**Chapter 27**

**Irrationally winding**

I don’t care for islands, especially very small ones.
—D.H. Lawrence

(R. Artuso and P. Cvitanović)

This chapter is concerned with the mode locking problems for circle maps: besides its physical relevance it nicely illustrates the use of cycle expansions away from the dynamical setting, in the realm of renormalization theory at the transition to chaos.

The physical significance of circle maps is connected with their ability to model the two-frequencies mode-locking route to chaos for dissipative systems. In the context of dissipative dynamical systems one of the most common and experimentally well explored routes to chaos is the two-frequency mode-locking route. Interaction of pairs of frequencies is of deep theoretical interest due to the generality of this phenomenon; as the energy input into a dissipative dynamical system (for example, a Couette flow) is increased, typically first one and then two of intrinsic modes of the system are excited. After two Hopf bifurcations (a fixed point with inward spiralling stability has become unstable and outward spirals to a limit cycle) a system lives on a two-torus. Such systems tend to mode-lock: the system adjusts its internal frequencies slightly so that they fall in step and minimize the internal dissipation. In such case the ratio of the two frequencies is a rational number. An irrational frequency ratio corresponds to a quasiperiodic motion - a curve that never quite repeats itself. If the mode-locked states overlap, chaos sets in. The likelihood that a mode-locking occurs depends on the strength of the coupling of the two frequencies.

Our main concern in this chapter is to illustrate the “global” theory of circle maps, connected with universality properties of the whole irrational winding set. We shall see that critical global properties may be expressed via cycle expansions involving “local” renormalization critical exponents. The renormalization theory of critical circle maps demands rather tedious numerical computations, and our intuition is much facilitated by approximating circle maps by number-theoretic models. The models that arise in this way are by no means mathematically trivial, they turn out to be related to number-theoretic abysses such as the Riemann conjecture, already in the context of the “trivial” models.

### 27.1 Mode locking

The simplest way of modeling a nonlinearly perturbed rotation on a circle is by 1-dimensional circle maps \( x \rightarrow x' = f(x) \), restricted to the one dimensional torus, such as the sine map

\[
x_{n+1} = f(x_n) = x_n + \Omega - \frac{k}{2\pi} \sin(2\pi x_n) \mod 1.
\]

(27.1)

\( f(x) \) is assumed to be continuous, have a continuous first derivative, and a continuous second derivative at the inflection point (where the second derivative vanishes). For the generic, physically relevant case (the only one considered here) the inflection is cubic. Here \( k \) parametrizes the strength of the nonlinear interaction, and \( \Omega \) is the bare frequency.

The state space of this map, the unit interval, can be thought of as the elementary cell of the map

\[
\hat{x}_{n+1} = \hat{f}(\hat{x}_n) = \hat{x}_n + \Omega - \frac{k}{2\pi} \sin(2\pi \hat{x}_n).
\]

where \( \hat{\cdot} \) is used in the same sense as in chapter 25.

The winding number is defined as

\[
W(k, \Omega) = \lim_{n \to \infty} (\hat{x}_n - \hat{x}_0)/n.
\]

(27.3)

and can be shown to be independent of the initial value \( \hat{x}_0 \).
For $k = 0$, the map is a simple rotation (the shift map) see figure 27.1

$$x_{n+1} = x_n + \Omega \mod 1,$$

and the rotation number is given by the parameter $\Omega$.

$$W(k = 0, \Omega) = \Omega.$$ 

For given values of $\Omega$ and $k$ the winding number can be either rational or irrational. For invertible maps and rational winding numbers $W = P/Q$ the asymptotic iterates of the map converge to a unique attractor, a stable periodic orbit of period $Q$

$$f^Q(x_i) = x_i + P, \quad i = 0, 1, 2, \ldots, Q - 1.$$ 

This is a consequence of the independence of $x_0$ previously mentioned. There is also an unstable cycle, repelling the trajectory. For any rational winding number, there is a finite interval of values of the winding number for which the iterates of the circle map are attracted to the $P/Q$ cycle. This interval is called the $P/Q$ mode-locked cycle. Parametrizing mode lockings by the exponent $\mu$ rather than the width $\Delta$ will be convenient for description of the distribution of the mode-locking widths, as the exponents $\mu$ turn out to be of bounded variation. The stability of the $P/Q$ cycle is

$$\Delta_{P/Q} = Q^{\Omega_{P/Q}^\text{right}} - \Omega_{P/Q}^\text{left},$$

where $\Omega_{P/Q}^\text{right}$ ($\Omega_{P/Q}^\text{left}$) denote the biggest (smallest) value of $\Omega$ for which $W(k, \Omega) = P/Q$. Parametrizing mode lockings by the exponent $\mu$ rather than the width $\Delta$ will be convenient for description of the distribution of the mode-locking widths, as the exponents $\mu$ turn out to be of bounded variation. The stability of the $P/Q$ cycle is

$$\Delta_{P/Q} = \frac{\partial W}{\partial x_0} = f'(x_0)f'(x_1)\cdots f'(x_{Q-1})$$

