Group theory for Feynman diagrams in non-Abelian gauge theories

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A simple and systematic method for the calculation of group-theoretic weights associated with Feynman diagrams in non-Abelian gauge theories is presented. Both classical and exceptional groups are discussed.

I. INTRODUCTION

The increased interest in non-Abelian gauge theories has in recent years led to the computation of many higher-order Feynman diagrams. Asymptotic form-factor and scattering amplitude calculations are of special interest, because they suggest that it might be possible to sum up diagrams with arbitrary numbers of soft gluons just as one can sum up soft-photon processes in QED. In such a program the analysis of the momentum integrals proceeds by the traditional techniques developed for QED calculations. The new aspect, characteristic of non-Abelian gauge theories, is the emergence of a group-theoretic weight (or weight, for short) associated with each Feynman diagram. The dramatic cancellations among various diagrams occur through interplay of their group-theoretic weights and their momentum-space integrals. So the study of weights becomes of interest, as it might suggest cancellation patterns needed for summations of diagrams.

In this paper we give a general method for computing group-theoretic weights, and give explicit rules for SU(n), SO(n), Sp(n), G2, E6, F4, and E8 symmetry groups. We restrict ourselves to the models with quarks in the defining (lowest dimensional) representation, but the method can be extended to higher representations. As long as global symmetry is assumed, we can compute weights not only in symmetric gauge theories, but also in those spontaneously broken gauge theories where all particles within a multiplet have the same mass.

The evaluation procedure is very simple. We think of the weight itself as a Feynman integral (over a discrete lattice), and introduce Feynman diagrammatic notation to replace the unwieldy algebraic expressions. Then we give two relations; the first eliminates all three-gluon vertices, and the second eliminates all internal gluon lines. The result is a sum over a unique set of irreducible group-theoretic tensors which form a natural basis for all Lie algebras. All this is accomplished without recourse to any explicit representation of the group generators and structure constants. As a by-product, we learn how to count quickly the number of invariant couplings for arbitrary numbers of quarks and gluons, thus avoiding involved reductions of outer products of representations by Young tableaux.

In most calculations, one looks for properties which arise solely from gauge invariance, and there the explicit numerical values of weights should really not be necessary. While in some such calculations it is appealing to express simple diagrams in terms of quadratic Casimir operators (so that the form of the expression is independent of the particular gauge group and the particular representation), for higher-order diagrams there is no simple way of relating weights to generalized Casimir operators, and such an approach becomes very cumbersome. Then the explicit expressions for weights might be both suggestive and useful as checks for the cancellations among various diagrams. Another application of explicit weight expressions is 1/n expansions for which the above evaluation method gives simple and direct estimates.

Possibly, a novel aspect of this paper is its treatment of exceptional groups. It is known that exceptional groups arise from invariance of norms defined on octonion spaces, but the demonstration is rather difficult (it involves Jordan algebras over octonionic matrices). We skirt the complexities of this underlying structure by giving a formulation of exceptional groups purely in terms of the geometrical properties of their defining representations. Intuition so developed might be of use to quark-model builders. We give the following example: Because SU(3) has a cubic invariant 3, it is possible to build a three-quark color singlet with desirable phenomenological properties. Are there any other groups that could accommodate three-quark color singlets? It turns out that the defining representations of G2, F4, and E8 are among groups with such invariants. A systematic discussion of such invariants shall be given elsewhere.

In the past, most weight calculations have involved SU(n) and, even more specifically, SU(3). This has led to the development of methods specific to SU(n). For the sake of completeness and comparison, we pursue this traditional line for a while and find ourselves at an impasse.
The organization of the paper is as follows. In Sec. II., we state the evaluation rules. In Sec. III., we introduce diagrammatic notation and derive various relationships true for all Lie groups, while particular groups are defined in Sec. IV. An example of weight evaluation is given in Sec. V. In Sec. VI., we discuss group-theoretic tensor bases and relations between basis tensors for specific representations, while higher representations are touched upon in Sec. VII. Full Feynman rules are stated in Appendix A. Appendix B is a long discussion of an older method of weight evaluation, specific to SU(n). For readers interested only in models with classical symmetry groups, Figs. 1–3 summarize all that is needed for weight computation.

II. RULES FOR THE EVALUATION OF GROUP-THEORETIC WEIGHTS

For our model we take a Yang–Mills theory with massive quarks of \( n \) colors and \( N \) massless gluons, defined by the classical Lagrangian density

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{q}(i\gamma^\mu - m)q,
\]

\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + C_{ijk} A_\mu^j A_\nu^k,
\]

\[
D_\mu^a = \partial_\mu^a - i A_\mu^a(T_1)^a_{\beta},
\]

\[
a, b = 1, 2, \ldots, n, \quad i, j = 1, 2, \ldots, N
\]

where the \( n \) complex quark fields \( q_\alpha \) transform as the defining (lowest-dimensional cogredient) representation of a compact simple \( N \)-dimensional Lie group \( \mathfrak{g} \), and the \( N \) Hermitian gluon fields \( A_\mu^a \) transform as its adjoint (regular) representation. In Yang–Mills theory the coupling constant \( e \) of the usual QED is generalized to quark–gluon coupling matrices \( (T_1)^a_\beta \). They are generators of \( \mathfrak{g} \), close a Lie algebra

\[
[T_i, T_j] = iC_{ijk} T_k,
\]

\[
\text{Tr} T_i = 0,
\]

and can be chosen to satisfy a normalization condition

\[
\text{Tr}(T_i T_j) = \delta_{ij}.
\]

For example, for \( SU(n) \), the conventional choice is \( T_i = \frac{1}{2} g_\lambda a_\lambda \) and \( a = \frac{1}{2} g^2 \). In this paper \( a \) shall remain arbitrary throughout. \( \sqrt{a} \) can be thought of as the overall coupling constant for a simple group \( \mathfrak{g} \), and powers of \( \sqrt{a} \) count the number of vertices in a diagram. That the gluon self-couplings \( -iC_{ijk} \) also scale as \( \sqrt{a} \) is evident from (2.2).

The Lagrangian (2.1) generates the usual Feynman diagrams. There is no mixing between the spacetime and the gauge group \( \mathfrak{g} \), and the amplitudes associated with a diagram \( G \) factorizes into \( W_c \mathcal{M}_c \), where \( W_c \) is the group-theoretic weight consisting of various \( (T_1)^a_\beta \) and \( C_{ijk} \), and \( \mathcal{M}_c \) arises from the integrals over internal momenta and is similar to QED Feynman amplitudes. Even though \( \mathcal{M}_c \) will not concern us in this paper, we give the rules for its computation in Appendix A. We note that while in momentum space there are four-gluon vertices, for \( W_c \) there exist only three-gluon couplings, because the group-theoretic factors in a four-gluon vertex have the form \( C_{ijk} C_{klm} \).

The group-theoretic weight \( W_c \) is a product of the following factors (all repeated indices are summed over):

(a) for each internal quark line, a factor \( b = 1, 2, \ldots, n \),

(b) for each internal gluon or ghost line, a factor \( b_{ij}, i, j = 1, 2, \ldots, N \),

(c) for each quark–quark–gluon vertex, a factor \( (T_1)^a_{\beta} \)

(d) for each three-gluon or ghost–ghost–gluon vertex, a factor \( -iC_{ijk} \),

(e) for the four-gluon vertex, the factors

- \( C_{imj} C_{knl} + C_{imk} C_{jnl} \) (multiplying \( B_{ij} B_{kl} \)),
- \( C_{imj} C_{knl} + C_{imk} C_{jnl} \) (multiplying \( B_{ij} B_{kl} \)),
- \( C_{imj} C_{knl} + C_{imk} C_{jnl} \) (multiplying \( B_{ij} B_{kl} \)),

where gluon group and Lorentz indices are paired as \((i, \lambda), (j, \mu), (k, \nu), \) and \((l, \xi)\) (see also Fig. 24).

