  $\psi(a)$  over a vector space can be factorized into its projections over the irreducible subspaces of a group  $G$

$$\psi(a) = \sum_m \mathbb{P}_m \psi(a), \quad (55)$$

where the sum runs over the irreducible representations of  $G$  and the projection operator onto the  $m$ th irreducible subspace, for a continuous group, is

$$\mathbb{P}_m = d_m \int_G d\mu(g) \chi_m(g(\theta)) \mathbb{D}(\theta). \quad (56)$$

Here,  $d_m$  is the dimension of the representation,  $d\mu(g)$  is the normalized Haar measure,  $\chi_m(g)$  is the character of  $m$ th irreducible representation, and  $\mathbb{D}(\theta)$  is the operator that transforms a scalar field defined on the state space as  $\mathbb{D}(\theta)\rho(a) = \rho(D(\theta)^{-1}a)$ . For our specific case of a single  $\text{SO}(2)$  symmetry,

$$d_m \rightarrow 1, \quad (57)$$

$$\int_G d\mu(g) \rightarrow \oint \frac{d\theta}{2\pi}, \quad (58)$$

$$\chi_m(g(\theta)) \rightarrow e^{-im\theta}. \quad (59)$$

Because the projection operator (56) decomposes scalar fields defined over the state space into their irreducible subspaces under action of  $G$ , it can be used to factorize the evolution operator. Thus, the kernel of the evolution operator transforms under the action of  $\mathbb{D}(\theta)$  as

$$\begin{aligned} \mathbb{D}(\theta) \mathcal{L}^t(a', a) &= \mathcal{L}^t(D(\theta)^{-1}a', a), \\ &= \mathcal{L}^t(a', D(\theta)a), \\ &= \delta(a' - D(\theta)f^t(a)) e^{\beta \Omega^t(a)}, \end{aligned} \quad (60)$$

where the second step follows from the equivariance of the system under consideration. Relative periodic orbits contribute to  $\mathbb{P}_m \mathcal{L}^t = \mathcal{L}^t_m$  since when its kernel is modified as in Eq. (60), the projection involves an integral over the group parameters that is non-zero when  $\theta = -\theta_p$ , the phase shifts of the relative periodic orbits. By computing the trace of  $\mathcal{L}^t_m$ , which in addition to the integral over state space, now involves another integral over the group parameters, one obtains the  $m$ th irreducible subspace contribution to the classical trace as

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_{m,\alpha}} = \sum_p T_p \sum_{r=1}^{\infty} \frac{\chi_m(g^r(\theta_p)) e^{r(\beta\Omega_p - sT_p)}}{|\det(1 - \hat{M}_p^r)|}. \quad (61)$$

The reduced trace formula (61) differs from the classical trace formula (54) by the group character term, which is evaluated at the relative periodic orbit phase shifts, and the reduced monodromy matrix  $\hat{M}$ , which is the  $(d - N - 1) \times (d - N - 1)$  reduced Jacobian for the relative periodic orbit evaluated on a Poincaré section in the reduced state space. The eigenvalues of  $\hat{M}$  are those of the relative periodic orbit Jacobian (6) excluding the marginal ones, i.e., the ones corresponding to time evolution and evolution along the continuous symmetry directions.

Since we are only interested in the leading eigenvalue of the evolution operator, we only consider contributions to the trace (54) from the projections (61) of the 0th irreducible subspace. For the SO(2) case at hand, these can be written explicitly as

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_{0,\alpha}} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta\Omega_p - sT_p)}}{|\det(1 - \hat{M}_p^r)|}. \quad (62)$$

This form differs from the classical trace formula (54) only by the use of the reduced monodromy matrix instead of the full monodromy matrix since the 0th irreducible representation of SO(2) has character 1. For this reason, cycle expansions,<sup>45</sup> which we cover next, are applicable to Eq. (62) after the replacement  $M \rightarrow \hat{M}$ .

### C. Cycle expansions

While the classical trace formula (54) and its factorization for systems with continuous symmetry (61) manifest the essential duality between the spectrum of an observable and that of the periodic orbits and relative periodic orbits, in practice, they are hard to work with since the eigenvalues are located at the poles of Eqs. (54) and (61). The dynamical zeta function (65), which we derive below, provides a perturbative expansion form that enables us to order terms in decreasing importance while computing spectra for the two-mode system. As stated earlier, Eq. (62) is equivalent to Eq. (54) via substitution  $M \rightarrow \hat{M}$ , so this derivation works for either.

