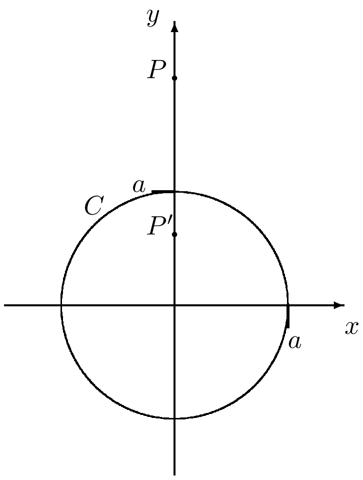
Problem 1



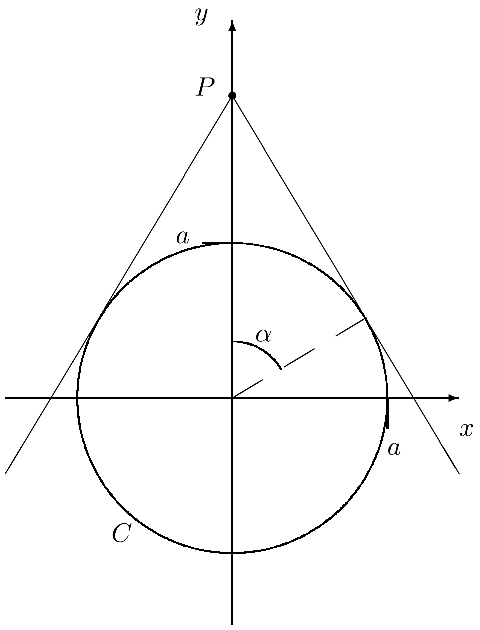
- Define a circumference C of radius a and a point P outside the circumference.
- Find the equations of the tangents to C that pass through P.
- Now consider a point P' inside C, and evaluate the equations found before. (Check what happens when you make P and P' colinear and you go from P to P' in the $(x, \Re\mathrm{e}\,(y)\,, \Im\mathrm{m}\,(y))$ subspace.

Solution

The equations of the tangents to C are given by

$$\begin{aligned} y - p &= \mp\frac{\sqrt{P^2 - a^2}}{a}x, \\ &= \mp\left(\left(\frac{P}{a}\right)^2 - 1\right)^{\frac{1}{2}} x. \\ &= \mp x \tan\alpha \end{aligned}$$

with α being the angle between the P the origin and the intercept between the tangent and the circunference, as shown in the figure.



We must study what happens when P gets closer to a. As P becomes smaller and approaches a, $\left(\frac{P}{a}\right)^2 - 1$ gets closer to 0, this means that y gets closer to the constant value P. Another way to see it, is that α gets closer to 0 making $\tan\alpha = -\tan\alpha$, so the two tangent lines collapse into one.

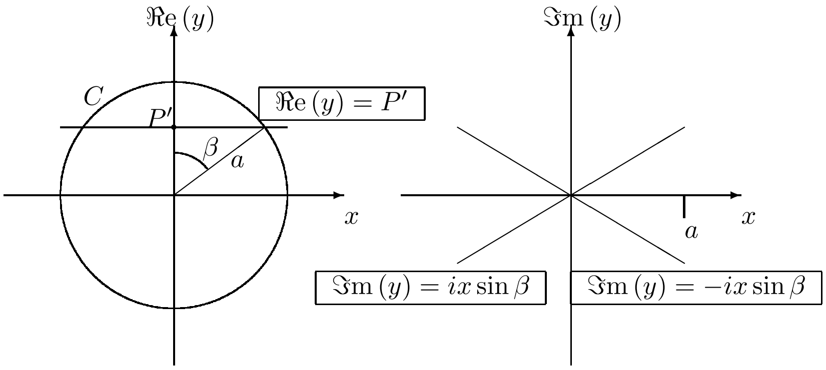
It is then interesting to see what happens when we go one step beyond and study the case when $P' < a$, as the equation for the tangents obtained in the first part of the solution, if applied in this regime, is no longer real,

$$
\begin{aligned}
y - P &= \mp i \left(1 - \left(\frac{P}{a}\right)^2\right)^{\frac{1}{2}} x \\
&= \mp i x \sin\beta
\end{aligned}
$$

We can then study the behavior of y with respect to x assuming that x remains real,

$$
\begin{aligned}
y &= P \mp i x \sin\beta \\
\Re\mathrm{e}\,(y) &= P \pm \Im\mathrm{m}\,(x) \sin\beta \\
\Im\mathrm{m}\,(y) &= \mp \Re\mathrm{e}\,(x) \sin\beta
\end{aligned}
$$

The figure presents the case where x is strictly real, and shows the dependence of y with x.