The weight \( W_c \) for an arbitrary Feynman amplitude \( G \) is evaluated in two steps:

1. Reexpress all three-gluon vertices \( -iC_{ijk} \) in terms of the defining representation:

(a) if \( \mathfrak{g} \) is \( SU(n) \) or \( E_n \)

\[
iC_{ijk} = \frac{1}{d} \text{Tr}(T_i T_j T_k - T_k T_j T_i);
\]

(b) if \( \mathfrak{g} \) is \( SO(n) \), \( Sp(n) \), \( G_2 \), \( F_4 \), or \( E_7 \)

\[
iC_{ijk} = \frac{2}{d} \text{Tr}(T_i T_j T_k).
\]

2. Eliminate all internal gluon lines

\[
\cdots (T_1)^a_\beta \cdots (T_1)^a_\beta \cdots
\]

by replacing them with gluon projection operators:

\[
\frac{1}{d} (T_i T_j T_k - T_k T_j T_i) = \begin{cases} \delta^{abc} - \frac{1}{n} \delta^{abc} \delta_{\alpha \beta} & \text{for } SU(n), \\ \frac{1}{2} (\delta^{abc} - \delta^{abc} \delta_{\alpha \beta}) & \text{for } SO(n), \end{cases}
\]

where \( n \) even, \( f^{abc} = -f^{cba} \), \( f^{abc} f^{def} = \delta^{cd}_{\alpha} \).
III. LIE ALGEBRA IN DIAGRAMMATIC NOTATION

A group-theoretic weight $W_G$ can be visualized as a Feynman diagram in which the internal lines represent sums over all colors of the associated particles, and vertices represent their couplings. There is never any need to label the lines and vertices; the equivalent points on the paper represent the same index in all terms of a diagrammatic equation.\(^{38}\) Besides automatically keeping track of indices, diagrams make it easier to recognize the symmetries of more complicated expressions.

In this section algebraic relations shall be transcribed into diagrammatic equations which apply to any semisimple Lie algebra with quarks in any representation. The diagrammatic Feynman rules are given in Fig. 2. Figure 3 summarizes the basic relations of a semisimple Lie algebra. Note that Fig. 3(a) fixes the sign convention for $-iC_{ijk}$; indices circle the vertex in anticlockwise direction. If the direction of the quark line were reversed, the right-hand side would change sign.

Figures 3(e) and 3(f) count the numbers of quarks and gluons, respectively: $\delta_a^m = N_\alpha$, $\delta_b^N = N$. The above

\[
\begin{align*}
1_{(T_j^a)_{\alpha}} &= \begin{cases} 
\frac{1}{2} (\delta_a^c \delta^c_b - \delta_a^b \delta^c_c) - \frac{1}{\alpha} \delta^c_{ab} f^{cde} f_{cde} & \text{for } G_2, \\
\frac{1}{2} (\delta_a^c \delta^c_b - \delta_a^b \delta^c_c) + \frac{1}{3\alpha} \delta^c_{ab} d_{cde} d_{cde} & \text{for } E_6, \\
\frac{1}{2} (\delta_a^c \delta^c_b - \delta_a^b \delta^c_c) - \frac{7}{3\alpha} \delta^c_{ab} d_{cde} d_{cde} - \frac{1}{\alpha} \delta^c_{ab} d_{cde} d_{cde} & \text{for } F_4, \\
\frac{1}{2} (\delta_a^c \delta^c_b + \delta^c_{ab} f^{cde} f_{cde} - \frac{2}{\alpha} \delta^c_{ab} d_{cde} d_{cde}) & \text{for } E_7, \\
\frac{1}{2} (\delta_a^c \delta^c_b + \delta^c_{ab} f^{cde} f_{cde} - \frac{2}{\alpha} \delta^c_{ab} d_{cde} d_{cde}) & \text{for } E_8.
\end{cases}
\end{align*}
\]

(We do not know how to evaluate $E_8$.)

The rules for the exceptional groups are supplemented by the identities of Sec. IV which define the associated invariants. Graphically, the above rules are summarized in Fig. 1. Section V gives an example of how the rules are used in a typical computation.

\[\text{Fig. 1. Weight evaluation rules for the defining representations of all simple groups except } E_8. \text{ (a) Elimination of a three-gluon coupling } -4C_{ijk}, \text{ (b) elimination of an internal gluon line. Further rules for exceptional groups are given in Sec. IV.}\]
definitions already enable us to perform some simple calculations. For example, to calculate the quadratic Casimir operator for the quark representation, Fig. 4(a), we form a trace (join the external quark lines) and use Figs. 3(c), 3(e), and 3(f), as outlined in Fig. 5, to obtain

$$C_F = \frac{1}{n} \sum_n N_n.$$  \hspace{1cm} (3.1)

In other words, if we know the gluon projection operators [as those listed in (2.7) through (2.13)], we can compute the dimension of the algebra by tracing the normalization relation (2.4):

$$N = \frac{1}{a} \text{Tr}(T_i T_j).$$  \hspace{1cm} (3.2)

Existence of the gluon Casimir operator $C_A$ [see Fig. 4(b)] is a necessary and sufficient condition that the algebra is semisimple. For compact groups $C_A > 0$. If the group is simple,$^{35,42}$

$$\text{Tr}(T_i T_j) = i \text{Tr}(C_i C_j).$$  \hspace{1cm} (3.3)

[where $i$ is called the index of the representation, and the adjoint (or regular) representation of $\mathfrak{g}$ is constructed from matrices $(C_1)_i j = -iC_i T_j)]$. Figs. 3(c) and 4(b) are compatible. For a semisimple group, this is generally not true. Joining gluon indices in commutators Figs. 3(a) and 3(d) leads to relations in Figs. 4(c) and 4(d). Similarly, the relation Fig. 4(e) follows from the commutation relation Fig. 3(a).

The antisymmetry of $C_{i j}$ leads to vanishing of nonplanar diagrams of Fig. 6 as well as all diagrams that contain these as subdiagrams. This follows from the commutation relations of Fig. 3, but it is easily seen as a consequence of the skewness of $C_{i j}$, Fig. 2(d). For example, interchange of vertices $1 \leftrightarrow 2$ in Fig. 6(a), and $1 \leftrightarrow 2, 3 \leftrightarrow 4,$ and $5 \leftrightarrow 6$ in Fig. 6(d) gives a factor $(-1)^3$ from skewness of $C_{i j}$, while the diagrams are mapped into themselves. The obscure diagram of Fig. 6(d) is related to the Peterson graph$^{39}$ in graph theory, while Fig. 6(a) is related to the nonplanar Kuratowski graph.$^{39,44}$

One quickly runs out of relations achievable by Lie algebra manipulations. For example, at this
IV. WEIGHT EVALUATION

Our objective is to express the group-theoretic weight of an arbitrary diagram as

$$W = \sum_{m,n} C_m T^{(m)} T^{(n)} W_n,$$  \tag{4.1}

where $T^{(m)}$ are some basis tensors which carry the external particles' indices, and $C_m$ are real coefficients. If $T^{(m)}$ are independent, $C_m$ can be computed by solving a set of linear equations

$$W_n = C_n T^{(n)}, \quad m, n = 1, 2, \ldots, \beta$$  \tag{4.2}

where $T^{(m)} = T^{(m)} \cdot T^{(n)}$, $W = T^{(m)} \cdot W$, and $T^{(m)}$ is obtained from $T^{(m)}$ by a reversal of all quark lines, and the product is formed by a contraction of all pairs of corresponding indices. For example, any gluon self-energy weight can be expressed in terms of a single basis $T^{11} = \delta_{ij}$, $W_{ij} = C_i \delta_{ij}$ (in this case $W^2 = W_{ij}$ and $T^{11} = N$).