We start by defining the "spectral determinant"

$$\det(s - \mathcal{A}) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{r(\beta\Omega_p - sT_p)}}{|\det(1 - M_p^r)|}\right), \quad (63)$$

whose logarithmic derivative  $((d/ds)\ln \det(s - \mathcal{A}))$  gives the classical trace formula (54). The spectral determinant (63) is easier to work with since the spectrum of  $\mathcal{A}$  is now located at the zeros of Eq. (63). The convergence of Eq. (63) is, however, still not obvious. More insight is gained by approximating  $|\det(1 - M_p^r)|$  by the product of expanding Floquet multipliers and then carrying out the sum over  $r$  in Eq. (63). This approximation yields

$$\begin{aligned} |\det(1 - M_p)| &= |(1 - \Lambda_{e,1})(1 - \Lambda_{e,2}) \cdots \\ &\quad (1 - \Lambda_{c,1})(1 - \Lambda_{c,2}) \cdots| \\ &\approx \prod_e |\Lambda_e| \equiv |\Lambda_p|, \end{aligned} \quad (64)$$

where  $|\Lambda_{e,i}| > 1$  and  $|\Lambda_{c,i}| < 1$  are expanding and contracting Floquet multipliers, respectively. By making this approximation, the sum over  $r$  in Eq. (63) becomes the Taylor expansion of natural logarithm. Carrying out this sum brings the spectral determinant (63) to a product (over prime cycles) known as the dynamical zeta function

$$1/\zeta = \prod_p (1 - t_p) \text{ where, } t_p = \frac{1}{|\Lambda_p|} e^{\beta\Omega_p - sT_p} z^{n_p}. \quad (65)$$

Each "cycle weight"  $t_p$  is multiplied by the "order tracking term"  $z^{n_p}$ , where  $n_p$  is the topological length of the  $p$ th prime cycle. This polynomial ordering arises naturally in the study of discrete time systems where the Laplace transform is replaced by  $z$ -transform. Here, we insert the powers of  $z$  by hand, to keep track of the ordering, and then set its value to 1 at the end of calculation. Doing so allows us to write the dynamical zeta function (65) in the "cycle expansion" form by grouping its terms in powers of  $z$ . For complete binary symbolic dynamics, where every binary symbol sequence is accessible, the cycle expansion reads

$$1/\zeta = 1 - t_0 - t_1 - (t_{01} - t_0t_1) - [(t_{011} - t_0t_1t_1) + (t_{001} - t_0t_1t_0)] - \cdots, \quad (66)$$

$$= 1 - \sum_f t_f - \sum_n \hat{c}_n, \quad (67)$$

where we labeled each prime cycle by its binary symbol sequence. In Eq. (67), we grouped the contributions to the zeta function into two groups: "fundamental" contributions  $t_f$  and "curvature" corrections  $c_n$ . The curvature correction terms are denoted explicitly by parentheses in Eq. (66) and correspond to "shadowing" combinations where combinations of shorter cycle weights, also known as "pseudocycle" weights, are subtracted from the weights of longer prime cycles. Since the cycle weights in Eq. (65) already decrease exponentially with increasing cycle period, the cycle expansion (66) converges even faster than exponentially when the terms corresponding to longer prime cycles are shadowed.

For complete binary symbolic dynamics, the only fundamental contributions to the dynamical zeta function are from the cycles with topological length 1, and all longer cycles appear in the shadowing pseudocycle combinations. More generally, if the symbolic dynamics is a subshift of finite type,<sup>8</sup> with the grammar of admissible sequences described by a finite set of pruning rules, and the flow is uniformly hyperbolic, cycle expansions of spectral determinants are guaranteed to converge super-exponentially.<sup>46</sup> A generic unimodal map symbolic dynamics is not a subshift of finite type. However, we have shown in Section IV that the Poincaré return map for the two-mode system (Fig. 5(d)) diverges at  $s \approx 0.98$  and approximated it as if its tip was located at the furthest point visited by an ergodic trajectory.

This brings the question of whether we can approximate the map in Fig. 5(d) in such a way that corresponding symbolic dynamics has a finite grammar of pruning rules? The answer is yes.

