billiards
stadium billiard
billiard!stadium
specular reflection

Chapter 6

Billiards

We owe it to a book to withhold judgment until we reach page 100.
Henrietta McNutt, George Johnson's seventh-grade English teacher

Do in detail the disk scattering dynamics; LiversageFig.1

The dynamics that we have the best intuitive grasp on, and find easiest to grapple with both numerically and conceptually, is the dynamics of billiards. For billiards, discrete time is altogether natural; a particle moving through a billiard suffers a sequence of instantaneous kicks, and executes simple motion inbetween, so there is no need to contrive a Poincaré section. We have already used this system in sect. 1.3 as the intuitively most accessible example of chaos. Here we define billiard dynamics more precisely, anticipating the applications to come.

6.1 Billiard dynamics

A billiard is defined by a connected region $Q \subset \mathbb{R}^D$, with boundary $\partial Q \subset \mathbb{R}^{D-1}$ separating Q from its complement $\mathbb{R}^D \setminus Q$. The region Q can consist of one compact, finite volume component (in which case the billiard phase space is bounded, as for the stadium billiard figure 6.1), or can be infinite in extent, with its complement $\mathbb{R}^D \setminus Q$ consisting of one or several finite or infinite volume components (in which case the phase space is open, as for the 3-disk pinball game figure 1.1). In what follows we shall most often restrict our attention to *planar billiards*. A point particle of mass m and momentum $p_n = mv_n$ moves freely within the billiard, along a straight line, until it encounters the boundary. There it reflects specularly (*specular* = mirrorlike), with no change in the tangential component of momentum, and instantaneous reversal of the momentum component normal to the boundary,

put arrows everywhere on figure 6.1, denote ball position by (s_n, ϕ_n, τ_n)

check OED

$$p' = p - 2(p \cdot \hat{n})\hat{n}\,, \tag{6.1}$$

section!Poincar"'e
Poincar"'e!section
Poincar"'e!section!3-disk
three-disk@3-disk!geometry

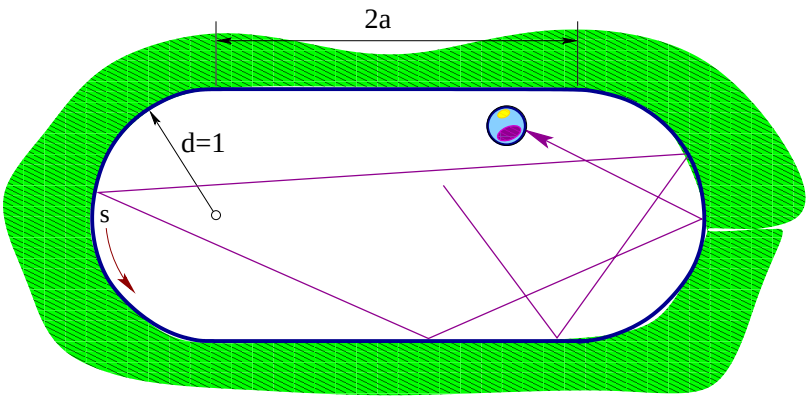

Figure 6.1: The stadium billiard is a 2-dimensional domain bounded by two semi-circles of radius $d = 1$ connected by two straight walls of length $2a$. At the points where the straight walls meet the semi-circles, the curvature of the border changes discontinuously; these are the only singular points of the flow. The length a is the only parameter.