Notice that the angle β starts-off being equal to 0, when $P' = a$, and its value increases as P' approaches 0 and gets its maximum value of $\pi/2$ when $P' = 0$. This means that the slope of the "tangent" starts-off at 0 in the boundary $P' = a$ and grows as $\sin\beta$ up to 1 when $P' = 0$.

Problem 2

Consider a repulsive central potential of the form $V(r) = \frac{k}{r}$, consider also a particle initially at a distance r_0 from the scattering center.

If the particle is known to have a total energy E, it is posible to find the initial direction it should be pointing to, in order to reach an specified coordinate.

The simplest example would be, if the initial position is given by (r_0, θ_0) and you want to get to (r_f, θ_0), you know you would simply have to align the particle's momentum with the scattering center.

Of course, it is not posible to get to all posible coordinates classically, mainly due to energy restrictions.

- Find an expression to find the correct angle you need to get from an initial position $(r_0, \theta_0 = 0)$ to a final position (r_f, θ_f).
- Find a way to determine what part of the (r, θ) space is classically forbidden for the particle.
- Evaluate the expression found in the first part within the forbiden area, allowing the coordinates to take on complex values.

Solution

In this system, the energy, E, is known to be conserved, since there are no forces that do work. The angular momentum, $\vec{\ell}$, is also conserved because $V(r)$ is central. But there is another quantity conserved for the particular case of a potential of the form $V(r) = \frac{k}{r}$, that is the Runge-Lenz vector.

Let us define the Runge-Lenz vector as

$$\vec{A} \equiv \vec{p} \times \vec{L} + mk\hat{\mathrm{r}}.$$

We need to prove that $\vec{A}$ is a conserved quantity, for that consider

$$\begin{aligned}
\vec{p} &= m\dot{\vec{r}}, \\
\vec{\ell} &= \vec{r} \times \vec{p}, \\
\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} &= F(r)\frac{\vec{r}}{r}
\end{aligned}$$

using these, and the fact that $\vec{\ell}$ is conserved we can write

$$\begin{aligned}
\dot{\vec{p}} \times \vec{\ell} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{p} \times \vec{\ell}\right) &= F(r)\frac{\vec{r}}{r} \times (\vec{r} \times m\dot{\vec{r}}), \\
&= \frac{mF(r)}{r}\left[\vec{r}\,(\vec{r} \cdot \dot{\vec{r}}) - \dot{\vec{r}}(r^2)\right], \\
&= \frac{mF(r)}{r}\left[\vec{r}\left(\vec{r} \cdot (\dot{r}\hat{\mathrm{r}} + r\hat{\theta})\right) - \dot{\vec{r}}(r^2)\right], \\
&= -mF(r)r^2\left[\frac{\dot{\vec{r}}}{r} - \frac{\dot{r}\,\vec{r}}{r^2}\right], \\
&= -mF(r)r^2\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vec{r}}{r}\right)\right],
\end{aligned}$$

but for our particular $V(r)$, $F(r) = \frac{k}{r^2}$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{p} \times \vec{\ell} + mk\hat{\mathrm{r}}\right] = 0 \implies \vec{A} = const.$$

We now proceed to define the eccentricity vector, $\vec{\varepsilon}$, as

$$\vec{\varepsilon} \equiv \frac{\vec{A}}{mk}.$$

To obtain the orbit equation it suffices to calculate $\vec{\varepsilon} \cdot \vec{r}$,

$$\begin{aligned}
\vec{\varepsilon} \cdot \vec{r} &= \frac{1}{mk}\left(\vec{p} \times \vec{\ell}\right) \cdot \vec{r} + r, \\
&= \frac{1}{mk}\left(\vec{r} \times \vec{p}\right) \cdot \vec{\ell} + r, \\
&= \frac{\ell^2}{mk} + r = \varepsilon r \cos\theta.
\end{aligned}$$

then

$$r(\theta) = \frac{\ell^2/mk}{-1 + \varepsilon\cos(\theta - \theta_0)},$$

where θ_0 is just a clock-wise rotation of the whole orbit. It is also simple to calculate ε as,