For a stable cycle $[\Delta_{P/Q}]$ lies between 0 (the superstable value, the “center” of the stability interval) and 1 (the $\Omega_{P/Q}^\text{right}$ endpoints of (27.5)). For the shift map (27.4), the stability intervals are shrunk to points. As $\Omega$ is varied from 0 to 1, the iterates of a circle map either mode-lock, with the winding number given by a rational number $P/Q \in (0, 1)$, or do not mode-lock, in which case the winding number is irrational. A plot of the winding number $W$ as a function of the shift parameter $\Omega$ is a convenient visualization of the mode-locking structure of circle maps. It yields a monotonic “devil’s staircase” of figure 27.2 whose self-similar structure we are to unravel. Circle maps with zero slope at the inflection point $x_c$ (see figure 27.3)

$$f'(x_c) = 0, \quad f''(x_c) = 0.$$
CHAPTER 27. IRRATIONALLY WINDING

A Farey series is characterized by the property that if \( \frac{P_i}{Q_i} \) and \( \frac{P_i'}{Q_i'} \) are consecutive terms of \( F_Q \), then

\[
P_i Q_i' - P_i' Q_i = 1.
\]

The number of terms in the Farey series \( F_Q \) is given by

\[
\Phi(Q) = \sum_{n=1}^{Q} \phi(Q) = \frac{3Q^2}{\pi^2} + O(Q \ln Q).
\]

(27.6)

Here the Euler function \( \phi(Q) \) is the number of integers not exceeding and relatively prime to \( Q \). For example, \( \phi(1) = 1, \phi(2) = 1, \phi(3) = 2, \ldots, \phi(12) = 4, \phi(13) = 12, \ldots \).

From a number-theorist’s point of view, the **continued fraction partitioning** of the unit interval is the most venerable organization of rationals, preferred already by Gauss. The continued fraction partitioning is obtained by ordering rationals corresponding to continued fractions of increasing length. If we turn this ordering into a way of covering the complementary set to mode-lockings in a circle map, then the first level is obtained by deleting \( \Delta_{[1,1]}, \Delta_{[1,2]}, \ldots, \Delta_{[a_0,1]} \) mode-lockings; their complement are the **covering** intervals \( \ell_1, \ell_2, \ldots, \ell_{a_0}, \) which contain all windings, rational and irrational, whose continued fraction expansion starts with \( [a_1, \ldots] \) and is of length at least 2. The second level is obtained by deleting \( \Delta_{[1,2]}, \Delta_{[1,3]}, \ldots, \Delta_{[2,2]}, \Delta_{[2,3]}, \ldots, \Delta_{[n,a]}, \ldots \) and so on.

The 1st level continued fraction partition \( S_1 = \{a_1a_2 \cdots a_k\} \) is defined as the monotonically increasing sequence of all rationals \( \frac{P_i}{Q_i} \) between 0 and 1 whose continued fraction expansion is of length \( n \):

\[
\frac{P_i}{Q_i} = [a_1, a_2, \ldots, a_k] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_k}}}, \quad a_i \geq 2, \quad a_k \geq 2, \quad \text{add up to } \sum_{i=1}^{k} a_i = n + 2.
\]

The object of interest, the set of the irrational winding numbers, is in this partitioning labeled by \( S_n = \{a_1a_2a_3 \cdots \}, \quad a_i \in \mathbb{Z}^+, \) i.e., the set of winding numbers with infinite continued fraction expansions. The continued fraction labeling is particularly appealing in the present context because of the close connection of the Gauss shift to the renormalization transformation \( R \), discussed below. The Gauss map

\[
T(x) = \begin{cases} 
\frac{1}{x} & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

(27.7)

\( [\cdots] \) denotes the integer part.) acts as a shift on the continued fraction representation of numbers on the unit interval

\[
x = [a_1, a_2, a_3, \ldots] \rightarrow T(x) = [a_2, a_3, \ldots].
\]

(27.8)

into the “mother” interval \( \ell_{a_1a_2 \cdots} \).

However natural the continued fractions partitioning might seem to a number theorist, it is problematic in practice, as it requires measuring infinity of mode-lockings even at the first step of the partitioning. Thus numerical and experimental use of continued fraction partitioning requires at least some understanding of the asymptotics of mode-lockings with large continued fraction entries.

The **Farey tree partitioning** is a systematic bisection of rationals: it is based of those continued fractions whose entries \( a_k \geq 2 \). Each rational in \( \ell_n \) and \( \ell_{1/1} \), and then recursively bisecting intervals by means of Farey medians.

