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Contributors

No man but a blockhead ever wrote except for money
—Samuel Johnson

This book is a result of collaborative labors of many people over a span of several decades. Coauthors of a chapter or a section are indicated in the byline to the chapter/section title. If you are referring to a specific coauthored section rather than the entire book, cite it as (for example):


Do not cite chapters by their numbers, as those change from version to version. Chapters without a byline are written by Predrag Cvitanović. Friends whose contributions and ideas were invaluable to us but have not contributed written text to this book, are credited in the acknowledgments.

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Bo Li
Solutions 31.2, 31.1, 32.1
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I feel I never want to write another book. What's the good! I can eke living on stories and little articles, that don't cost a tithe of the output a book costs. Why write novels anymore!
—D.H. Lawrence

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F. Haake’s heartfelt lament on page 377 was uttered at the end of the first conference presentation of cycle expansions, in 1988. G.P. Morriss advice to students as how to read the introduction to this book, page 6, was offered during a 2002 graduate course in Dresden. K. Huang’s C.N. Yang interview quoted on page 337 is available on ChaosBook.org/extras. T.D. Lee remarks on as to who is to blame, page 37 and page 269, as well as M. Shub’s helpful technical remark on page 476 came during the Rockefeller University December 2004 “Feigenbaum Fest.” Quotes on pages 37, 127, and 334 are taken from a book review by J. Guckenheimer [1.1].

Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable α to Ω sets, P. Cvitanovic asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle, The answer was transmitted back: “The master says: ‘It is holomorphic in a strip.’” Hence His Master’s Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master’s voice. As a matter of fact, the answer is the book. We are still chewing on it.

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