$W_n$ and $T^{(m)}$ are weights of diagrams with no external legs, which we shall refer to as vacuum weights. From (4.2) it is clear that vacuum weights carry all the information needed for weight evaluation. They also have a direct combinatoric significance. We have already noted that single-loop vacuum weights count the number of ways in which a loop can be colored (Figs. 3(e) and 3(f)). For arbitrary weights a hint is given by SO(3), where the weight of a gluon vacuum diagram is simply the number of ways of coloring the lines of the diagram with the three colors meeting at each vertex.\textsuperscript{44}

In general, a vacuum weight is a combinatoric number generated by some more complicated "graph coloring rule."

How is this "coloring rule" built into vacuum weights? If we eliminate gluon self-couplings by (2.5), we note that the remaining couplings $(T_i)_{\alpha\beta}^e$ always appear in the combination $(T_i)_{\alpha\beta}^e (T_j)_{\gamma\delta}^e$. It is this combination that must implement the "coloring rule." What is its significance? As $(T_i)_{\alpha\beta}^e (T_j)_{\gamma\delta}^e$ transforms as the adjoint representation (see Behrends et al.,\textsuperscript{52} Sec. V A for a demonstration), $(T_i)_{\alpha\beta}^e (T_j)_{\gamma\delta}^e$ picks out the part of the quark-antiquark product that transforms as a gluon. Repeated applications of $(1/a)(T_i)_{\alpha\beta}^e (T_j)_{\gamma\delta}^e$ reduce to a single application through the normalization convention (2.4); hence, we shall refer to $(1/a)(T_i)_{\alpha\beta}^e (T_j)_{\gamma\delta}^e$ as the gluon projection operator. The problem of weight evaluation is solved once such projection operators are known.

A gluon projection operator is also a weight (diagrammatically, a Born term in quark-gluon scattering), and, according to (4.1), we can express it in terms of quark-quark scattering basis tensors. To construct a complete set of these, we need to know all invariants of the particular quark representation. There is no simple way to em-
merate the invariants of an arbitrary representation; let us instead concentrate on models with quarks in the defining representation (the lowest-dimensional cogredient representation \(n' \leq n\)). All higher representations can be constructed from the defining representation; in particular, the adjoint representation emerges as the \((T_i)^a_i q = A \iff \cdots\). Furthermore, in the defining representation a classical group has a simple geometrical interpretation [such as length preservation for SO(n)]. The main thrust of this section will be to use such invariance properties to characterize the exceptional groups as well.

A. Invariants of the defining representation

Motivated by the existence of invariants such as \(g_{ab}q_{cd}\) for SO(n), we study unitary transformations \(G_i\) which preserve an arbitrary polynomial

\[
P(Gq) = P(q),
\]

\[
P(q) = \epsilon^{ab\cdots f} q_{ab\cdots f}.
\]

Infinitesimal parametrization \(G = 1 + i \epsilon_i T_i\) gives us a differential statement of \(P(q)\) invariance,

\[
\frac{\delta P(G_q)}{\delta \epsilon_i} = 0,
\]

so that the generators (if a nontrivial group exists) must satisfy

\[
(T_i)^a_i g^{ab\cdots f} + (T_j)^a_j g^{ab\cdots f} + (T_k)^a_k g^{ab\cdots f} = 0.
\]

Contracting this with \((1/\alpha)(T_i)^a_i\) we obtain an invariance condition for gluon projection operators.

Suppose \(g_{abc}\) is an invariant tensor. Then \(g^{abc}\), \(g^{abc}g^{abc}\), and so forth automatically satisfy (4.4) and give us no new constraints on \(T_i\). Let us therefore concentrate on primitive invariant tensors (primitives). They are defined by the requirement that any invariant tensor can be expressed in terms of chains of their contractions (which, diagrammatically, can be disconnected or connected, but cannot contain loops). We assume that the number of primitives is finite [hence, the number of bases in (4.1) is also finite]. Any weight is expressible in terms of primitives; in particular, the gluon projection operator will be of the form

\[
\frac{1}{\alpha} (T_i)^a_i (T_j)^b_j = C_1 g_{ab} g_{cd} + C_2 g_{abcd} + \cdots,\]

Substituting this into invariance conditions (4.4), we obtain conditions on \(C_1, g^{ab\cdots f}\), which, as will be shown, suffice to determine the gluon projection operator up to the overall normalization \(a\). In case of exceptional groups, the invariance conditions are so constraining that they can be realized only in certain dimensions \(q\) (dimensional constraints already appear in classical groups; the symplectic invariant can be realized only in even dimensions).

Our intention is merely to demonstrate that if we know the invariants of the defining representation, we can construct the gluon projection operators and evaluate any weight. Hence, we shall simply state the primitive invariants for each defining representation and show the conditions they must satisfy. Again, as we are computing vacuum weights, we shall find that no explicit realizations of \(g_{ab\cdots f}\) are needed, only some identities which implement the "coloring rules."

All simple Lie algebras are generated by a small set of primitives which are either fully symmetric \((\sigma^{abc}\cdots c)\) or fully antisymmetric \((\epsilon^{abc}\cdots c)\). All defining representations preserve \(\delta_{ab}\) and the Levi-Civita tensor in \(n,\) dimensions, \(\epsilon^{abc}\cdots c\). Their further primitive invariant tensors are

- SU(n): \(\cdots\),
- SO(n): \(\delta_{ab}\),
- Sp(n): \(f_{ab}, n\) even
- \(G_2: \delta_{ab}, \epsilon_{abc}\).

![Diagram](image)

**FIG. 8.** (a) Diagrammatic notation for fully symmetric tensors \(d_{ab\cdots c}\), \(d_{ab\cdots c}\), and fully antisymmetric tensors \(f_{ab\cdots c}\). (b) Invariance conditions for gluon projection operators, (c) normalization convention for gluon projection operators, (d) normalization convention for cubic quark self-couplings.
\[ E_0 = \delta^{abc}, \]
\[ F_i = \delta_{ab}, d_{abc}, \]
\[ E_7 = f^{ab}, d^{abc}, \]
\[ E_8 = \delta_{ij}, C_{ijk}, \text{ unknown}. \]

Before we proceed with the discussion of individual groups, let us make a few observations that will apply to all cases. Owing to the full (anti) symmetry of \( f^{ab...c} \) and \( d^{ab...c} \), the invariance conditions can be stated very compactly (Fig. 8). \( f^{ab...c} \) and \( d^{ab...c} \) can be interpreted as quark self-couplings. Unlike quark-gluon couplings \( (T_i)_k^k \), whose scale is fixed relative to \( C_{ijk} \) by (2.2), they have no \textit{a priori} relation to gauge couplings, and to characterize their scale we introduce an arbitrary normalization \( \alpha \). For cubic couplings we can define \( \alpha \) by

\[ \alpha^{abc} d^{abc} = \alpha \delta_{abc}, \tag{4.6} \]
\[ \alpha^{abc} f^{abc} = \alpha \delta_{abc}. \tag{4.7} \]

\( \alpha \) for different groups need not be the same.