We define the \( n \)th Farey tree level \( T_n \) as the monotonically increasing sequence of those continued fractions \( [a_1, a_2, \ldots, a_k] \) whose entries \( a_i \geq 1, \quad i = 1, 2, \ldots, k \).

\[
T_2 = \left\{[4], [2, 2], [1, 1, 2], [1, 3]\right\} = \left\{\frac{1}{4}, \frac{3}{5}, \frac{3}{4} \right\}.
\]

(27.9)

The number of terms in \( T_n \) is \( 2^n \). Each rational in \( T_{n+1} \) has two “daughters” in \( T_n \), given by

\[
\left[\cdots, a - 1, 2, \cdots, a + 1\right]
\]

\[
\left[\cdots, a - 1, 2, \cdots, a\right]
\]
CHAPTER 27. IRRATIONALLY WINDING

27.2 Local theory: “Golden mean” renormalization

The way to pinpoint a point on the border of order is to recursively adjust the parameters so that at the recurrence times \( t = n_1, n_2, n_3, \ldots \) the trajectory passes through a region of contraction sufficiently strong to compensate for the accumulated expansion of the preceding \( n_i \) steps, but not so strong as to force the trajectory into a stable attracting orbit. The renormalization operation \( R \) implements this procedure by recursively magnifying the neighborhood of a point on the border in the dynamical space (by rescaling by a factor \( \alpha \)), in the parameter space (by shifting the parameter origin onto the border and rescaling by a factor \( \delta \)), and by replacing the initial map \( f \) by the \( n \)th iterate \( f^n \) restricted to the magnified neighborhood

\[
f_p(x) \to Rf_p(x) = 
\]

There are by now many examples of such renormalizations in which the new function, framed in a smaller box, is a rescaling of the original function, i.e., the fixpoint function of the renormalization operator \( R \). The best known is the period doubling renormalization, with the recurrence times \( n_i = 2^i \). The simplest circle map example is the golden mean renormalization, with recurrence times \( n_i = F_i \) given by the Fibonacci numbers (27.10). Intuitively, in this context a metric self-similarity arises because iterates of critical maps are themselves critical, i.e., they also have cubic inflection points with vanishing derivatives.

The renormalization operator appropriate to circle maps acts as a generalization of the Gauss shift (27.38); it maps a circle map (represented as a pair of functions \( (g, f) \), of winding number \( [a, b, c, \ldots] \) into a rescaled map of winding number \( [b, c, \ldots] \):

\[
R \left( \frac{\alpha}{f} \right) = \left( \frac{\alpha \rho^{-1} \circ f \circ \alpha^{-1}}{\rho \rho^{-1} \circ g \circ \alpha^{-1}} \right). \tag{27.12}
\]

Acting on a map with winding number \( [a, a, a, \ldots] \), \( R_a \) returns a map with the same winding number \( [a, a, a, \ldots] \), so the fixed point of \( R_a \) has a quadratic irrational winding number \( W = [a, a, a, \ldots] \). This fixed point has a single expanding eigenvalue \( \delta_\alpha \)

Similarly, the renormalization transformation \( R_{a_1} \cdots R_{a_n} R_{n_1} \equiv R_{a_1 a_2 a_3 \cdots a_n} \) has a fixed point of winding number \( W_p = [a_1, a_2, a_3, \ldots, a_p, a_1, a_2, \ldots] \), with a single expanding eigenvalue \( \delta_p \).

For short repeating blocks, \( \delta \) can be estimated numerically by comparing successive continued fraction approximants to \( \sqrt{2} \). The simple example of (27.13) is the sequence of Fibonacci number continued fraction approximants to the golden mean winding number \( W = [1, 1, 1, \ldots] = \sqrt{5} - 1 \).

What this means is that the “devil’s staircase” of figure 27.2 is self-similar under magnification by factor \( \delta_p \) around any quadratic irrational \( W_p \).

The fundamental result of the renormalization theory (and the reason why all this is so interesting) is that the ratios of successive \( P_r / Q_r \) mode-locked intervals converge to universal limits. The simplest example of (27.13) is the sequence of Fibonacci number continued fraction approximants to the golden mean winding number \( W = [1, 1, 1, \ldots] = \sqrt{5} - 1 \).
When global problems are considered, it is useful to have at least and idea on external scaling laws for mode-lockings. This is achieved, in a first analysis, by fixing the cycle length $Q$ and describing the range of possible asymptotics.

For a given cycle length $Q$, it is found that the narrowest interval shrinks with a power law

$$\Delta_{1/Q} \propto Q^{-3}$$  \hspace{1cm} (27.14)

For fixed $Q$ the widest interval is bounded by $P/Q = F_{n-1}/F_n$, the $n$th continued fraction approximant to the golden mean. The intuitive reason is that the golden mean winding sits as far as possible from any short cycle mode-locking.

The golden mean interval shrinks with a universal exponent

$$\Delta_{P/Q} \propto Q^{-2\mu_1}$$  \hspace{1cm} (27.15)

where $P = F_{n-1}$, $Q = F_n$ and $\mu_1$ is related to the universal Shenker number $\delta_1$ (27.13) and the golden mean (27.11) by

$$\mu_1 = \frac{\ln |\delta_1|}{2\ln \rho} = 1.08218\ldots$$  \hspace{1cm} (27.16)

The closeness of $\mu_1$ to 1 indicates that the golden mean approximant mode-lockings barely feel the fact that the map is critical (in the $k=0$ limit this exponent is $\mu = 1$).