As shown in Fig. 5(d), the cycles  $\overline{001}$  and  $\overline{011}$  pass quite close to the tip of the cusp. Approximating the map as if its tip is located exactly at the point where  $\overline{001}$  cuts give us what we are looking for: a single grammar rule, which says that the symbol sequence “00” is inadmissible. This can be made rigorous by the help of kneading theory, however, the simple result is easy to see from the return map in Fig. 5(d): Cover the parts of the return map, which are outside the borders set by the red dashed lines, the cycle  $\overline{001}$  and then start any point to the left of the tip and look at images. You will always land on a point to the right of the tip, unless you start at the lower left corner, exactly on the cycle  $\overline{001}$ . As we will show, this “finite grammar approximation” is reasonable since the orbits that visit outside the borders set by  $\overline{001}$  are very unstable, and hence, less important for the description of invariant dynamics.

The binary grammar with only rule that forbids repeats of one of the symbols is known as the “golden mean” shift,<sup>8</sup> because it has a topological entropy of  $\ln((1 + \sqrt{5})/2)$ . Binary itineraries of golden mean cycles can be easily obtained from the complete binary symbolic dynamics by substitution  $0 \rightarrow 01$  in the latter. Thus, we can write the dynamical zeta function for the golden mean pruned symbolic dynamics by replacing 0s in (66) by 01

$$1/\zeta = 1 - t_{01} - t_1 - (t_{011} - t_{01}t_1) - [(t_{0111} - t_{011}t_1) + (t_{01011} - t_{01}t_{011})] - \dots \quad (68)$$

Note that all the contributions longer than topological length 2 to the golden mean dynamical zeta function are in form of shadowing combinations. In Section VD, we will compare the convergence of the cycle averages with and without the finite grammar approximation, but before moving on to numerical results, we explain the remaining details of the computation.

While dynamical zeta functions are useful for investigating the convergence properties, they are not exact, and their computational cost is same as that of exact spectral determinants. For this reason, we expand the spectral determinant (63) ordered in the topological length of cycles and pseudo-cycles. We start with the following form of the spectral determinant (63):

$$\det(s - \mathcal{A}) = \prod_p \exp \left( - \sum_{r=1}^{n_p r \leq N} \frac{1}{r} \frac{e^{r(\beta\Omega_p - sT_p)}}{|\det(1 - M_p^r)|} z^{n_p r} \right), \quad (69)$$

where the sum over the prime cycles in the exponential becomes a product. We also inserted the order tracking term  $z$  and truncated the sum over cycle repeats at the expansion order  $N$ . For each prime cycle, we compute the sum in Eq. (69) and expand the exponential up to order  $N$ . We then multiply this expansion with the contributions from previous cycles and drop terms with order greater than  $N$ . This way, after setting  $z = 1$ , we obtain the spectral determinant truncated to cycles and pseudo-cycles of topological length up to  $n_p \leq N$ ,

$$F_N(\beta, s) = 1 - \sum_{n=1}^N Q_n(\beta, s). \quad (70)$$

In what follows, we shall drop the subscript,  $F_N \rightarrow F$ , but actual calculations are always done for a range of finite truncation lengths  $N$ . We are searching for the eigenvalues  $s(\beta)$  of the operator  $\mathcal{A}$  in order to compute the moments (52). These are located at the zeros of the spectral determinant, hence as a function of  $\beta$  they satisfy the implicit equation

$$F(\beta, s(\beta)) = 0. \quad (71)$$

By taking derivative of Eq. (71) with respect to  $\beta$  and applying chain rule, we obtain

$$\frac{ds}{d\beta} = - \frac{\partial F}{\partial \beta} / \frac{\partial F}{\partial s}. \quad (72)$$

Higher order derivatives can be evaluated similarly. Defining

$$\begin{aligned} \langle \Omega \rangle &= -\partial F / \partial \beta, \\ \langle T \rangle &= \partial F / \partial s, \quad \langle T^2 \rangle = \partial^2 F / \partial s^2, \\ \langle \Omega^2 \rangle &= -\partial^2 F / \partial \beta^2, \quad \langle \Omega T \rangle = \partial^2 F / \partial \beta \partial s, \end{aligned} \quad (73)$$

we write the cycle averaging formulas as

$$\langle \omega \rangle = \langle \Omega \rangle / \langle T \rangle, \quad (74)$$

$$\begin{aligned} \Delta &= \frac{1}{\langle T \rangle} \left( \langle \Omega^2 \rangle - 2 \frac{ds}{d\beta} \langle \Omega T \rangle + \left( \frac{ds}{d\beta} \right)^2 \langle T^2 \rangle \right) \\ &= \frac{1}{\langle T \rangle} \langle (\Omega - T \langle \omega \rangle)^2 \rangle, \end{aligned} \quad (75)$$

with everything evaluated at  $\beta = 0, s = s(0)$ .