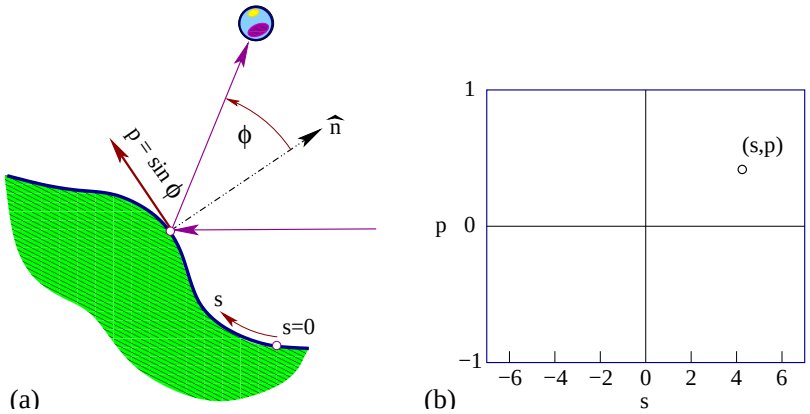

Figure 6.2: (a) A planar billiard trajectory is fixed by specifying the perimeter length parametrized by s and the outgoing trajectory angle ϕ, both measured counterclockwise with respect to the outward normal $\hat{n}$. (b) The Birkhoff phase space coordinate pair (s, p) fully specifies the trajectory, where $p = |p| \sin\phi$ is the momentum component tangential to the boundary (and we set $|p| = 1$ whenever convenient).

with $\hat{n}$ the unit vector normal to the boundary ∂Q at the collision point. The angle of incidence equals the angle of reflection, as illustrated in figure 6.2. A billiard is a Hamiltonian system with a $2D$-dimensional phase space $x = (q, p)$ and potential $V(q) = 0$ for $q \in Q$, $V(q) = \infty$ for $q \in \partial Q$.

Smilansky teaches: generating function; canonincal transformations; action angle variables. Problem: prove reflectin law by minimization of action

move 1, -1 on the p axis of figure 6.2 inward

A billiard flow has a natural Poincaré section defined by Birkhoff coordinates s_n, the arc length position of the nth bounce measured along the billiard boundary, and $p_n = |p| \sin\phi_n$, the momentum component parallel to the boundary, where ϕ_n is the angle between the outgoing trajectory and the normal to the boundary. We measure both the arc length s, and the parallel momentum p counterclockwise relative to the outward normal (see figure 6.2 as well as figure 1.6 (a)). In $D = 2$, the Poincaré section is a cylinder (topologically an annulus), figure 6.3, where the parallel momentum p ranges for $-|p|$ to $|p|$, and the s coordinate is cyclic along each connected component of ∂Q. The volume in the full phase space is preserved by the Liouville theorem (5.23). The Birkhoff coordinates $x = (s, p) \in \mathcal{P}$, are

label the two areas Q_1, Q_2 in figure 6.3. Draw corresponding rectangles?

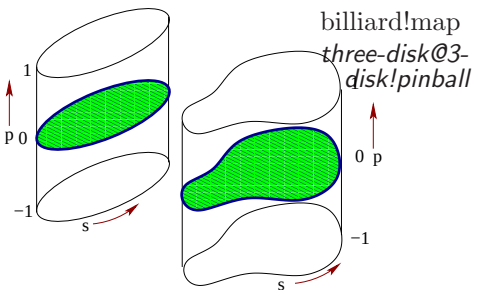

Figure 6.3: In $D = 2$ the billiard Poincaré section is a cylinder, with the parallel momentum p ranging over $p \in \{-1, 1\}$, and with the s coordinate is cyclic along each connected component of ∂Q. The rectangle figure 6.2 (b) is such cylinder unfolded, with periodic boundary conditions glueing together the left and the right edge of the rectangle.

billiard!map
three-disk@3-disk!pinball

the natural choice, because with them the the Poincaré return map preserves the phase space volume in the (s, p) parametrized Poincaré section (a perfectly good coordinate set (s, ϕ) does not do that).

6.8
page 100
sect. 6.2

Without loss of generality we set $m = |v| = |p| = 1$. Poincaré section condition eliminates one dimension, and the energy conservation $|p| = 1$ eliminates another, so the Poincaré section return map P is $(2D-2)$-dimensional.