To summarize: for critical maps the spectrum of exponents arising from the circle maps renormalization theory is bounded from above by the harmonic scaling, and from below by the geometric golden-mean scaling:

$$3/2 > \mu_{\text{geo}} \geq 1.08218\ldots$$  \hspace{1cm} (27.17)

### 27.3 Global theory: Thermodynamic averaging

Consider the following average over mode-locking intervals (27.5):

$$\Omega(\tau) = \sum_{Q=1}^{\infty} \sum_{P|Q} \Delta_{P/Q}^{T^\tau}.$$  \hspace{1cm} (27.18)

The sum is over all irreducible rationals $P/Q$, $P < Q$, and $\Delta_{P/Q}$ is the width of the parameter interval for which the iterates of a critical circle map lock onto a cycle of length $Q$, with winding number $P/Q$.

The qualitative behavior of (27.18) is easy to pin down. For sufficiently negative $\tau$, the sum is convergent; in particular, for $\tau = -1$, $\Omega(-1) = 1$, as for the critical circle maps the mode-lockings fill the entire $\Omega$ range [27.15]. However, as $\tau$ increases, the contributions of the narrow (large $Q$) mode-locked intervals $\Delta_{P/Q}$ get blown up to $1/\Delta^{T^\tau}_{P/Q}$, and at some critical value of $\tau$ the sum diverges. This occurs for $\tau < 0$, as $\Omega(0)$ equals the number of all rationals and is clearly divergent.

The sum (27.18) is infinite, but in practice the experimental or numerical mode-locked intervals are available only for small finite $Q$. Hence it is necessary to split up the sum into subsets $S_n = (i)$ of rational winding numbers $P/Q$, on the “level” $n$, and present the set of mode-lockings hierarchically, with resolution increasing with the level:

$$Z_n(\tau) = \sum_{i \in S_n} \Delta_i^{T^\tau}.$$  \hspace{1cm} (27.19)

The original sum (27.18) can now be recovered as the $z = 1$ value of a “generating” function $\Omega(z, \tau) = \sum_i e^{\tau \Omega_i(\tau)}$. As $z$ is anyway a formal parameter, and $n$ is a rather arbitrary “level” in some ad hoc partitioning of rational numbers, we bravely introduce a still more general, $P/Q$ weighted generating function for (27.18):

$$\Omega(q, \tau) = \sum_{Q=1}^{\infty} \sum_{P|Q} e^{-\tau P/Q} Q^{2\Phi(q/P)}.$$  \hspace{1cm} (27.20)

The sum (27.18) corresponds to $q = 0$. Exponents $\nu_{P/Q}$ will reflect the importance we assign to the $P/Q$ mode-locking, i.e., the measure used in the averaging over all mode-lockings. Three choices of of the $\nu_{P/Q}$ hierarchy that we consider here correspond respectively to the Farey series partitioning

$$\Omega(q, \tau) = \sum_{Q=1}^{\infty} \Phi(q) \sum_{(P|Q)} Q^{2\Phi(q/P)}.$$  \hspace{1cm} (27.21)

the continued fraction partitioning

$$\Omega(q, \tau) = \sum_{m=1}^{\infty} e^{-\tau m} \sum_{[a_1,\ldots,a_n]} Q^{2\Phi(q/a_n)},$$  \hspace{1cm} (27.22)

and the Farey tree partitioning

$$\Omega(q, \tau) = \sum_{d=1}^{m} 2^{-\tau d} \sum_{j=1}^{2^d} Q_j^{2\Phi(q/Q_j)}.$$  \hspace{1cm} (27.23)

We remark that we are investigating a set arising in the analysis of the parameter space of a dynamical system: there is no “natural measure” dictated by dynamics, and the choice of weights reflects only the choice of hierarchical presentation.
27.4 Hausdorff dimension of irrational windings

A finite cover of the set irrational windings at the “nth level of resolution” is obtained by deleting the parameter values corresponding to the mode-lockings in the subset \( S_n \); left behind is the set of complement covering intervals of widths

\[
\ell_i = \Omega^\text{min}_P/Q - \Omega^\text{max}_P/Q, \tag{27.24}
\]

Here \( \Omega^\text{min}_P/Q, \Omega^\text{max}_P/Q \) are respectively the lower (upper) edges of the mode-locking intervals \( \Lambda^i_P/Q, (\Lambda^i_P/Q) \) bounding \( \ell_i \) and \( i \) is a symbolic dynamics label, for example the entries of the continued fraction representation \( P/Q \) of \( i \) of one of the boundary mode-lockings, \( i = a_1a_2\cdots a_n \). \( \ell_i \) provide a finite cover for the irrational winding set, so one may consider the sum

\[
Z_n(\tau) = \sum_{\kappa \in S_n} \ell_i^{-\tau} \tag{27.25}
\]

The value of \( -\tau \) for which the \( n \to \infty \) limit of the sum (27.25) is finite is the Hausdorff dimension \( D_H \) of the irrational winding set. Strictly speaking, this is the Hausdorff dimension only if the choice of covering intervals \( \ell_i \) is optimal; otherwise it provides an upper bound to \( D_H \). As by construction the \( \ell_i \) intervals cover the set of irrational winding with no slack, we expect that this limit yields the Hausdorff dimension. This is supported by all numerical evidence, but a proof that would satisfy mathematicians is lacking.