B. Special unitary groups SU\((n)\)

The defining representation of SU\((n)\) is a set of all unitary \((U^\dagger U = 1)\) and unimodular \((detU = 1)\) \([n \times n]\) matrix transformations acting on an \( n \)-dimensional complex vector space \( (n \text{ quarks})\). The infinitesimal transformations can be parametrized by \( N = n^2 - 1 \) traceless Hermitian matrices \((T_i)_k^k\) which close a Lie algebra (2.2). The invariants are the Hermitian (sesquilinear\(^v\)) metric \( \delta^{ab}\) (which imposes the unitarity condition; \( \bar{q}q = q^a \delta^a_q d_a\) is preserved) and the Levi-Civita tensor in \( n \) dimensions, \( \epsilon^{ab...c} \). The contragredient Levi-Civita tensor acts as an inverse to the cogredient one in the sense that a direct product of the two can be expressed as a generalized Kronecker \( \delta \) function [see also (6.4)]

\[ \text{SU}(n) \]

\[ \begin{array}{c}
\text{(a)} \quad A = \begin{pmatrix}
\text{a} & \text{b} \\
\text{b} & \text{a}
\end{pmatrix}
\end{array} \]

\[ \begin{array}{c}
\text{(b)} \quad O = \begin{pmatrix}
\text{a} & \text{b} \\
\text{b} & \text{a}
\end{pmatrix}
\end{array} \]

\[ \Rightarrow \text{ b = -1/n} \]

\[ \epsilon^{abc}\delta^{def}\epsilon_{ghstu} = \delta^{abc}_{ghstu}. \tag{4.8} \]

Gluon projection operator expansion (4.5) is of the form

\[ \frac{1}{a}(T_i)_k^k(T_i)_k^k = A(\delta^{ab}_{\delta} b^a + b^a \delta_{\delta}), \tag{4.9} \]

which we give diagrammatically in Fig. 9(a). [Any possible \( \epsilon^{abc...c} \) terms reduce to the above two by (4.8)]. Substituting this expression into \( \epsilon^{abc...c} \) invariance condition Fig. 8(b), we obtain the equation Fig. 9(b), which, when contracted with \( \delta_{\delta}^a \) (in the only way possible, the incoming line with any outgoing line) yields \( b = -1/n \). We now see how a projection operator\(^{53}\) works; \( \delta_{\delta}^a \) removes the singlet from a quark-antiquark state, leaving \( N = n^2 - 1 \) gluons. Tracelessness of \( T_i \) ensures that the gluon does not connect to the vacuum (i.e., that the group is semisimple). From the normalization convention Fig. 8(c) \( \lambda = 1 \), and we can verify that the number of gluons is indeed \( N = n^2 - 1 \) by evaluating (3.2).

C. Special orthogonal groups SO\((n)\)

The defining representation of SO\((n)\) is a set of all orthogonal \((R^T R = 1)\) and unimodular \((detR = 1)\) \([n \times n]\) matrix transformations acting on an \( n \)-dimensional complex vector space \( (n \text{ quarks})\). The defining invariant is a symmetric tensor \( d^ab = d^ba \) (and its inverse \( d_{ab} = d_{ba} \)) introduced diagrammatically in Fig. 10(a). The remainder of Fig. 10 derives the gluon projection operator from the inner:

\[ \begin{array}{c}
\text{SO}(n) \quad d^a b = \begin{pmatrix}
\text{a} & \text{b} \\
\text{b} & \text{a}
\end{pmatrix},
\end{array} \]

\[ \begin{array}{c}
\text{(b)} \quad A = \begin{pmatrix}
\text{a} & \text{b} \\
\text{b} & \text{c}
\end{pmatrix}
\end{array} \]

\[ \begin{array}{c}
\text{(c)} \quad O = \begin{pmatrix}
\text{a} & \text{b} \\
\text{b} & \text{c}
\end{pmatrix}
\end{array} \]

\[ \Rightarrow \text{ b = 0, c = -1} \]

\[ \begin{array}{c}
\text{(d)} \quad \frac{1}{\alpha} \quad = \begin{pmatrix}
\text{a} & \text{b} \\
\text{b} & \text{a}
\end{pmatrix}
\end{array} \]

FIG. 10. (a) Diagrammatic notation for SO\((n)\)-invariant tensor \( d_{ab} \), (b) the most general form of the gluon projection operator for SO\((n)\), (c) \( d_{ab} \) invariance condition, (d) gluon projection operator for SO\((n)\).
defining representations. Construction of the gluon projection operator (Fig. 11) proceeds as in the SO(n) case.

E. Exceptional group G2 (Ref. 56)

The defining representation of G2 (n = 7) preserves a symmetric invariant δ_{ab} \[ G_2 \text{ is a subgroup of SO(7)}, \] and a fully antisymmetric cubic invariant \( f^{abc} \). It is possible to show that \( G_2 \) is the only nontrivial simple group that possesses such invariants, and that \( f^{abc} \) must satisfy the alternativity relations given in Fig. 12(b). By these relations two out of three tensors \( f^{abc} f^{def} = f^{acf} f^{bef} \) and \( f^{abc} f^{def} \) can always be eliminated in favor of the third and some combination of \( \delta_{ij} \)'s. As in the SO(n) case, \( \delta_{ab} \) invariance makes generators \( T_i \) antisymmetric, and the gluon projection operator (4.5) has the form given in Fig. 13(a). From the identity Fig. 13(b), we derive relation Fig. 13(c), which determines the gluon projection operator through invariance of \( f^{abc} \) [Fig. 8(b)]. Actually, Fig. 13(c) (through a few more applications of the alternativity relations) leads to a very strong statement that any chain of three \( f^{abc} \) can be reduced to a sum of terms linear in \( f^{abc} \) by the equation of Fig. 13(d). This guarantees that even though the projection operator (2.10) replaces internal quark lines by internal quark lines, the resulting weights can always be reduced to the bases (4.1). The gluon number, evaluated by (3.2), is indeed \( N = 14 \). Further relations are given in Fig. 14.

An explicit realization of tensors \( f^{abc} \) is given by octonions. In this framework \( G_2 \) is the auto-

D. Symplectic groups Sp(n)

The invariant preserved by the defining representation of \( \text{Sp}(n) \) is a skew-symmetric metric \( f^{ab} = -f^{ba} \) (and its inverse \( f_{ab} = -f^{ba} \)). An inverse exists only if \( f^{ab} \) is nonsingular, \( \det(f) \neq 0 \). The skew-symmetry of \( f^{ab} \) allows that only for even-dimensional

FIG. 11. (a) Diagrammatic notation for \( \text{Sp}(n) \)-invariant tensor \( f_{ab} \), (b) \( f_{ab} \) invariance condition, (c) gluon projection operator for \( \text{Sp}(n) \).

The variance of \( d^{ab} \) [Fig. 8(b)]. By diagonalizing \( d^{ab} \) and rescaling \( q^a \) fields, we can always find a representation where \( d_{ab} = \delta_{ab} \). There is no distinction between upper and lower indices (quark = antiquark, the representation is real), and in diagrams we can omit all \( d^{ab} \) tensors and all line arrows, and note that because of (4.4) the generators are antisymmetric: \( (\bar{T}_i)_ab = -(T_i)_{ba} \). They are clearly traceless, and it is easily verified that the Levi-Civita tensor \( \epsilon^{abc} \) in \( n \) dimensions is preserved as well.

In the conventional choice of SO(n) generators with only two nonzero elements \pm 1, the normalization is fixed by \( a = 2g^2 \).