The dynamics is given by the Poincaré return map

$$P : (s_n, p_n) \mapsto (s_{n+1}, p_{n+1}) \tag{6.2}$$

from the nth collision to the $(n+1)$st collision. [1] The discrete time dynamics map P is equivalent to the Hamiltonian flow (5.1) in the sense that both describe the same full trajectory. Let t_n denate the instant of nth collision. Then the position of the pinball $\in Q$ at time $t_n + \tau \leq t_{n+1}$ is given by $2D-2$ Poincaré section coordinates $(s_n, p_n) \in \mathcal{P}$ together with τ, the distance reached by the pinball along the nth section of its trajectory.

replace the reference to figure 1.6 (a) by ?

emphasize that the discrete/continuous time averages are the same

add figures from my papers

6.1
page 99

Example 6.1 3-disk game of pinball: *In case of bounces off a circular disk, the position coordinate $s = r\theta$ is given by angle $\theta \in [0, 2\pi]$. For example, for the 3-disk game of pinball of figure 1.4 and figure 1.6 we have two types of collisions:*

$$P_0 : \begin{cases} \phi' = -\phi + 2\arcsin p \\ p' = -p + \frac{a}{R}\sin\phi' \end{cases} \qquad \textit{back-reflection} \tag{6.3}$$

replace this by a smarter formula!

$$P_1 : \begin{cases} \phi' = \phi - 2\arcsin p + 2\pi/3 \\ p' = p - \frac{a}{R}\sin\phi' \end{cases} \qquad \textit{reflect to 3rd disk}. \tag{6.4}$$

make int exercise!

Here a = radius of a disk, and R = center-to-center separation. Actually, as in this example we are computing intersections of circles and straight lines, nothing more than high-school geometry is required. There is no need to compute arcsin*'s either - one only needs to compute a square root per each reflection, and the simulations can be very fast.*

rewrite this section in the way we really compute $\mathcal{L}$'s, R's and P_1's this impact parameter stuff is not the right thing!

Trajectory of the pinball in the 3-disk billiard is generated by a series of P_0's and P_1's. At each step on has to check whether the trajectory intersects the desired disk (and no disk inbetween). With minor modifications, the above formulas are valid for any smooth billiard as long as we replace a by the local curvature of the boundary at the point of collision.

6.2
page 99

[1] Mason: Poincare sections cannot handle flows like Shilnikov. Mention exceptional case, like when the section cannot decipher the type of the trajectory

stability!billiards
billiards!stability
Sinai-Bunimovich curvature
curvature!Sinai-Bunimovich

Figure 6.4: Variations in the phase space coordinates of a pinball between the $(k-1)$th and the kth collision. (a) δq_k variation away from the direction of the flow. (b) δz_k angular variation tranverse to the direction of the flow. (c) $\delta q_{\|}$ variation in the direction of the flow is conserved by the flow.

6.2 Stability of billiards

We turn next to the question of local stability of discrete time billiard systems. Infinitesimal equations of variations (4.2) do not apply, but the multiplicative structure (4.31) of the finite-time Jacobian matrices does. As they are more physical than most maps studied by dynamicists, let us work out the billiard stability in some detail.

On the face of it, a plane billiard phase space is 4-dimensional. However, one dimension can be eliminated by energy conservation, and the other by the fact that the magnitude of the velocity is constant. We shall now show how going to a local frame of motion leads to a [2×2] Jacobian matrix. In sect. 6.2.1 we show that due to the symplectic invariance the situation is even simpler; the stability of a 2-dimensional billiard flow is given by a single number, the Sinai-Bunimovich curvature.

⇓PRELIMINARY

⇑PRELIMINARY

Consider a 2-dimensional billiard with phase space coordinates $x = (q_1, q_2, p_1, p_2)$. Let t_k be the instant of the kth collision of the pinball with the billiard boundary, and $t_k^{\pm} = t_k \pm \epsilon$, ϵ positive and infinitesimal. With the mass and the velocity equal to 1, the momentum direction can be specified by angle θ: $x = (q_1, q_2, \sin\theta, \cos\theta)$. Now parametrize the 2-$d$ neighborhood of a trajectory segment by $\delta x = (\delta z, \delta\theta)$, where