The physically relevant statement is that for critical circle maps \( D_H = 0.870\ldots \) is a (global) universal number.

exercise 27.2

27.4.1 The Hausdorff dimension in terms of cycles

Estimating the \( n \to \infty \) limit of (27.25) from finite numbers of covering intervals \( \ell_i \) is a rather unilluminating chore. Fortunately, there exist considerably more elegant ways of extracting \( D_H \). We have noted that in the case of the “trivial” mode-locking problem (27.4), the covering intervals are generated by iterations of the Farey map (27.37) or the Gauss shift (27.38). The nth level sum (27.25) can be approximated by \( L_n \), where

\[
L_n(x, y) = \delta(x - f^{-1}(y))[f'(y)]^\tau
\]

This amounts to approximating each cover width \( \ell_i \) by \( |df^n|/dx \) evaluated on the \( i \)th interval. We are thus led to the following determinant

\[
\det(1 - z L_i) = \exp\left(-\sum_{p} \sum_{r=1}^{\infty} z^{p r} |\Lambda^i_r|^\tau \right)
\]

The sum (27.25) is dominated by the leading eigenvalue of \( L_n \); the Hausdorff dimension condition \( Z_\infty(-D_H) = O(1) \) means that \( \tau = -D_H \) should be such that the leading eigenvalue is \( z = 1 \). The leading eigenvalue is determined by the \( k = 0 \) part of (27.26); putting all these pieces together, we obtain a pretty formula relating the Hausdorff dimension to the prime cycles of the map \( f(x) \):

\[
0 = \prod_p \left( 1 - 1/|\Lambda^i_p|^{D_H} \right). \tag{27.27}
\]

For the Gauss shift (27.38) the stabilities of periodic cycles are available analytically, as roots of quadratic equations: For example, the \( x_n \) fixed points (quadratic irrationals with \( x_n = [a,a,a\ldots] \) infinitely repeating continued fraction expansion) are given by

\[
x_n = -a + \frac{\sqrt{a^2 + 4}}{2}, \quad \Lambda_a = -\left(\frac{a + \sqrt{a^2 + 4}}{2}\right)^2 \tag{27.28}
\]

and the \( x_{ab} = [a,b,a,b,a,b,\ldots] \) 2–cycles are given by

\[
x_{ab} = -ab + \frac{\sqrt{(ab)^2 + 4ab}}{2b}, \quad \Lambda_{ab} = \left(ab, a, b\right)^2 = \left(\frac{ab + 2 + \sqrt{ab(ab + 4)}}{2}\right)^2 \tag{27.29}
\]

We happen to know beforehand that \( D_H = 1 \) (the irrationals take the full measure on the unit interval, or, from another point of view, the Gauss map is not a repeller), so is the infinite product (27.27) merely a very convoluted way to compute the number 1? Possibly so, but once the meaning of (27.27) has been grasped, the corresponding formula for the critical circle maps follows immediately:

\[
0 = \prod_p \left( 1 - 1/|\delta^i_p|^{D_H} \right). \tag{27.30}
\]

The importance of this formula relies on the fact that it expresses \( D_H \) in terms of universal quantities, thus providing a nice connection from local universal exponents to global scaling quantities: actual computations using (27.30) are rather involved, as they require a heavy computational effort to extract Shenker’s scaling \( \delta^i_p \) for periodic continued fractions, and moreover dealing with an infinite alphabet requires control over tail summation if an accurate estimate is to be sought. In table 27.1 we give a small selection of computed Shenker’s scalings.
27.5 Thermodynamics of Farey tree: Farey model

We end this chapter by giving an example of a number theoretical model motivated by the mode-locking phenomenology. We will consider it by means of the thermodynamic formalism of chapter K, by looking at the free energy.

Consider the Farey tree partition sum (27.23); the narrowest mode-locked interval (27.15) at the rth level of the Farey tree partition sum (27.23) is the golden mean interval

$$\Delta_{r-1,r} \propto |\delta_r|^{-n}. \quad (27.31)$$

It shrinks exponentially, and for $r$ positive and large it dominates $q(r)$ and bounds $dq(r)/dr$:

$$q_{\text{max}} = \frac{\ln |\delta_r|}{\ln 2} = 1.502642 \ldots$$

(27.32)

However, for $r$ large and negative, $q(r)$ is dominated by the interval (27.14) which shrinks only harmonically, and $q(r)$ approaches 0 as

$$\frac{q(r)}{r} \rightarrow \frac{3 \ln n}{n \ln 2} \rightarrow 0. \quad (27.33)$$

So for finite $n$, $q_{\text{max}}(r)$ crosses the $r$ axis at $-r = D_n$, but in the $n \rightarrow \infty$ limit, the $q(r)$ function exhibits a phase transition; $q(r) = 0$ for $r < -D_n$, but is a non-trivial function of $r$ for $-D_n \leq r$. This non-analyticity is rather severe: to get a clearer picture, we illustrate it by a few number-theoretic models (the critical circle maps case is qualitatively the same).