$$\begin{aligned}
\varepsilon^2 &= \frac{(\vec{p} \times \vec{\ell}) \cdot (\vec{p} \times \vec{\ell})}{(mk)^2} + 1 + 2\frac{(\vec{p} \times \vec{\ell}) \cdot \hat{\mathrm{r}}}{mk}, \\
&= \frac{1}{(mk)^2}\left[\ell^2 p^2 - (\vec{L} \cdot \vec{p})\right] + 1 + \frac{2\ell^2}{mkr}, \\
&= \left(\frac{\ell p}{mk}\right)^2 + \frac{2\ell^2}{mkr} + 1, \\
&= \frac{2\ell^2}{mk^2}\left(\frac{p^2}{2m} + \frac{k}{r}\right) + 1 \\
\varepsilon^2 &= 1 + \frac{2\ell^2 E}{mk^2}.
\end{aligned}$$

Before continuing, we can renormalize the problem by setting $k = m = r_0 = 1$, then the orbit equation can be rewritten in the more convenient way

$$r(\theta) = r^* \frac{(\varepsilon^2 - 1)}{-1 + \varepsilon\cos(\theta - \theta_0)}$$

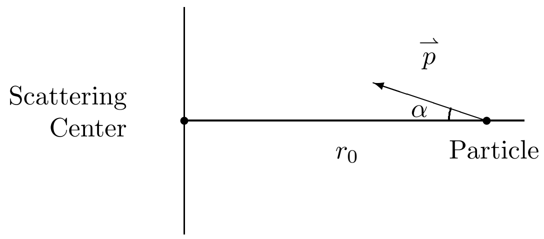
where

$$\begin{aligned}
r^* &= \frac{1}{2E}, \\
\varepsilon^2 &= 1 + 2\ell^2 E, \\
&= 1 + 2E[p_0 \sin^2\alpha], \\
&= 1 + 2E[2(E-1)]\sin^2\alpha \\
&= 1 + 2E(E-1)(1 - \cos 2\alpha).
\end{aligned}$$

to yield

$$r(\theta, \alpha) = \frac{(E-1)(1 - \cos 2\alpha)}{-1 + \varepsilon(\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0)}$$

where α is given by the figure.



From the orbit equation, and the initial condition $r(\theta = 0) = r_0 = 1$, we can write

$$\begin{aligned}
\cos\theta_0 &= \frac{r^*(\varepsilon^2 - 1) + 1}{\varepsilon}, \\
&= \frac{2(E-1)\sin^2\alpha + 1}{\varepsilon} \\
&= \frac{(E-1)(1-\cos 2\alpha) + 1}{\varepsilon} \\
&= \frac{E - (E-1)\cos 2\alpha}{\varepsilon} \\
\sin\theta_0 &= \sqrt{1 - \frac{[E-(E-1)\cos 2\alpha]^2}{1 + 2E(E-1)(1-\cos 2\alpha)}} \\
&= \pm\frac{(E-1)\sin 2\alpha}{\varepsilon}
\end{aligned}$$

these expressions for the sin and cos of θ_0 let us write the orbit equation as

$$r(\theta, \alpha) = \frac{(E-1)(1-\cos 2\alpha)}{-1 + E\cos\theta - (E-1)\cos(\theta + 2\alpha)},$$

which represents the orbit for the particle with energy E, initial position ($r_i = 1, \theta_i = 0$) and an initial firing angle α.

It is of our interest to find what is the forbidden region that no trajectory would be able to reach, the *caustic*, regardless of the initial firing angle α. To find that, we can calculate the minimum distance from the scatterer to any of the posible orbit for a given θ. The best way to calculate this funtion is to evaluate the partial differential of r with respect to α and set it equal to zero, this will provide us with the $\alpha^\star$ that corresponds to the orbit that approaches the scatterer the most, as function of the angle θ, then we can use this $\alpha^\star(\theta)$ in our orbit equation and that would yield the equation of the caustic, $r^\star$. The partial of r with respect to α is

$$\begin{aligned}
\frac{\partial r(\theta,\alpha)}{\partial\alpha} &= \frac{(E-1)}{(-1 + E\cos\theta - (E-1)\cos(\theta+2\alpha))^2} \times \\
&\quad [(-1 + E\cos\theta - (E-1)\cos(\theta+2\alpha))2\sin 2\alpha - (1-\cos 2\alpha)2\sin(\theta+2\alpha)] \\
&= \frac{2(E-1)}{(-1 + E\cos\theta - (E-1)\cos(\theta+2\alpha))^2} \times \\
&\quad [-\sin 2\alpha + E\cos\theta\sin 2\alpha - (E-1)\cos(\theta+2\alpha)\sin 2\alpha \\
&\quad -(E-1)\sin(\theta+2\alpha) + (E-1)\cos 2\alpha\sin(\theta+2\alpha)]
\end{aligned}$$