FIG. 12. (a) Diagrammatic notation for the tensor \( f_{abc} \) for the exceptional group \( G_2 \), (b) the "alternativity" relation which relates contractions of pairs of \( f_{abc} \), (c) the invariance condition for \( f_{abc} \).
morphism group of octonions, i.e., it is a set of all $[7 \times 7]$ real matrices $G_{ab}$ such that the transformation
\[ e_i^a G_{ab} e_b = a, b = 1, 2, \ldots, 7 \]
preserves the octonionic multiplication rule
\[ e_a e_b = -e_b + f_{abc} e_c, \quad \tag{4.10} \]
where $f_{abc}$ are given explicitly in Ref. 58; for our purposes, it is sufficient to note that octonions satisfy the alternativity condition if
\[ [xyz] = (xy)z - x(yz), \]
\[ [xyz] = [zxy] = [yzx] = -[yxz], \]
where $x, y, z$ are arbitrary octonions. The alternativity relation Fig. 12(b) follows from the multiplication rule (4.10) and the alternativity condition. Equation (4.10) also fixes the normalization (4.7) $\alpha = -6$. Then $-\alpha$ is simply the number of distinct colorings of diagram Fig. 8(d) allowed by the octonion multiplication rule.

F. Exceptional group $E_6$

The defining representation of $E_6$ ($n = 27$) preserves a fully symmetric cubic invariant $d_{abc}$ (and its inverse $e^{abc}$).\(^{15-21,60-63}\) No condition relating $d_{abc}d_{ade}$ type tensors exists and the only nontrivial relation\(^{64}\) on $d_{abc}$ tensors is a trilinear Springer relation\(^{60}\) [Fig. 15(b)] which arises from the requirement of $d_{abc}$ invariance [Fig. 8(b)]. This relation enables us to compute the gluon projection operator [whose general form is given by Fig.

FIG. 15. (a) The most general form of the gluon projection operator for $E_6(27)$. (b) Springer's relation. Together with the invariance condition for the gluon projection operator, it fixes the constants in (a).

Evaluation of (3.2) yields the dimension of the algebra of $E_6$, $N = 78$.

Springer's relation arises from the characteristic equation for $[3 \times 3]$ Hermitian octonion matrices. The gluon projection operator (2.11) was actually first constructed by Freudenthal\(^{65}\) in a very different notation (as a derivation of a Jordan algebra). His normalization convention is $\alpha = \frac{3}{2}$.

FIG. 16. Diagrammatic notation for the tensor $d_{abc}$ for the exceptional group $F_4(26)$, (b) “characteristic” relation which relates contractions of pairs of $d_{abc}$, (c) expansion of this identity [which follows from (b)] leads to (d) a relation between contractions of three $d_{abc}$. Antisymmetrization in top legs and symmetrization in bottom legs yields (e) the Jordan identity which together with the invariance condition for $d_{abc}$ fixes the gluon projection operator for $F_4(26)$. 
G. Exceptional group $F_4$

The defining representation of $F_4$ ($n = 26$) preserves both $d_{abc}$ and $\delta_{ab}$. To derive $F_4$, it is necessary to assume that a relation between bi-linear combinations $d_{abc}d_{cde}$ exists. The only non-trivial relation of such type is the characteristic relation of $G_2$. The glion projection operator is constructed the way it was constructed for $G_2$. The identity of Fig. 16(c) leads us to the Jordan identity of Fig. 16(e), which together with the $d_{abc}$ invariance [Fig. 8(b)] fixes the projection operator up to an overall normalization. The normalization convention [Fig. 8(c)] then yields the glion projection operator given in (2.12). There are $N = 52$ gluons. Further relations are given in Fig. 17.

An explicit realization of tensors $d_{abc}$ is given by octonion matrices. In this framework, $F_4$ is the isomorphism group of the exceptional simple Jordan algebra of traceless Hermitian $[3 \times 3]$ matrices $x$ with octonion matrix elements. The non-associative multiplication rule for elements $x$ can be written as

$$x = x_\alpha e_\alpha, \quad \alpha = 1, 2, \ldots, 26$$

$$\text{Tr}e_\alpha = 0, \quad e_\alpha \text{ is a } [3 \times 3] \text{ basis matrix,}$$

$$e_\alpha e_\beta = e_\beta e_\alpha = \frac{\delta_{\alpha\beta}}{3} + d_{abc} e_\alpha e_\beta,$$

$$\text{Tr}1 = 3, \quad 1 \text{ is a } [3 \times 3] \text{ unit matrix.}$$

Transformations of $F_4$ preserve the quadratic form $\text{Tr}(x^2)$ [the length in 26-dimensional space, so that $F_4$ is a subgroup of SO(26)], as well as a fully symmetric cubic form

$$\text{Tr}(xyz) = \text{Tr}(yxz) = \text{Tr}(yzx) = d_{abc} x^a y^b z^c.$$

H. Exceptional group $E_7$

The defining representation of $E_7$ ($n = 56$) preserves a skew-symmetric tensor $f^{ab}$ [$E_7$ is a subgroup of Sp(56)] and a fully symmetric quartic invariant $e^{abcd}$ of $[E_7]$. The gluon projection operator can have the general form of Fig. 18(c). The invariance of $d_{abc}$ gives the Brown relation of $[E_7]$, which enables us to compute Fig. 18(e), impose the normalization condition Fig. 8(c), and derive (2.13). The evaluation of the gluon number gives $N = 133$. 

![Diagram](image-url)
In the explicit realization of tensors $d_{abot}$ by octonion matrices, the conventional normalization is $\alpha = \frac{1}{2}$.

I. Exceptional group $E_8$

The defining representation of $E_8$ ($n = N = 248$) is also the adjoint representation, so our method of reducing everything to the lowest-dimensional representation is of no help. Still, if the invariants of the defining representation of $E_8$ were known, we would be able to reduce higher-order weight diagrams to a basic set just as for all other simple groups. Known invariants are $\delta_{ab}$ and $C_{abc}$, and other invariants are certainly higher than quartic. The Tits construction, which relates $SU(n) \rightarrow E_8$, $SO(n) \rightarrow F_4$, and $Sp(n) \rightarrow E_6$, suggests (extrapolating octonions $\rightarrow E_8$) that the $E_8$ invariant is a fully symmetric octet $d_{abdefgh}$. We do not know whether this is true and we hope we shall never need to know.

We should also point out that we have not proved that our identities for $F_4$, $E_6$, and $E_7$ suffice to evaluate any weight. We have only verified this for all vacuum weights up to 4 loops ($F_4$ and $E_6$) and 3 loops ($E_7$).

V. ILLUSTRATIVE EXAMPLES

Evaluation of any $\tilde{W}_0$ is now almost trivial, especially for classical groups. We just proceed applying systematically the rules of Fig. 1, first eliminating all three-gluon vertices, and then removing all internal gluon lines. Removal of each gluon line reduces $\tilde{W}_0$ into a sum of weights of lower order. Eventually we end up with a set of irreducible tensor bases, each multiplied by some polynomial in $n$ (n is the number of quark colors).

