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Renormalization Description of Transitions to Chaos

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12.1. Introduction

The aim of this chapter, and the following one, is to review the developments which have greatly increased our understanding of chaotic dynamics during the past decade, and given us new concepts and modes of thought that, we hope, will have far-reaching repercussions in many different fields.

Once it was believed that given the initial conditions, we knew what a deterministic system would do far into the future. That was immodest. Today it is widely appreciated that given infinitesimally different starting points, we often end up with wildly different outcomes. Even with the simplest conceivable equations, almost any nonlinear system will exhibit chaotic behavior.

Confronted today with a potentially turbulent nonlinear dynamical system, we analyze it through a sequence of three distinct steps. First, we determine the intrinsic *dimension* of the system—the minimum number of degrees of freedom necessary to capture its essential dynamics. If the system is very turbulent (its attractor is of high dimension) we are, at present, out of luck. We know only how to deal with the transitional regime between regular motions and weak turbulence. In this regime the attractor is of low dimension, the number of relevant parameters is small, and we can proceed to the second step; we classify all the motions of the system by a hierarchy whose successive layers require increased precision and patience on the part of the observer. We call this classification the *symbolic dynamics* of the system: the following chapter shows how to do this for the Hamiltonian systems, and in this chapter we take the period n-tuplings and circle maps as instructive examples.

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Though the dynamics might be complex beyond belief, it is still generated by a simple deterministic law and, with some luck and intelligence, our labeling of possible motions will reflect this simplicity. If the rule that gets us from one level of the classification hierarchy to the next does not depend on the level, the resulting hierarchy is self-similar, and we can proceed with the third step: investigate the *scaling structure* of the dynamical system. This is the subject of the present chapter.

The scaling structure of a dynamical system is encoded into the *scaling functions*. The purpose of scaling functions is twofold:

For an experimentalist, they are the theorist's prediction of the motions expected in a given parameter and phase-space range. Given the observed motions up to a given level, the symbolic dynamics predicts what motions should be seen next, and the scaling functions predict where they should be seen, and what precision is needed for their observation.

For a theorist, the scaling functions are the tool which resolves asymptotically the fine structure of a chaotic dynamical system and proves that the system is indeed chaotic, and not just a regular motion of period exhausting the endurance of an experimentalist. Furthermore (and that is theoretically very sweet), the scalings often tend to *universal* limits. In such cases, the finer the scale, the better the theorist's prediction! So what *a priori* appears to be an arbitrarily complex dynamics turns out to be something very simple, and common to many apparently unrelated phenomena.

This is the essence of the recent progress here: large classes of nonlinear systems exhibit transitions to chaos which are *universal* and *quantitatively* measurable in a variety of experiments. This advance can be compared to past advances in the theory of solid-state phase transitions (cf. Chapter 10); for the first time we can predict and measure "critical exponents" for turbulence. But the breakthrough consists not so much in discovering a new set of scaling numbers, as in developing a new way to do physics. Traditionally we use regular as zeroth-order approximations to physical systems, and account for weak nonlinearities perturbatively. We think of a dynamical system as a smooth system whose evolution we can follow by integrating a set of differential equations. The new insight is that the zeroth-order approximations to strongly nonlinear systems should be quite different. They show an amazingly rich structure which is not at all apparent in their formulation in terms of differential equations. However, these systems do show self-similar structures which can be encoded by universality equations of a type which we will develop here.

As there is already much good literature on this subject, there is no point in reproducing it here. Instead, we provide the references which cover the same ground. However, it must be realized that the essence of this subject is incommunicable in print; here the intuition is developed by computing.

An introduction to the subject is provided by either of the two general reprint collections: one edited by Cvitanović,[1] and the other by Hao Bai-Lin.[2] Even though the selections overlap in only a few articles, both collections