Since we are interested in the case, $\dfrac{\partial r}{\partial\alpha} = 0$, we need to solve,

$$\begin{aligned}
[-\sin 2\alpha^\star &+ E\cos\theta\sin 2\alpha^\star - (E-1)\cos(\theta+2\alpha^\star)\sin 2\alpha^\star \\
&- (E-1)\sin(\theta+2\alpha^\star) + (E-1)\cos 2\alpha^\star\sin(\theta+2\alpha^\star)] = 0
\end{aligned}$$

this reduces to

$$\begin{aligned}
(E-1)\sin\theta &= (1-\cos\theta)\sin 2\alpha^\star + (E-1)\cos\theta\cos 2\alpha^\star, \\
&= C\cos\delta\cos 2\alpha^\star + C\sin\delta\sin 2\alpha^\star, \\
&= C\cos(2\alpha^\star - \delta).
\end{aligned}$$

The parameters C and δ can be found through

$$\begin{aligned}
\tan\delta &= \left(\frac{1-\cos\theta}{(E-1)\sin\theta}\right), \\
C^2 &= (1-\cos\theta)^2 + ((E-1)\sin\theta)^2, \\
&\vdots \\
&= 2(1-\cos\theta) + E(E-2)\sin^2\theta,
\end{aligned}$$

and we obtain an expression for $\alpha^\star$

$$\alpha^\star = \frac{1}{2}\left[\cos^{-1}\left(\frac{(E-1)\sin\theta}{\sqrt{2(1-\cos\theta)+E(E-2)\sin^2\theta}}\right) + \tan^{-1}\left(\frac{1-\cos\theta}{(E-1)\sin\theta}\right)\right].$$

Some manipulation will help us find an easier expression for $\alpha^\star$, for instance consider

$$\begin{aligned}
y &= \tan x, \\
x &= \tan^{-1} y \\
\cos[\tan^{-1} y] &= \cos x = \frac{1}{(1+y^2)^{1/2}} \\
\sin[\tan^{-1} y] &= \sin x = \frac{y}{(1+y^2)^{1/2}}
\end{aligned}$$

similarly

$$\begin{aligned}
g &= \cos h \\
\cos^{-1} g &= h \\
\sin[cos^{-1} g] &= \sin h = \sqrt{1-g^2}
\end{aligned}$$

so, if we call $a = (E-1)\sin\theta$, $b = 1-\cos\theta$ and $C^2 = a^2 + b^2$, we have

$$\begin{aligned}
\cos 2\alpha^\star &= \cos\left[\cos^{-1}\left(\frac{a}{A}\right) + \tan^{-1}\left(\frac{b}{a}\right)\right], \\
&= \frac{a}{A}\frac{1}{\sqrt{1+\left(\frac{b}{a}\right)^2}} - \sqrt{1-\left[\frac{a}{A}\right]^2}\frac{\frac{b}{a}}{\sqrt{1+\left(\frac{b}{a}\right)^2}} \\
&= \frac{a^2}{A^2} - \frac{b^2}{A^2}.
\end{aligned}$$

To the last line we can add zero as $+a^2 - a^2$ or $+b^2 - b^2$, and we then have,