As an example, we evaluate the $SO(n)$ quadratic

\begin{equation}
SO(n) = \left\{ \begin{array}{l}
(a) \quad \alpha = \left( \frac{2}{3} \right)^2 \\
(b) \quad = 4 \\
(c) \quad = 4 - \frac{2}{3} - \frac{2}{3}
\end{array} \right.
\end{equation}

FIG. 19. A sample diagrammatic computation: quadratic Casimir operator for the adjoint representation of $SO(3)$. (a) $C_{ab}$ are replaced by the defining representation, (b) internal gluons are replaced by gluon projection operators, and (c) the expression is expanded and evaluated.

\begin{equation}
\begin{array}{cccc}
SU(n) & n^2 - 1 & 2n & \begin{array}{c}
\frac{1}{3}
\end{array} \\
SO(n) & \frac{n(n-1)}{2} & \frac{n(n-1)}{2} & \begin{array}{c}
\frac{1}{2}
\end{array} \\
Sp(n) & \frac{n(n+1)}{2} & \frac{n(n+1)}{2} & \begin{array}{c}
\frac{1}{2}
\end{array} \\
G_2(7) & 14 & 4 & \begin{array}{c}
0
\end{array} \\
F_4(26) & 52 & 3 & \begin{array}{c}
0
\end{array} \\
E_6(27) & 78 & 4 & \begin{array}{c}
0
\end{array} \\
E_7(56) & 133 & 3 & \begin{array}{c}
0
\end{array}
\end{array}
\end{equation}

FIG. 20. A tabulation of some simple weight evaluations.

Casimir operator for the adjoint representation (gluons) in Fig. 19. We find that

\begin{equation}
C_A = \alpha (n - 2).
\end{equation}

Other such results are tabulated in Fig. 20. Of course, dimensions and Casimir operators (or representation indices) are all tabulated in the literature and our algorithm is unnecessary for their evaluation. However, we can now calculate the weight of any diagram. A typical example would be computation of all the weights that appear in the $SU(n)$ quark-quark scattering calculation, or the order of the first nonleading term in $1/n$ expansion for various groups.

VI. RELATIONS BETWEEN BASIS TENSORS

The procedure outlined in Secs. I–V always leads us to a unique set of tensors $(T^a)^b_c$ and traces over $T^a$. In other words, we are expressing all $\tilde{W}_0$ in terms of the defining representation. Let us illustrate this by writing all irreducible bases $T^{(a)}$ for quark-quark scattering weights [see (4.1)]:

\begin{equation}
\begin{array}{l}
SU(n): \quad 0^a_0^b, 0^b_0^a, (\beta = 2) \\
SO(n): \quad 0^a_0^b, 0^b_0^a, 0^c_0^b, 0^c_0^d, (\beta = 3) \\
Sp(n): \quad 0^a_0^b, 0^b_0^a, f^a_0^c f^c_0^a, (\beta = 3) \\
G_2(7): \quad 0^a_0^b, 0^a_0^b, 0^a_0^b, 0^a_0^b, f^a_0^c f^c_0^d (\beta = 4)
\end{array}
\end{equation}

and so forth. These bases appear naturally in our approach, but they are by no means the only possible choice. For example, we can replace the “color exchange” base $0^a_0^b$ by the “color flip” base $-f^a_0^c f^c_0^a$ using relations (2.7)–(2.13). As another example, we write down all irreducible tensor invariants for a process with $r$ external gluons and no external quarks, the set of all dis-
distinct traces over $r T_i$ matrices (Fig. 21).

$\beta_r$, the number of all distinct tensors of rank $r$, is the number of ways in which $r T_i$ matrices can be grouped into traces over their products, with the restriction that $\text{Tr}(T_i) = 0$. $\beta_r$ can be calculated in a number of arduous ways, such as by Young tableaux,\textsuperscript{66} or by the method of Appendix B. However, it turns out that $\beta_r$ had already been calculated in 1708,\textsuperscript{67,68} and is known as a number of derangements, or subfactorial

$$\beta_\mathcal{R} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right). \quad (6.1)$$

Not all tensor bases thus enumerated are necessarily independent, because they might be related through the invariants of the defining representation. $\beta_n$ was calculated from a single condition, tracelessness. Thus, traces over $T_i$ form natural bases for all simple Lie groups, $\text{SU}(n)$ in particular. For $\text{SO}(n)$, $\text{Sp}(n)$, $G_2$, $F_4$, and $E_7$, the clockwise and anticlockwise directions of loops in Fig. 21 are related by $\delta_{ab}, f_{ab}$ invariance, and the number of independent bases is reduced:

$$P(x) = \det |A - ix| = \sum_{k=0}^{n^2} (-x)^{n-k} \frac{1}{k!} A_{a_1}^{a_2} A_{a_2}^{a_3} \cdots A_{a_k}^{a_k}, \quad (6.3)$$

where

$$\delta_{\mu_1 \mu_2 \cdots \mu_r} = \begin{vmatrix} \delta_{a_1 b_1} & \cdots & \delta_{a_1 b_r} \\ \vdots & \ddots & \vdots \\ \delta_{a_r b_1} & \cdots & \delta_{a_r b_r} \end{vmatrix} \quad (6.4)$$

is the generalized Kronecker $\delta$. Identity $P(A) = 0$ yields the characteristic equation for $A$:

$$0 = \sum_{k=0}^{n^2} (-x)^{n-k} \frac{1}{k!} A_{a_1}^{a_2} A_{a_2}^{a_3} \cdots A_{a_k}^{a_k}, \quad (6.5)$$

Now if we substitute $A = a_i T_i$, where $T_i$ are generators of the group $\mathcal{G}$, for each $n$ we obtain various relations between tensor invariants. As an example, we work out the $n = 4$ case diagrammatically in Fig. 22(a). The indices are symmetrized because the whole expression is multiplied by a symmetric factor $a_i a_j a_k a_l$, summed over all $i, j, k, l$. More familiar relationships are worked out explicitly for $\text{SU}(2)$ and $\text{SU}(3)$ in Figs. 22(b) and 22(c). The $\text{SU}(3)$ relationship can be rewritten in terms of $d_{ijk}$ tensors, the form of which has been originally derived by Macfarlane et al.\textsuperscript{27} Higher $\text{SU}(n)$ relationships have been worked out in Ref. 29. Such relations do not affect the cor-
rectness of our general procedure for weight evaluation.

VII. HIGHER REPRESENTATIONS

In Sec. IV we have constructed gluon projection operators from the invariants of the quark representation. This approach is by no means restricted to the defining representation; in Appendix B we shall give an example of a calculation in terms of the invariants of the adjoint representation. That calculation will exemplify the difficulties arising in the study of higher representations; it is not easy to find a complete set of invariants for an arbitrary representation, and even when those are found, the evaluation of weights can still be difficult.

However, we already have a simpler solution for one higher representation; we know how to compute weights of diagrams with all particles in the adjoint representation. We evaluate them by rewriting them in terms of the defining representation. This suggests that we should attempt to express the particular higher representation in terms of the defining representation; once that is accomplished, the weights can be evaluated by the methods of Sec. IV. In principle, we always know how to construct any representation from the defining one by the Young symmetrization procedure.

As an example we construct the antisymmetric second-rank tensor representation of SU(n). The projection operator \( \frac{1}{2} (T^a_{ij} T^a_{ij} - T^a_{ij} T^a_{ij} + T^a_{ij} T^a_{ij} - T^a_{ij} T^a_{ij}) \) picks out the antisymmetric part of a two-quark state \( q_A q_I \), and the generator of SU(n) in this representation is\(^{49}\)

\[
(T_{ij})^{a} = \frac{1}{2} \left[ (T^a_{ij} T^a_{ij} - T^a_{ij} T^a_{ij} + T^a_{ij} T^a_{ij} - T^a_{ij} T^a_{ij}) \right],
\]

where \( a, b, \ldots, 1, 2, \ldots, n \), and \( T_i \) are the generators of the defining representation of SU(n) (Sec. IV B). This is a nice example of how compact the diagrammatic notation is\(^{70}\) (Fig. 23) compared to tensor notation. To check this construction we compute the dimension [Fig. 3(f)] and the index (3.3) and verify\(^{58,49,69}\) that

\[
\frac{n(n - 1)}{2},
\]

\[
\frac{1}{a} \text{Tr}(T_i T_j) = n - 2.
\]

Further examples of projection operators for higher representations are given by Behrends et al.\(^{52}\)

We should also mention that there already exist algorithms for computing weights of arbitrary representations. For example, Agrawala and Belinfante\(^{71}\) have developed a computer program for evaluation of SU(n) invariants.