An approximation to the “trivial” Farey level thermodynamics is given by the “Farey model,” in which the intervals $\ell_{\ell/Q}$ are replaced by $Q^{-2}$:

$$Z_n(r) = \sum_{\ell=1}^{2^n} Q_{2^n}^r. \quad (27.34)$$

Here $Q_i$ is the denominator of the $i$th Farey rational $P_i/Q$. For example (see figure 27.4),

$$Z_2(1/2) = 4 + 5 + 5 + 4. \quad (27.34)$$

By the annihilation property (27.38) of the Gauss shift on rationals, the $n$th Farey level sum $Z_n(-1)$ can be written as the integral

$$Z_n(-1) = \int dx \delta(f^n(x)) = \sum_{a_{n-1} = 0} 1/[f'_n a_{n-1}(0)],$$

and in general

$$Z_n(r) = \int dx L_r^0(0, x),$$

with the sum restricted to the Farey level $a_1 + \ldots + a_k = n + 2$. It is easily checked that $f'_n a_{n-1}(0) = (-1)^k Q_{n-1}^{a_k-a_{n-1}}$, so the Farey model sum is a partition generated by the Gauss map preimages of $x = 0$, i.e., by rationals, rather than by the quadratic irrationals as in (27.26). The sums are generated by the same transfer operator, so the eigenvalue spectrum should be the same as for the periodic orbit expansion, but in this variant of the finite level sums we can evaluate $q(r)$ exactly for $r = k/2$, $k$ a nonnegative integer. First, one observes that $Z_1(0) = 2^n$. It is easy to check that $Z_n(1/2) = \sum Q_i = 2 \cdot 3^n$. More surprisingly, $Z_n(3/2) = \sum Q_i^2 = 54 \cdot 7^{n-1}$. A few of these “sum rules” are listed in the table 27.2, they are consequence of the fact that the denominators on a given level are Farey sums of denominators on preceding levels.

A bound on $D_H$ can be obtained by approximating (27.34) by

$$Z_n(r) = n^{2r} + 2^n \rho^{2nr}. \quad (27.35)$$

In this approximation we have replaced all $\ell_{\ell/Q}$, except the widest interval $\ell_{1/n}$, by the narrowest interval $\ell_{1/n}$ (see (27.15)). The crossover from the harmonic dominated to the golden mean dominated behavior occurs at the $r$ value for which the two terms in (27.35) contribute equally:

$$D_n = \hat{D} + O\left(\ln n \ln 2\right), \quad \hat{D} = \frac{\ln 2}{2 \ln \rho} = .72 \ldots \quad (27.36)$$

For negative $r$ the sum (27.35) is the lower bound on the sum (27.25), so $\hat{D}$ is a lower bound on $D_H$.

From a general perspective the analysis of circle maps thermodynamics has revealed the fact that physically interesting dynamical systems often exhibit mixtures of hyperbolic and marginal stabilities. In such systems there are orbits that...
CHAPTER 27. IRRATIONALLY WINDING

<table>
<thead>
<tr>
<th>1/2</th>
<th>(2\times(2/3)\times(2/5))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1/3)</td>
</tr>
<tr>
<td>2</td>
<td>((5 + \sqrt{7})/2)</td>
</tr>
<tr>
<td>3</td>
<td>(7)</td>
</tr>
<tr>
<td>4</td>
<td>((5 + \sqrt{7})/2)</td>
</tr>
<tr>
<td>5</td>
<td>(7 + 4\sqrt{5})</td>
</tr>
<tr>
<td>6</td>
<td>(26.20249\ldots)</td>
</tr>
</tbody>
</table>

Table 27.2: Partition function sum rules for the Farey model.

The above maps look innocent enough, but note that what is being partitioned is not the dynamical space, but the parameter space. The flow described by (27.37) and by its non-trivial circle-map generalizations will turn out to be a renormalization group flow in the function space of dynamical systems, not an ordinary flow in the state space of a particular dynamical system.

The Farey tree has a variety of interesting symmetries (such as “flipping heads and tails” relations obtained by reversing the order of the continued-fraction entries) with as yet unexploited implications for the renormalization theory: some of these are discussed in ref. [27.13].

An alternative labeling of Farey denominators has been introduced by Knauf [27.7] in context of number-theoretical modeling of ferromagnetic spin chains: it allows for a number of elegant manipulations in thermodynamic averages connected to the Farey tree hierarchy.

stay ’glued’ arbitrarily close to stable regions for arbitrarily long times. This is a generic phenomenon for Hamiltonian systems, where elliptic islands of stability coexist with hyperbolic homoclinic tangles. Thus the considerations of chapter 24 are important also in the analysis of renormalization at the onset of chaos.

Résumé

The mode locking problem, and the quasiperiodic transition to chaos offer an opportunity to use cycle expansions on hierarchical structures in parameter space: this is not just an application of the conventional thermodynamic formalism, but offers a clue on how to extend universality theory from local scalings to global quantities.

Commentary

Remark 27.1 The physics of circle maps. Mode–locking phenomenology is reviewed in ref. [27.6], a more theoretically oriented discussion is contained in ref. [27.3]. While representative of dissipative systems we may also consider circle maps as a crude approximation to Hamiltonian local dynamics: a typical island of stability in a Hamiltonian 2-dimensional map is an infinite sequence of concentric KAM tori and chaotic regions. In the crudest approximation, the radius can here be treated as an external parameter \( \Omega \), and the angular motion can be modeled by a map periodic in the angular variable [27.9, 27.12]. By losing all of the ‘island-within-island’ structure of real systems, circle map models skirt the problems of determining the symbolic dynamics for a realistic Hamiltonian system, but they do retain some of the essential features of such systems, such as the golden mean renormalization [13.5, 27.9] and non-hyperbolicity in form of sequences of cycles accumulating toward the borders of stability. In particular, in such systems there are orbits that stay “glued” arbitrarily close to stable regions for arbitrarily long times. As this is a generic phenomenon in physically interesting dynamical systems, such as the Hamiltonian systems with coexisting elliptic islands of stability and hyperbolic homoclinic tangles, development of good computational techniques is here of utmost practical importance.