By conservation of probability, we expect that for an invariant measure  $\rho_0(a)$ , the eigenvalue  $s(0)$  is 0. However, we did not make this substitution in our derivation of the cycle averaging formulas since, in practice, our approximations to the spectral determinant are always based on a finite number of periodic orbits. This makes the solution of  $F_N(0, s(0)) = 0$  small, but not exactly 0. This eigenvalue has a special meaning: It indicates how well the periodic orbits cover the strange attractor. Following this interpretation, we define  $\gamma = -s(0)$  as the “escape rate”: the rate at which the dynamics escape the region that is covered by the periodic orbits. Specifically, for our finite grammar approximation, the escape rate tells us how frequently the ergodic flow visits the part of the Poincaré map that we cut off by applying our finite grammar approximation.

We defined  $\langle T \rangle$  in Eq. (73) as a shorthand for a partial derivative, however, we can also develop an interpretation for it by looking at the definitions of the dynamical zeta function (65) and the spectral determinant (63). In both series, the partial derivative with respect to  $s$  turns them into a sum weighted by the cycle periods; with this intuition, we define  $\langle T \rangle$  as the “mean cycle period.”

These remarks conclude our review of periodic orbit theory and its extension to the equivariant dynamical systems. We are now ready to present our numerical results and discuss their quality.

#### D. Numerical results

We constructed the spectral determinant (70) to different orders for two observables: phase velocity  $\dot{\theta}$  and the leading Lyapunov exponent. Remember that  $\Omega_p$  appearing in Eq. (69) is the integrated observable, so in order to obtain the moments of phase velocity and the leading Lyapunov exponent from Eqs. (74) and (75), we respectively put in  $\Omega_p = \theta_p$ , the phase shift of the prime cycle  $p$ , and  $\Omega_p = \ln|\Lambda_{p,e}|$ , the logarithm of its expanding Floquet multiplier of  $\Lambda_{p,e}$ .

In Section III D, we explained that SO(2) phase shifts correspond to the drifts in configuration space. We define the corresponding diffusion coefficient as

$$D = \frac{1}{2d} \lim_{t \rightarrow \infty} \frac{1}{t} \langle \theta(t)^2 - \langle \theta(t) \rangle^2 \rangle, \quad (76)$$

where  $d = 1$  since the configuration space is one dimensional.

Tables II and III, respectively, show the cycle averages of the escape rate  $\gamma$ , mean period  $\langle T \rangle$ , leading Lyapunov exponent  $\lambda$ , mean phase velocity  $\langle \dot{\theta} \rangle$ , and the diffusion coefficient  $D$  with and without the finite grammar approximation. In the latter, we input all the relative periodic orbits we have found into expansion (69), whereas in the former, we discarded the cycles with symbol sequence "00."

In Section V C, we motivated the finite grammar approximation by claiming that it would lead to faster convergence of dynamical averages due to the nearly exact shadowing combinations of the golden mean zeta function (68). This claim is supported by the data in Tables II and III. Take, for example, the Lyapunov exponent. This converges to 7 digits for the 12th order expansion when using the finite grammar approximation in Table II, but only converges to 4 digits at this order in Table III. Other

TABLE II. Cycle expansion estimates for the escape rate  $\gamma$ , average cycle period  $\langle T \rangle$ , Lyapunov exponent  $\lambda$ , average phase velocity  $\langle \dot{\theta} \rangle$ , and the diffusion coefficient  $D$ , using cycles up to length  $N$  in the golden mean approximation (68) of the symbolic dynamics.