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APPENDIX A: COMPLETE FEYNMAN RULES FOR \( W_G \)

With the definition of the group-theoretic weight \( W_G \) given in Sec. II, the rules for \( M_G \) are easily

\[
\begin{array}{c|c}
\text{Factors for} & W_G & M_G \\
\hline
- & \text{quark} & \text{quark} \\
\hline
i \gamma^\mu & -i \gamma^\mu & -i \gamma^\mu \\
\hline
i \gamma^\mu i \gamma^\nu & -i \gamma^\mu i \gamma^\nu & -i \gamma^\mu i \gamma^\nu \\
\hline
i \gamma^\mu i \gamma^\nu i \gamma^\rho & -i \gamma^\mu i \gamma^\nu i \gamma^\rho & -i \gamma^\mu i \gamma^\nu i \gamma^\rho \\
\hline
\end{array}
\]

FIG. 24. Factors for the group-theoretic weights \( W_G \) and Feynman momentum integrals \( M_G \) in the Feynman gauge.
constructed by consulting some standard reference, such as Aber's and Lee. In this appendix we state the full rules for unrenormalized Feynman amplitudes in (unbroken) non-Abelian gauge theories as an extension of the rules for constructing Feynman-parametric integrals given previously. Factors of rule 5, of Ref. 73, are now replaced by the factors of Fig. 24. Additionally \( M_q \) gets a factor \(-1\) for each quark or ghost loop.

**APPENDIX B: EVALUATION OF SU(\(n\)) WEIGHS USING \(f\)-AND \(\kappa\)-TENSOR BASES**

In this appendix we extend the SU(3) method of Dittner\textsuperscript{28} to SU(\(n\)). The generalized Gell-Mann \([n \times n] \lambda\) matrices together with \(I, \tilde{I}, \) and \(\lambda\) span all complex matrices,\textsuperscript{37} so we can write a multiplication rule for \(\lambda\) matrices as

\[
\text{SU}(n): \quad \lambda_i \lambda_j = (\bar{\alpha} + i b) \delta_{ij} \mu + (d_{ijk} + if_{ijk}) \lambda_k. \quad (B1)
\]

This relation, which has no obvious analogs for other simple groups, is the departure point for most of the earlier attempts at weight evaluation.\textsuperscript{29-36} The tensors \(\delta_{ij}, d_{ijk}, \) and \(f_{ijk}\) are numerically invariant in the sense that they are the same for all equivalent representations \(\lambda_i \rightarrow u^\dagger \lambda_i u, \) \(u^* u = 1.\)

They are real by definition. \(b = 0\) because of the Hermiticity of \(\lambda_i,\) while \(\bar{\alpha}\) is related to the arbitrary normalization of Eq. (2.4), \(\alpha = (\nu^2/4)\bar{\alpha}.\)

According to Sec. IV, we can evaluate any weight if we know how to evaluate vacuum weights. There \(\lambda_i\) matrices always appear in traces, \(\text{TR}(\lambda_1 \lambda_2 \cdots \lambda_m),\) and they can be eliminated by the repeated application of the \(\lambda\)-multiplication rule [depicted in Fig. 25(b)]. The problem of weight evaluation for SU(\(n\)) is then reduced to the problem of evaluation of vacuum weights built solely from the adjoint representation invariant tensors \(\delta_{ij}, f_{ijk},\) and \(d_{ijk}.\)

Dittner solves this by setting up a chain of sets of linear equations of type (4.2), which make it possible (in principle) to compute weights with \(k + 1\) loops once all vacuum weights with up to \(k\) loops

\[
\begin{align*}
\text{SU}(n) & \\
(a) & \quad d_{ijk} = d_{ijk} \\
(b) & \quad \frac{1}{n} + \frac{1}{2} + \frac{1}{2} \\
(c) & \quad \frac{1}{2} \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] \\
(d) & \quad \frac{1}{3} \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 + \lambda_3 \end{array} \right]
\end{align*}
\]

**FIG. 25.** (a) Notation for the (fully symmetric) numerical tensor \(d_{ijk},\) (b) multiplication rule for SU(\(n\)) matrices \(T_1 = \frac{1}{2} \lambda_1,\) (c) decomposition of three external gluon loop into real and imaginary parts, (d) \(d_{ijk}\) as its real part.

\[
\begin{align*}
& \begin{array}{cccc}
& r & a_r & 1 \\
& 0 & 1 & 0 \\
& 2 & 1 & \end{array} \\
& \begin{array}{cccc}
& 3 & 2 & 15 \\
& 4 & 15 & \end{array} \\
& \begin{array}{cccc}
& 5 & 140 & \end{array} \\
& \begin{array}{cccc}
& 6 & 1915 & \end{array}
\end{align*}
\]

**FIG. 26.** Construction of all simple \(d\) and \(f\) tensors with \(r\) external gluons.

\[
\begin{align*}
& \begin{array}{cccc}
& r & a_{r-1} & \bar{a}_r \\
& 3 & 1 & 2 \\
& 4 & 2 & 12 \\
& 5 & 5 & 120 \\
& 6 & 14 & 1680
\end{array}
\end{align*}
\]

**FIG. 27.** Catalan's trees.
are known. To achieve this, it is necessary to construct independent bases for processes with \( r = 2, 3, \ldots \) external gluon legs.

The simplest set of tensors for each \( r \) is easily constructed (see Fig. 26). To enumerate them, we start a systematic construction by drawing all Catalan\(^{69,74}\) trees in Fig. 27, whose number is Catalan’s number (the number of ways in which a product of \( n \) numbers can be evaluated)

\[
a_{r+1} = \frac{(2r+3)!}{(r+1)! (r+2)!}, \quad a_0 = 0. \tag{B2}
\]

By \((r - 1)!\) permutations of all branches, and factor 2 for each crotch (\( f \) or \( d \) tensor), we obtain the number of all distinct connected tensors

\[
\bar{a}_r = 2^{r-2}(2r-3)! \quad \bar{a}_1 = 0, \quad \bar{a}_2 = 1 \tag{B3}
\]

where \((2n - 1)!!\) is the product of the first \( n \) odd integers, \( 711 = 7 \times 5 \times 3 \times 1 \). To relate \( \bar{a}_r \) to the \( a_r \), the number of all distinct tensors (connected and unconnected) we introduce generating functions

\[
A(t) = \sum_{r=1}^{\infty} a_r t^r, \tag{B4}
\]

\[
\bar{A}(t) = \sum_{r=1}^{\infty} \bar{a}_r t^r = \frac{1}{12}[1 + 6t + (1 - 4t)^{3/2}] \tag{B5}
\]

The numbers of connected and disconnected graphs are related in the usual fashion,

\[
A(t) = e^{\bar{A}(t)}. \tag{B6}
\]

By differentiation with respect to \( t \), this can be restated as

\[
\alpha_r = \sum_{k=0}^{r} \binom{r-1}{k} \bar{a}_{r-k} a_k, \tag{B7}
\]

which enables us to calculate recursively \( \alpha_r \), listed in Fig. 26.