Remark 27.2 Critical mode–locking set The fact that mode-lockings completely fill the unit interval at the critical point has been proposed in refs. [27.3, 27.14]. The proof that the set of irrational windings is of zero Lebesgue measure is given in ref. [27.15].

Remark 27.3 Counting noise for Farey series. The number of rationals in the Farey series of order \( Q \) is \( \phi(Q) \), which is a highly irregular function of \( Q \) incrementing \( Q \) by anything from 2 to \( Q \) terms. We refer to this fact as the “Euler noise.”

The Euler noise poses a serious obstacle for numerical calculations with the Farey series partitions; it blocks smooth extrapolations to \( Q \to \infty \) limits from finite \( Q \) data. While this in practice renders inaccurate most Farey-sequence partitioned averages, the finite \( Q \) Hausdorff dimension estimates exhibit (for reasons that we do not understand) surprising numerical stability, and the Farey series partitioning actually yields the best numerical value of the Hausdorff dimension (27.25) of any methods used so far; for example the computation in ref. [27.16] for critical sine map (27.1), based on \( 240 \leq Q \leq 250 \) Farey series partitions, yields \( D_H = 0.7012 \pm 0.0001 \). The quoted error refers to the variation of \( D_H \) over this range of \( Q \); as the computation is not asymptotic, such numerical stability can underestimate the actual error by a large factor.

Remark 27.4 Farey tree presentation function. The Farey tree rationals can be generated by backward iterates of \( 1/2 \) by the Farey presentation function [27.17]:

\[
\begin{align*}
    f_0(x) &= x/(1 - x) \quad 0 \leq x < 1/2 \\
    f_1(x) &= (1 - x)/x \quad 1/2 < x \leq 1.
\end{align*}
\]

(27.37)

The Gauss shift (27.7) corresponds to replacing the binary Farey presentation function branch \( f_0 \) in (27.37) by an infinity of branches

\[
\begin{align*}
    f_k(x) &= f_0 \circ f_0^{k-1}(x) = \frac{1}{a} - a, \quad \frac{1}{a-1} < x \leq \frac{1}{a} \\
    f_0 \circ f_0 \circ f_0(x) &= f_0 \circ f_0 \circ f_0(x).
\end{align*}
\]

(27.38)

A rational \( x = [a_1, a_2, \ldots, a_k] \) is annihilated by the \( k \)th iterate of the Gauss shift, \( f_0 \circ \cdots \circ f_0(x) = 0 \). The above maps look innocent enough, but note that what is being partitioned is not the dynamical space, but the parameter space. The flow described by (27.37) and by its non-trivial circle-map generalizations will turn out to be a renormalization group flow in the function space of dynamical systems, not an ordinary flow in the state space of a particular dynamical system.

The Farey tree has a variety of interesting symmetries (such as “flipping heads and tails” relations obtained by reversing the order of the continued-fraction entries) with as yet unexploited implications for the renormalization theory: some of these are discussed in ref. [27.13].
Remark 27.5 Circle map renormalization. The idea underlying golden mean renormalization goes back to Shenker [27.12]. A renormalization group procedure was formulated in refs. [27.8, 27.10, 27.11], where moreover the uniqueness of the relevant eigenvalue is claimed. This statement has been confirmed by a computer-assisted proof [27.18], and in the following we will always assume it. There are a number of experimental evidences for local universality, see refs. [27.19, 27.20].

On the other side of the scaling tale, the power law scaling for harmonic fractions (discussed in refs. [27.2, 27.13]) is derived by methods akin to those used in describing intermittency [27.24]: 1/Q cycles accumulate toward the edge of 0/1 mode-locked interval, and as the successive mode-locked intervals 1/Q, 1/(Q − 1) lie on a parabola, their differences are of order Q−½.

Remark 27.6 Farey series and the Riemann hypothesis. The Farey series thermodynamics is of a number theoretical interest, because the Farey series provide uniform coverings of the unit interval with rationals, and because they are closely related to the deepest problems in number theory, such as the Riemann hypothesis [27.25, 27.26]. The distribution of the Farey series rationals across the unit interval is surprisingly uniform—indeed, so uniform that in the pre-computer days it has motivated a compilation of an entire handbook of Farey series [27.27]. A quantitative measure of the non-uniformity of the distribution of Farey rationals is given by displacements of Farey rationals for F/Q ∈ F from uniform spacing:

$$\delta_i = \frac{i}{\Phi(Q)} - \frac{P_i}{Q}, \text{ } i = 1, 2, \ldots, \Phi(Q)$$