N	$\gamma$	$\langle T \rangle$	$\lambda$	$\langle \dot{\theta} \rangle$	D
1	0.249829963	3.6415122	0.10834917	0.0222352	0.000000
2	-0.011597609	5.8967605	0.10302891	-0.1391709	0.143470
3	0.027446312	4.7271381	0.11849761	-0.1414933	0.168658
4	-0.004455525	6.2386572	0.10631066	-0.2141194	0.152201
5	0.000681027	5.8967424	0.11842700	-0.2120545	0.164757
6	0.000684898	5.8968762	0.11820050	-0.1986756	0.157124
7	0.000630426	5.9031596	0.11835159	-0.1997353	0.157345
8	0.000714870	5.8918832	0.11827581	-0.1982025	0.156001
9	0.000728657	5.8897511	0.11826873	-0.1982254	0.156091
10	0.000728070	5.8898549	0.11826788	-0.1982568	0.156217
11	0.000727891	5.8898903	0.11826778	-0.1982561	0.156218
12	0.000727889	5.8898908	0.11826780	-0.1982563	0.156220

TABLE III. Cycle expansion estimates of the escape rate  $\gamma$ , average cycle period  $\langle T \rangle$ , Lyapunov exponent  $\lambda$ , average phase velocity  $\langle \dot{\theta} \rangle$ , and the diffusion coefficient  $D$  using all cycles found up to length  $N$ .

N	$\gamma$	$\langle T \rangle$	$\lambda$	$\langle \dot{\theta} \rangle$	D
1	0.249829963	3.6415122	0.10834917	0.0222352	0.000000
2	-0.011597609	5.8967605	0.10302891	-0.1391709	0.143470
3	0.022614694	4.8899587	0.13055574	-0.1594782	0.190922
4	-0.006065601	6.2482261	0.11086469	-0.2191881	0.157668
5	0.000912644	5.7771642	0.11812034	-0.2128347	0.168337
6	0.000262099	5.8364534	0.11948918	-0.2007615	0.160662
7	0.000017707	5.8638210	0.12058951	-0.2021046	0.160364
8	0.000113284	5.8511045	0.12028459	-0.2006143	0.159233
9	0.000064082	5.8587350	0.12045664	-0.2006756	0.158234
10	0.000093124	5.8536181	0.12035185	-0.2007018	0.158811
11	0.000153085	5.8417694	0.12014700	-0.2004520	0.158255
12	0.000135887	5.8455331	0.12019940	-0.2005299	0.158465

observables compare similarly in terms of their convergence in both cases. Note, however, that the escape rate in Table II converges to  $\gamma = 0.000727889$ , whereas in Table III it gets smaller and smaller with an oscillatory behavior. This is due to the fact that in the finite grammar approximation, we threw out the part of attractor that corresponds to the cusp of the return map in Fig. 5(d) above the point cut by  $\overline{001}$ .

In order to compare with the cycle averages, we numerically estimated the leading Lyapunov exponent of the two-mode system using the method of Wolf *et al.*<sup>47</sup> This procedure was repeated 100 times for different initial conditions, yielding a numerical mean estimate of  $\bar{\lambda} = 0.1198 \pm 0.0008$ . While the finite grammar estimate  $\lambda_{FG} = 0.1183$  is within 0.6% range of this value, the full cycle expansion agrees with the numerical estimate. This is not surprising since in the finite grammar approximation, we discard the most unstable cycles to obtain faster convergence, and so can expect a slight underestimate of the Lyapunov exponent.

## VI. CONCLUSIONS AND DISCUSSION

In this tutorial, we have studied a simple dynamical system that exhibits chaos and is equivariant under a continuous symmetry transformation. We have shown that reducing this symmetry simplifies the qualitative dynamics to a great extent and enables one to find all relative periodic orbits of the systems via standard techniques such as Poincaré sections and return maps. In addition, we have shown that one can extract quantitative information from the relative periodic orbits by computing cycle averages.

We motivated our study of the two-mode system by the resemblance of its symmetry structure to that of spatially extended systems. The steps outlined here are, in principle, applicable to physical systems that are described by  $N$ -Fourier mode truncations of PDEs such as 1D Kuramoto-Sivashinsky,<sup>48</sup> 3D pipe flows,<sup>19</sup> etc.

In Section IV, we showed that the dynamics of our two-mode model can be completely described by a unimodal return map of the Poincaré section that we constructed after

continuous symmetry reduction. In a high-dimensional system, finding such an easy symbolic dynamics or any symbolic dynamics at all is a challenging problem on its own. In Ref. 49, the authors found that for the desymmetrized (confined in the odd subspace) 1D spatio-temporally chaotic Kuramoto-Sivashinsky system a bimodal return map could be obtained after reducing the discrete symmetry of the problem. However, we do not know any study that has been able to simplify turbulent fluid flow to such an extent.