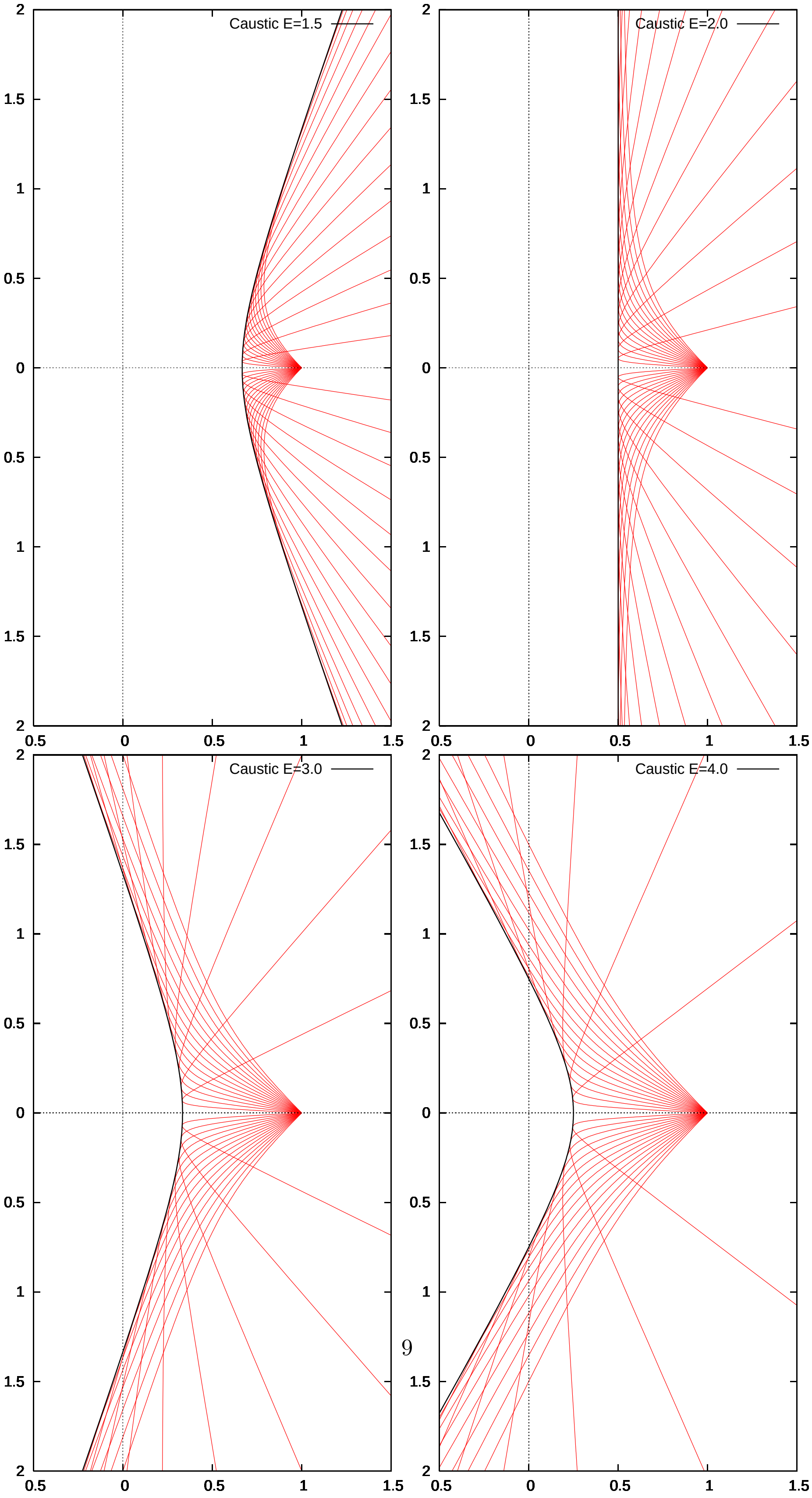
$$\begin{aligned}
\cos 2\alpha^\star &= 1 - 2\sin^2\alpha &&= 2\cos^2\alpha - 1, \\
&= 1 - 2\left(\tfrac{b}{A}\right)^2 &&= 2\left(\tfrac{a}{A}\right)^2 - 1,
\end{aligned}$$

so we have

$$\begin{aligned}
\alpha^\star &= \cos^{-1}\left[\frac{(E-1)\sin\theta}{\sqrt{2(1-\cos\theta) + E(E-2)\sin^2\theta}}\right] \\
&= \sin^{-1}\left[\frac{1-\cos\theta}{\sqrt{2(1-\cos\theta) + E(E-2)\sin^2\theta}}\right]
\end{aligned}$$

These two solutions are valid for positive θs, for negative *theta*s we can use the $\cos^{-1}$ one, in the domain $[-\pi, 0]$.

Some figures of the caustic for different energies are given below.


Caustic E=1.5
Caustic E=2.0
Caustic E=3.0
Caustic E=4.0
2
1.5
1
0.5
0
0.5
1
1.5
2
0.5
0
0.5
1
1.5