⇓PRELIMINARY
draw the figure!
⇑PRELIMINARY

$$\delta z = \delta q_1 \cos\theta - \delta q_2 \sin\theta\,, \tag{6.5}$$

$\delta\theta$ is the variation in the direction of the pinball motion. Due to energy conservation, there is no need to keep track of $\delta q_{\|}$, variation along the flow, as that remains constant. $(\delta q_1, \delta q_2)$ is the coordinate variation transverse to the kth segment of the flow, figure 6.4. From the Hamilton's equations of motion for a free particle, $dq_i/dt = p_i$, $dp_i/dt = 0$, we obtain the equations of motion (4.1) for the linearized neighborhood

⇓PRIVATE
⇑PRIVATE

$$\frac{d}{dt}\delta\theta = 0, \quad \frac{d}{dt}\delta z = \delta\theta\,. \tag{6.6}$$

Let $\delta\theta_k = \delta\theta(t_k^+)$ and $\delta z_k = \delta z(t_k^+)$ be the local coordinates immediately after the kth collision, and $\delta\theta_k^- = \delta\theta(t_k^-)$, $\delta z_k^- = \delta z(t_k^-)$ immediately before. Integrating the free flight from t_{k-1}^+ to t_k^- we obtain

volume preservation

draw figure

$$\begin{aligned} \delta z_k^- &= \delta z_{k-1} + \tau_k \delta\theta_{k-1}\,, \qquad \tau_k = t_k - t_{k-1} \\ \delta\theta_k^- &= \delta\theta_{k-1}\,, \end{aligned} \tag{6.7}$$

and the stability matrix (4.30) for the kth free flight segment is

$$\mathbf{J}_T(x_k) = \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix}\,. \tag{6.8}$$

At incidence angle ϕ_k (the angle between the outgoing particle and the outgoing normal to the billiard edge), the incoming transverse variation δz_k^- projects onto an arc on the billiard boundary of length $\delta z_k^-/\cos\phi_k$. The corresponding incidence angle variation $\delta\phi_k = \delta z_k^-/\rho_k \cos\phi_k$, ρ_k = local radius of curvature, increases the angular spread to

doodle a little picture here - then the relation becomes obvious, it is a "footprint" of flashlight shining on a surface at an angle.

$$\begin{aligned} \delta z_k &= -\delta z_k^- \\ \delta\theta_k &= -\delta\theta_k^- - \frac{2}{\rho_k \cos\phi_k}\delta z_k^-\,, \end{aligned} \tag{6.9}$$

so the Jacobian matrix associated with the reflection is

Tanner changed signs here, Predrag changed them back

$$\mathbf{J}_R(x_k) = -\begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}\,, \qquad r_k = \frac{2}{\rho_k \cos\phi_k}\,. \tag{6.10}$$

The full Jacobian matrix for n_p consecutive bounces describes a beam of trajectories defocused by $\mathbf{J}_T$ along the free flight (the τ_k terms below) and defocused/refocused at reflections by $\mathbf{J}_R$ (the r_k terms below)

replace n_p fundamental domain - count symmetry wall reflections

$$\mathbf{J}_p = (-1)^{n_p} \prod_{k=n_p}^{1} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}\,, \tag{6.11}$$

where τ_k is the flight time of the kth free-flight segment of the orbit, $r_k = 2/\rho_k \cos\phi_k$ is the defocusing due to the kth reflection, and ρ_k is the radius of curvature of the billiard boundary at the kth scattering point (for our 3-disk game of pinball, $\rho = 1$). As the billiard dynamics is phase space volume preserving, $\det \mathbf{J} = 1$, and the eigenvalues are given by (5.18).

6.4 page 100

repalce figure 6.5

This is still another example of the Jacobian matrix chain rule (4.35) for discrete time systems, rather similar to the Hénon map stability (4.36). Stability of every flight segment or reflection taken alone is a shear with two unit eigenvalues,

$$\det \mathbf{J}_T = \det \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix}\,, \quad \det \mathbf{J}_R = \det \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}\,, \tag{6.12}$$

Sinai-Bunimovich curvature
curvature!Sinai-Bunimovich

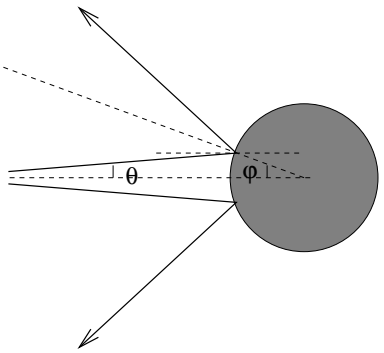

Figure 6.5: Defocusing of a beam of nearby trajectories at a billiard collision. (A. Wirzba)

Figure 6.6: A horocycle.

but acting in concert in the intervowen sequence (6.11) they can lead to a hyperbolic deformation of the infinitesimal neighborhood of a billiard trajectory.