In Section V, we showed that symbolic dynamics and their associated grammar rules greatly affect the convergence of cycle averaging formulas. In general, finding a finite symbolic description of a flow is rarely as easy as it is in our model system. There exist other methods of ordering cycle expansion terms, for example, ordering pseudo-cycles by their stability and discarding terms that are above a threshold.<sup>50</sup> In this case, one expects the remaining terms to form shadowing combinations and converge exponentially. Whichever method of term ordering is employed, the cycle expansions are only as good as the least unstable cycle that one fails to find. Symbolic dynamics solves both problems at once since it puts the cycles in order by topological length so that one cannot miss any accessible cycle and shadowing combinations occur naturally. The question one might then ask is: When there is no symbolic dynamics, how can you make sure that you have found all the periodic orbits of a flow up to some cycle period?

In searching for cycles in high-dimensional flows, one usually looks at the near recurrences of the ergodic flow and then runs Newton searches starting near these recurrences to find if they are influenced by a nearby exactly recurrent solution. Such an approach does not answer the question we just asked with full confidence, although one might argue that the dynamically important cycles influence the ergodic flow, leading to recurrences, and thus, cycles found this way are those that are relevant for computing averages.

To sum up, we have shown that periodic orbit theory successfully extends to systems with continuous symmetries. When dealing with high dimensional systems, one still needs to think about some of the remaining challenges outlined above. Once these are overcome, it should become possible to extract quantitative information about turbulence by using exact unstable solutions of the Navier-Stokes equations.

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## APPENDIX A: MULTIPLE SHOOTING METHOD FOR FINDING RELATIVE PERIODIC ORBITS

Let us assume that we have a set of good guesses for a set of state space points, flight times, and 1D symmetry group parameter increments  $\{a_i^{(0)}, \tau_i^{(0)}, \theta_i^{(0)}\}$  such that the points  $\{a_i^{(0)}\}$  lie close to the relative periodic orbit  $p$  such that

$$a_{i+1}^{(0)} \approx D(-\theta_i^{(0)})f^{\tau_i^{(0)}}(a_i^{(0)}) \quad \text{cyclic in } i = 1, \dots, n. \quad (\text{A1})$$

Here, the period and the shift of the relative periodic orbit  $p$  are  $T_p \approx \sum \tau_i$  and  $\theta_p \approx \sum \theta_i$ . The Lagrangian description of the flow is then  $a(\tau) = f^\tau(a(0))$ . We want to determine corrections  $(\Delta a_i, \Delta \tau_i, \Delta \theta_i)$  so that

$$a_{i+1} + \Delta a_{i+1} = D(-\theta_i - \Delta \theta_i)f^{\tau_i + \Delta \tau_i}(a_i + \Delta a_i) \quad \text{cyclic in } i = 1, \dots, n. \quad (\text{A2})$$

To linear order in

$$(\Delta a_i^{(m+1)}, \Delta \tau_i^{(m+1)}, \Delta \theta_i^{(m+1)}) = (a_i^{(m+1)} - a_i^{(m)}, \tau_i^{(m+1)} - \tau_i^{(m)}, \theta_i^{(m+1)} - \theta_i^{(m)}) \quad (\text{A3})$$

the improved Newton guess  $(a_i^{(m+1)}, \tau_i^{(m+1)}, \theta_i^{(m+1)})$  is obtained by minimizing the effect of perturbations along the spatial, time, and phase directions,

$$a'_{i+1} - D_{i+1}f^{\tau_i}(a_i) = D_{i+1}(J_{i+1}\Delta a_i + v_{i+1}\Delta \tau_i - t_{i+1}\Delta \theta_i), \quad (\text{A4})$$

where, for brevity,  $a_i^{(m+1)} = a_i^{(m)} + \Delta a_i^{(m)} = a'_i$ ,  $a_i^{(m)} = a_i$ ,  $D(-\theta_i) = D_{i+1}$ ,  $v(a_i(\tau_i)) = v_{i+1}$ ,  $J^{\tau_i}(a_i) = J_{i+1}$ ,  $t(a_i(\tau_i)) = Ta_i(\tau_i) = t_{i+1}$ , etc. For sufficiently good initial guesses, the improved values converge under Newton iterations to the exact values  $(\Delta a_i, \Delta \tau_i, \Delta \theta_i) = (\Delta a_i^{(\infty)}, \Delta \tau_i^{(\infty)}, \Delta \theta_i^{(\infty)})$  at a super-exponential rate.

