Appendix C

Finding cycles

(C. Chandre)

C.1 Newton-Raphson method

C.1.1 Contraction rate

Consider a d-dimensional map \( x' = f(x) \) with an unstable fixed point \( x_\ast \).

The Newton-Raphson algorithm is obtained by iterating the following map

\[
x' = g(x) = x - (J(x) - 1)^{-1}(f(x) - x),
\]

The linearization of \( g \) near \( x_\ast \) leads to

\[
x_\ast + \epsilon' = x_\ast + \epsilon - (J(x_\ast) - 1)^{-1}(f(x_\ast) + J(x_\ast)\epsilon - x_\ast - \epsilon) + O(\|\epsilon\|^2),
\]

where \( \epsilon = x - x_\ast \). Therefore,

\[
x' - x_\ast = O(\|x - x_\ast\|^2).
\]

After \( n \) steps and if the initial guess \( x_0 \) is close to \( x_\ast \), the error decreases super-exponentially

\[
g^n(x_0) - x_\ast = O(\|x_0 - x_\ast\|^{2^n}).
\]

C.1.2 Computation of the inverse

The Newton-Raphson method for finding \( n \)-cycles of \( d \)-dimensional mappings using the multi-shooting method reduces to the following equation

\[
\begin{pmatrix}
1 & -Df(x_1) & \cdots & -Df(x_{n-1}) \\
-Df(x_1) & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-Df(x_{n-1}) & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\delta_1 \\
\delta_2 \\
\vdots \\
\delta_n
\end{pmatrix}
= \begin{pmatrix}
F_1 \\
F_2 \\
\vdots \\
F_n
\end{pmatrix},
\]

where \( Df(x) \) is the \([d \times d]\) Jacobian matrix of the map evaluated at the point \( x \), and \( \delta_m = x_m - x_\ast \) and \( F_m = x_m - f(x_{m-1}) \) are \( d \)-dimensional vectors. By some straightforward algebra, the vectors \( \delta_m \) are expressed as functions of the vectors \( F_m \):

\[
\delta_m = -\sum_{k=1}^{m} \beta_{k,m-1} F_k - \beta_{1,m-1} (1 - \beta_{1,m})^{-1} \sum_{k=1}^{m} \beta_{k,m} F_k.
\]

for \( m = 1, \ldots, n \), where \( \beta_{k,m} = D f(x_{m-1}) D f(x_{m-2}) \cdots D f(x_k) \) for \( k < m \) and \( \beta_{k,m} = 1 \) for \( k \geq m \). Therefore, finding \( n \)-cycles by a Newton-Raphson method with multiple shooting requires the inverting of a \([d \times d]\) matrix \( 1 - D f(x_n) D f(x_{n-1}) \cdots D f(x_1) \).

C.2 Hybrid Newton-Raphson / relaxation method

Consider a d-dimensional map \( x' = f(x) \) with an unstable fixed point \( x_\ast \). The transformed map is the following one:

\[
x' = g(x) = x + \gamma C(f(x) - x),
\]

where \( \gamma > 0 \) and \( C \) is a \( d \times d \) invertible constant matrix. We note that \( x_\ast \) is also a fixed point of \( g \). Consider the stability matrix at the fixed point \( x_\ast \)

\[
A_f = \left. \frac{dg}{dx} \right|_{x=x_\ast} = 1 + \gamma C(A_f - 1).
\]

The matrix \( C \) is constructed such that the eigenvalues of \( A_f \) are of modulus less than one. Assume that \( A_f \) is diagonalizable: In the basis of diagonalization, the matrix writes:

\[
\tilde{A}_f = 1 + \gamma C(\tilde{A}_f - 1),
\]
where $A_f$ is diagonal with elements $\mu_i$. We restrict the set of matrices $C$ to diagonal matrices with $C_0 = \epsilon_i$, where $\epsilon_i = \pm 1$. Thus $A_f$ is diagonal with eigenvalues $\gamma_i = 1 + \gamma \epsilon_i (\mu_i - 1)$. The choice of $\gamma$ and $\epsilon_i$ is such that $|\gamma_i| < 1$. It is easy to see that if $\Re(\mu_i) < 1$ one has to choose $\epsilon_i = 1$, and if $\Re(\mu_i) > 1$, $\epsilon_i = -1$. If $\lambda$ is chosen such that

$$0 < \gamma < \min_{i=1,...,d} \frac{2|\Re(\mu_i) - 1|}{|\mu_i - 1|^2},$$

all the eigenvalues of $A_f$ have modulus less than one. The contraction rate at the fixed point for the map $g$ is then $\max_i |1 + \gamma \epsilon_i (\mu_i - 1)|$. If $\Re(\mu_i) = 1$, it is not possible to stabilize $x_0$ by the set of matrices $\gamma C$.

From the construction of $C$, we see that $2^d$ choices of matrices are possible. For example, for 2-dimensional systems, these matrices are

$$C \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

For 2-dimensional dissipative maps, the eigenvalues satisfy $\Re(\mu_1) \Re(\mu_2) \leq \det Df < 1$. The case $(\Re(\mu_1) > 1, \Re(\mu_2) > 1)$ which is stabilized by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has to be discarded. The minimal set is reduced to three matrices.

### C.2.1 Newton method with optimal section

(F. Christiansen)

In some systems it might be hard to find a good starting guess for a fixed point. This can happen, for example, if the topology and/or the symbolic dynamics of the flow is not well understood. By changing the Poincaré section one might get a better initial guess in the sense that $x$ and $f(x)$ are closer together. We illustrate this in figure C.1. The figure shows a Poincaré section, $y = 0$, an initial guess $x$, the corresponding $f(x)$ and pieces of the trajectory near these two points.

If Newton iteration does not converge for the initial guess $x$ we might have to work very hard to find a better guess, particularly if this is in a high-dimensional system (high-dimensional in this context might mean a Hamiltonian system with 3 or more degrees of freedom). Clearly, we could easily obtain a much better guess by simply shifting the Poincaré section to $y = 0.7$ where the distance $x - f(x)$ would be much smaller. Naturally, one cannot so easily determine by inspection the best section for a higher dimensional system. Rather, the way to proceed is as follows: We want to have a minimal distance between our initial guess $x$ and its image $f(x)$. We therefore integrate the flow looking for a minimum in the distance $d(t) = \|f(t) - x\|$. $d(t)$ is now a minimum with respect to variations in $f'(x)$, but not necessarily with respect to $x$. We therefore integrate $x$ either forward or backward in time. Doing this minimizes $d$ with respect to $x$, but now it is no longer minimal with respect to $f'(x)$. We therefore repeat the steps, alternating between correcting $x$ and $f'(x)$. In most cases this process converges quite rapidly. The result is a trajectory for which the vector $(f(x) - x)$ connecting the two end points is perpendicular to the flow at both points. We can now define a Poincaré section as the hyper-plane that goes through $x$, $(x' - x) \cdot v(x) = 0$.

The image $f(x)$ lies in the section. This section is optimal in the sense that a close return on the section is a local minimum of the distance between $x$ and $f'(x)$. More important, the part of the stability matrix that describes linearization perpendicular to the flow is exactly the stability of the flow in the section when $f(x)$ is close to $x$. With this method, the Poincaré section changes with each Newton iteration. Should we later want to put the fixed point on a specific Poincaré section, it will only be a matter of moving along the trajectory.