However, tensors so constructed are redundant, and if we attempt to use them to expand an arbitrary tensor with \( r \) external gluons, we would not be able to calculate the expansion coefficients, because the determinant of the system of \( \alpha_r \) equations vanishes for \( r > 3 \).

So our next task is to find all the relations between \( \alpha_r \) tensors. These stem from the associativity of \( T_i \) matrices. For example, \( \text{Tr}(T_j T_k T_i) \) can be evaluated in two ways, by pairing matrices either as \( \text{Tr}(T_j T_i) (T_k T_i) \) or \( \text{Tr}(T_j T_k) (T_i T_k) \), and then using Fig. 25(b). The two evaluations give the relationship of Fig. 28(a). There are \((4-1)! = 6\) distinct connected tensor bases (Fig. 21) with four \( T_i \) each, giving us \( \bar{\gamma}_r = 6 \) relationships. We cast those in the form familiar from the literature, \(^{26,28}\) three equations for the real parts [Fig. 28(b)] and three for the imaginary parts [Fig. 28(c)]. Figure 28(c) states that \( \delta_{i,ik} \) are invariant [see (4.4) and remember that \( (T_i)_{ik} = -i \delta_{ik} \) for the adjoint representation of \( SU(n) \)]. The second and third lines of Fig. 28(b) are two versions of the \( SU(n) \) generalization, \(^{26,27}\) of the \( SU(2) \) relationship

\[
\epsilon_{ijk} \delta_{ikm} = \delta_{ij} \delta_{jm} - \delta_{im} \delta_{jm}. \tag{B8}
\]

Glancing back at the gluon projection operator for \( F \) [Fig. 1(b)], we realize that this is the gluon projection operator for models with quarks in the adjoint representation of \( SU(n) \).

The number of associativity relations for arbitrary \( r \) is again related to Catalan’s number, which is nothing but the number of associativity patterns

\[
\bar{\gamma}_r = (r-1)! (a_{r+1} - 1), \quad r \geq 2. \tag{B9}
\]

For each \( r \) there are

\[
\beta_r = \bar{\alpha}_r - \bar{\gamma}_r = (r-1)! \quad r \geq 2 \tag{B10}
\]

independent connected tensors. The total number of independent tensors \( \beta_r \) is given by

\[
B(t) = \sum_{r=2}^{\infty} \frac{\beta_r t^r}{r!}, \tag{B11}
\]
is obtained. Now it is necessary to solve these equations—for the details, we refer the reader to Dittner’s papers. 28 To illustrate the form of the results, we give the reduction of a gluon “box” diagram in two (of many possible) choices of $f$, $d$ bases [Fig. 29(a)]. For comparison with the method of Sec. VI, we also evaluate the same diagram in $T_4$ bases, Fig. 29(b).

To summarize, for $SU(n)$ the knowledge of the invariants of the adjoint representation leads to a feasible method of weight evaluation. However, compared with the evaluation via the defining representation, it suffers from numerous drawbacks. It introduces a tensor $d_{ijk}$ that does not appear in the original interaction Lagrangian, and leads to arbitrariness in the choice of tensor bases (note that the $T_4$ bases are unique). Finally, it involves solving large sets of linear equations; already for $r = 4$ we found it convenient to do the algebra on a computer. 13 By contrast, if we use the defining representation, evaluation never requires solving any equations (for classical groups, at least): It is a systematic procedure of eliminating internal gluons one by one until only irreducible tensors are left. If there are $d_{ijk}$ couplings in the model, they are easily incorporated into our scheme by Fig. 25(d).

---

11These are not to be confused with Cartan’s weight diagrams.
23P. Cvitanović (unpublished).
(1965).
32R. Rockmore, Phys. Rev. D 11, 620 (1975) [the method of this paper is applicable only to SU(3)].
36For a semisimple algebra the symmetric bilinear form $\text{Tr}(T_i T_j)$ is nonsingular and it can always be brought to the convenient form (2.4). In the language of Cartan’s diagrams, $\frac{1}{2}$ sets the length scale for root vectors. Any representation can be used for normalization of the Lie algebra. In mathematics this is usually done by fixing the value of the quadratic Casimir operator for the adjoint representation.
38Diagrammatic notation appears frequently in group-theoretic problems. Canning (Ref. 17) has used diagrammatic equations for SU(6) which are identical to ours, and similar notation has been developed by Penrose (Refs. 39–40), J. Mandula (unpublished), Yeung (Ref. 44), and Cahalan and Knight (Ref. 17). Diagrammatic methods for coupling coefficients for arbitrary representations (related to Wigner’s $3n^3$ coefficients) have a long tradition in atomic spectroscopy, nuclear shell theory, and many other areas (see Ref. 41).
40C. Murphy, Proc. Camb. Philos. Soc. 71, 211 (1972). The reduction of diagrams with four external gluons attempted in this paper is valid only for SO(9).
43J. Patera and D. Sankoff, Tables of Branching Rules for Representations of Simple Lie Algebras (Univ. de Montréal, Montreal, Quebec, Canada, 1973).
44Group-theoretic weights have an amusing graph-theoretic interpretation for SO(6). If we consider a planar vacuum diagram (no external lines) with normalization $a = 2$, then $w_4$ is the number of ways of coloring the lines of the graph with three colors (see Ref. 39). This, in turn, is related to the chromatic polynomials, Heawood’s conjecture, and even the four-color problem. [See R. C. Read, J. Combinatorial Theory 4, 52 (1969) and O. Ore, The Four-Color Problem (Academic, New York, 1967)].
46In Ref. 7, this is not manifest because the weights are computed explicitly for SU(6) by a method discussed here in Appendix B. However, Higgs particles contribute only as a correction to the three-gluon vertex which is proportional to $\epsilon_4$ (Fig. 4b) for any group, and the cancellations between remaining diagrams follow from Lie algebra commutation relations alone. [See also P. S. Yeung, Phys. Rev. D 13, 2306 (1976)].
48Sometimes the defining representation is referred to as the vector representation (see Refs. 13 and 49), the principal linear representation (see Ref. 50), or the fundamental $n$-tuplet. However, note that in Cartan’s terminology a group of rank $r$ has $r$ fundamental representations, and that sometimes higher representations are called “vector” (Ref. 52, p. 25).
51For this reason mathematicians refer to $T_i$ as “derivations.”
53Such projection operators are sometimes called completeness relations. For SU(6) they were given by Macfarlane et al. (Ref. 27), and for SO(6) by Cheng et al. (Ref. 49).
55This arises because $Sp(n)$ is a complex representation of the quaternionic norm invariance group in $n/2$ dimensions. $f^{ab}$ is a representation of an imaginary unit $i$ for a quaternion written as $C_1 + i C_2$, $C_1$ complex. Skew-symmetry arises from $i^2 = -1$, and inverse from $i = -1$.
(unpublished).

57 É. Cartan, Oeuvres Completes (Gauthier-Villars, Paris, 1952).
59 The same relation has been obtained by R. E. Behrends et al. (Sec. V D of Ref. 52) without octonions. Use of octonions greatly simplifies the derivation.
66 We thank R. Pearson for carrying out a Young tableau calculation to check our numbers. $\beta_r$ is the number of times the singlet appears in the decomposition of a product of $r$ adjoint representations: $N \times N \times \cdots \times N = \beta_1 + \cdots$.
70 Diagrammatic representations of Young symmetrizers are discussed by Penrose (Ref. 39).
74 Catalan, J. M. Pure Appl. 3, 508 (1838).