In order to deal with the marginal multipliers along the time and group orbit directions, one needs to apply a pair of constraints, which eliminate variations along the marginal directions on the relative periodic orbit's 2D torus. These can be formulated as a local Poincaré section orthogonal to the flow and a local slice orthogonal to the group orbit at each point along the orbit,

$$\langle v(a_i) | \Delta a_i \rangle = 0, \quad \langle t(a_i) | \Delta a_i \rangle = 0. \quad (\text{A5})$$

We can rewrite everything as one matrix equation

$$A\Delta = E, \quad (\text{A6})$$

where

$$A = \begin{pmatrix} D_2 J_2 & D_2 v_2 & -TD_2 f^{\tau_1}(a_1) & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ v(a_1) & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ t(a_1) & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & D_3 J_3 & D_3 v_3 & -TD_3 f^{\tau_2}(a_2) & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & v(a_2) & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & t(a_2) & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & D_1 J_1 & D_1 v_1 & -TD_1 f^{\tau_1}(a_1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & v(a_n) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & t(a_n) & 0 & 0 \end{pmatrix}, \tag{A7}$$

$$\Delta = (\Delta a_1, \Delta \tau_1, \Delta \theta_1, \Delta a_2, \Delta \tau_2, \Delta \theta_2, \dots, \Delta a_n, \Delta \tau_n, \Delta \theta_n)^T, \tag{A8}$$

$$E = (a_2 - D_2 f^{\tau_1}(a_1), 0, 0, a_3 - D_3 f^{\tau_2}(a_2), 0, 0, \dots, a_1 - D_1 f^{\tau_n}(a_n), 0, 0)^T. \tag{A9}$$

We then solve Eq. (A6) for  $\Delta$  and update our initial guess by adding the vector of the computed  $\Delta$  values to it and iterate.

**APPENDIX B: PERIODIC SCHUR DECOMPOSITION**

Here, we briefly summarize the periodic eigen decomposition<sup>51</sup> needed for the evaluation of Floquet multipliers for two-mode periodic orbits. Due to the non-hyperbolicity of the return map of Fig. 5(d), Floquet multipliers can easily differ by 100s of orders of magnitude even in a model as simple as the two-mode system.

We obtain the Jacobian of the relative periodic orbit as a multiplication of short-time Jacobians from the multiple shooting computation of Appendix A, so that

$$\begin{aligned} \hat{J} &= D_n J_n D_{n-1} J_{n-1} \dots D_1 J_1 \\ &= \hat{J}_n \hat{J}_{n-1} \dots \hat{J}_1, \end{aligned}$$

where,  $\hat{J}_i = D_i J_i \in \mathbb{R}^{4 \times 4}$ ,  $i = 1, 2, \dots, n$ . (B1)

This Jacobian is the same as the definition in Eq. (6) since  $J_i$  and  $D_i$  commute with each other and are multiplicative in time and phase, respectively. In order to determine the eigenvalues of  $\hat{J}$ , we bring each term appearing in the product (B1) into periodic, real Schur form as follows:

$$\hat{J}_i = Q_i R_i Q_i^T, \tag{B2}$$

where  $Q_i$  are orthogonal matrices that satisfy the cyclic property:  $Q_0 = Q_n$ . After this similarity transformation, we can define  $R = R_n R_{n-1} \dots R_1$  and re-write the Jacobian as

$$\hat{J} = Q_n R Q_n^T. \tag{B3}$$

The matrix  $R$  is, in general, block-diagonal with  $1 \times 1$  blocks for real eigenvalues and  $2 \times 2$  blocks for the complex pairs. It also has the same eigenvalues as  $\hat{J}$ . In our case, it is

diagonal since all Floquet multipliers are real for relative periodic orbits of the two-mode system. For each relative periodic orbit, we have two marginal Floquet multipliers corresponding to the time evolution direction and the continuous symmetry direction, as well as one expanding and one contracting eigenvalue.

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