The Riemann hypothesis states that the zeros of the Riemann zeta function lie on the line s = 1/2 + ir line in the complex s plane, and would seem to have nothing to do with physicists’ real mode-locking widths that we are interested in here. However, there is a real-line version of the Riemann hypothesis that lies very close to the mode-locking problem. According to the theorem of Franel and Landau [27.28, 27.25, 27.26], the Riemann hypothesis is equivalent to the statement that

$$\sum_{Q \in \mathcal{Q}} |k| = o(Q^{1+\epsilon})$$

for all $\epsilon$ as $Q \to \infty$. The mode-lockings $\Delta_{Q/0}$ contain the necessary information for constructing the partition of the unit interval into the $\ell_i$ covers, and therefore implicitly contain the $\delta_i$ information. The implications of this for the circle-map scaling theory have not been worked out, and is not known whether some conjecture about the thermodynamics of irrational windings is equivalent to (or harder than) the Riemann hypothesis, but the danger lurks.

Remark 27.7 Farey tree partitioning. The Farey tree partitioning was introduced in refs. [27.29, 27.30, 27.13] and its thermodynamics is discussed in detail in refs. [27.16, 27.17]. The Farey tree hierarchy of rationals is rather new, and, as far as we are aware, not previously studied by number theorists. It is appealing both from the experimental and from the golden-mean renormalization point of view, but it has a serious drawback of lumping together mode-locking intervals of wildly different sizes on the same level of the Farey tree.

Remark 27.8 Local and global universality. Numerical evidences for global universal behavior have been presented in ref. [27.3]. The question was reexamined in ref. [27.16], where it was pointed out how a high-precision numerical estimate is in practice very hard to obtain. It is not at all clear whether this is the optimal global quantity to test but at least the Hausdorff dimension has the virtue of being independent of how one partitions mode-lockings and should thus be the same for the variety of thermodynamic averages in literature.

The formula (27.30), linking local to global behavior, was proposed in ref. [27.1]. The derivation of (27.30) relies only on the following aspects of the “hyperbolicity conjecture” of refs. [27.13, 27.21, 27.22, 27.23]:

1. $\ell_i$ grow exponentially with $n_i$, the length of the continued fraction block $p_i$. According to (27.14), $\lim_{n \to \infty} \delta_p = n^\delta$. In the calculation of ref. [27.1] the explicit values of the asymptotic exponents and prefactors were not used, only the assumption that the growth of $\delta_p$ with $n$ is not slower than a power of $n$.

Remark 27.9 Farey model. The Farey model (27.33) has been proposed in ref. [27.16]; though it might seem to have been pulled out of a hat, the Farey model is as sensible description of the distribution of rationals as the periodic orbit expansion (27.26).

Remark 27.10 Symbolic dynamics for Hamiltonian rotational orbits. The rotational codes of ref. [27.40] are closely related to those for maps with a natural angle variable, for example for circle maps [27.37, 27.39] and cat maps [27.41]. Ref. [27.40] also offers a systematic rule for obtaining the symbolic codes of “islands around islands” rotational orbits [27.43]. These correspond, for example, to orbits that rotate around orbits that rotate around the elliptic fixed point; thus they are defined by a sequence of rotation numbers.

A different method for constructing symbolic codes for “islands around islands” was given in refs. [27.46, 27.44]; however in these cases the entire set of orbits in an island was assigned the same sequence and the motivation was to study the transport implications for chaotic orbits outside the islands [27.43, 27.45].

Exercises

27.1 Mode-locked intervals. Check that when $k \neq 0$ the interval $\Delta_{0/k}$ have a non-zero width (look for instance at simple fractions, and consider $k$ small). Show that for small $k$ the width of $\Delta_{0/k}$ is an increasing function of $k$. 

Exercises
for \( s \in (1, 2) \) \((S(s)\) being expressed by a converging sum). You may use either more detailed estimate for \( \zeta(s,a) \) (via Euler summation formula) or keep on sub-
tracting leading contributions [27.34].

27.6. \( \alpha_n \) and \( \alpha_{\infty} \). Look at the integration region and how it scales by plotting it for increasing values of \( n \).

27.7. Estimates of the Riemann zeta function. Try to approximate numerically the Riemann zeta function for \( s = 2, 4, 6 \) using different acceleration algorithms: check your results with refs. [27.35, 27.36].

27.8. Farey tree and continued fractions I. Consider the Farey tree presentation function \( f : [0, 1] \to [0, 1] \), such that if \( I = [0, 1/2] \) and \( J = [1/2, 1] \), \( f(I) = [1 - x] / x \). Show that the corresponding induced map is the Gauss map \( g(x) = 1 / x - 1 / \lfloor x \rfloor \).

27.9. Farey tree and continued fraction II. (Lethal weapon II). Build the simplest piecewise linear approximation to the Farey tree presentation function (hint: substitute first the rightmost, hyperbolic branch with a linear one): consider then the spectral determinant of the induced map \( \hat{g} \), and calculate the first two eigenvalues besides the probability conservation one. Compare the results with the rigorous bound deduced in ref. [24.17].

For further reading, you may consult the References section at the end of the text.


[27.34] P. Dahlqvist, unpublished notes.