8.1 page 124

As a concrete application, consider the 3-disk pinball system of sect. 1.3. Analytic expressions for the lengths and eigenvalues of $\overline{0}$, $\overline{1}$ and $\overline{10}$ cycles follow from elementary geometrical considerations. Longer cycles require numerical evaluation by methods such as those described in chapter 17.

8.2 page 124
6.3 page 99
chapter 17

6.2.1 Sinai-Bunimovich curvatures

explain that Liouville holds, but phase space is lost thorugh escape

add the billiard mean-time of flight section from Sune article

⇓PRELIMINARY

need to draw a figure of a horocycle!

Imagine a set of projectiles leaving a point (q_1, q_2) in all directions, parametrized by angle θ; they generate a "horocycle" in the configuration space, a set of all points reached by time t, see figure 6.6. A $\delta\theta$ wedge of angles stretches into a horocycle arc $\delta z = t\delta\theta$, and $\kappa = \delta\theta/\delta z = 1/t$ is the local curvature of the horocycle.

Eqs. (6.7) and (6.9) can be rewritten as

$$\begin{aligned} \frac{\delta z_k^-}{\delta\theta_k^-} &= \frac{\delta z_{k-1}}{\delta\theta_{k-1}} + \tau_k \\ \frac{\delta\theta_k}{\delta z_k} &= \frac{\delta\theta_k^-}{\delta z_k^-} + \frac{2}{\rho_k \cos\phi_k}\,, \end{aligned} \tag{6.13}$$

leading to the continued fraction recursion for the curvature κ immediately after the kth bounce

billiard!general
restitution coefficient

$$\kappa_k = r_k + \frac{1}{\tau_k - \frac{1}{\kappa_{k-1}}}\,, \qquad \kappa_k = -\frac{\delta\theta_k}{\delta z_k}\,. \tag{6.14}$$

Somehow this came out backwards in time?

add exercise relating **J** and κ

⇑PRELIMINARY

⇓PRIVATE

6.3 General billiard flows

Consider a trajectory $x(t)$ obtained by integrating vector field $v(x)$

$$x_k^- = T(x_{k-1}) = x_{k-1} + \int_{t_{k-1}^+}^{t_k^-} d\tau\, v(x(\tau))\,. \tag{6.15}$$

Here $t_k^+ = t_k + \epsilon$ denotes the time infinitesimally after the instant of the kth kick, $t_k^- = t_k - \epsilon$ the time infinitesimally before it, and $x_k^- = x(t_k^-)$, $x_k = x(t_k^+)$. The trajectory is smooth almost all of the time, except for a sequence of instantaneous kicks (such as hard wall billiard colisions),

$$x_k = R(x_k^-)\,. \tag{6.16}$$

Map T is the integral (6.15) of the smooth flow evolution between two consecutive kicks, from t_{k-1}^+ to t_k^-. R describes the instantaneous kick change of velocity at time t_k^- to velocity at t_k^+. The discrete "time" k is an integer, the number of applications of the $R \circ T$ map, where we streamline the notation by denoting a map composition by "$\circ$"

$$x_k = R \circ T \cdots R \circ T(x_0) = (R \circ T)^k(x_0)\,. \tag{6.17}$$

Example 6.2 When the ball loses its bounce. *A point particle with position and momentum $x(t) = (q(t), p(t))$ moves within the billiard, along a given integrable trajectory (free flight, gravity, constant electric or magnetic field, harmonic oscillator, ...) until it encounters a boundary. There it reflects following the Newton's law of restitution, with no change in the tangential component of momentum, and instantaneous reversal of the momentum component normal to the boundary, in the $p, \hat{n}$ plane:*

Consider a $D = 2$ *planar billiard*" needed?

$$R: \begin{cases} q = q^- \\ p = p^- - (1+\gamma)(p^- \cdot \hat{n})\hat{n} \end{cases}, \qquad 0 < \gamma \leq 1\,, \tag{6.18}$$

with $\hat{n}$ the unit vector normal to the billiard boundary ∂Q at the collision point, and γ the restitution coefficient.

To avoid confusion: for Hamiltonian flight segments, the flow field v in (6.15) combines the Newtonian velocities and accelerations, $v(x) = (\dot{q}(x), \dot{p}(x))$.

Due to the collisional loss of energy, an inelastic billiard is not a Hamiltonian system. For $\gamma < 1$ the phase space volume is not preserved, the Liouville theorem (5.23)

section!Poincaré
Poincaré!section
billiard!map
stability!billiards
billiards!stability

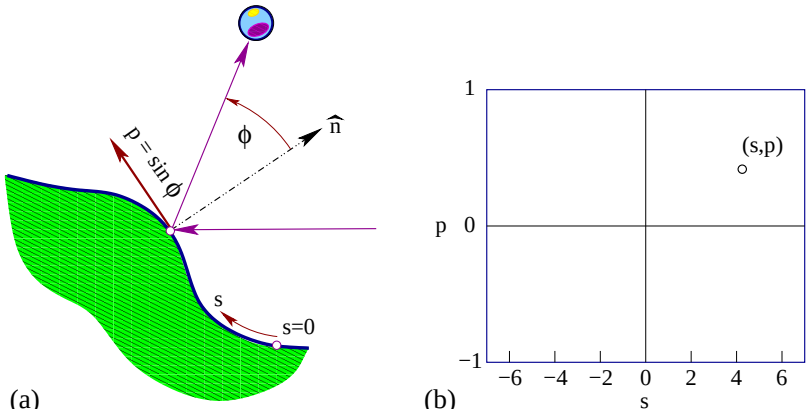

Figure 6.7: (a) A trajectory of a general billiard is fixed by specifying the perimeter wall arclength, the outgoing wall-normal momentum, and the outgoing wall-parallel momentum $(s, p_\parallel, p_\perp)$. For $\gamma < 1$ the ball loses its bounce, $\phi^+ < \phi^-$. [REDRAW] (b) [REPLACE by a perspective 3-d section.]

does not apply, and sustaning motion requires an external driving mechanism that can compensate for the collisional energy loss. Poincaré section condition eliminates one dimension, hence a 3-d Poincaré section for a plane billiard flow can be defined by marking s_k, the arc length position of the kth kick measured along the billiard boundary, and the outgoing wall-normal and wall-parallel momenta $(s_k, p_{\perp,k}, p_{\parallel,k})$. Alternatively, on can can keep track of the wall-parallel momentum $p_{\parallel,k}$ and the energy shell $E_k = p_k^2/2m + V(q_k)$ that the particle is on after kth billiard wall reflection.

We turn next to the question of local stability of a general billiard flow.

If $x(t)$ reaches the wall at time t_k, a neighboring trajectory (see figure 4.4) reflects off the billiard wall time δt later, following the reflection law (6.18):

$$x(t_k^+ + \delta t) + \delta x(t_k^+ + \delta t) = R(x(t_k^- + \delta t) + \delta x(t_k^- + \delta t))\,.$$

To linear order

$$\begin{aligned} x_k^+ + \frac{dx(t_k^+)}{dt}\delta t + \delta x_k^+ &= R\left(x_k^- + \frac{dx(t_k^-)}{dt}\delta t + \delta x_k^-\right) \\ &= x_k^+ + \frac{\partial R(x_k^-)}{\partial x}(\delta x_k^- + v_k^-\delta t)\,,. \end{aligned}$$

Hence

$$\delta x_k^+ = \mathbf{J}_R(x_k^-)\delta x_k^- - \left(v_k^+ - \mathbf{J}_R(x_k^-)v_k^-\right)\delta t \tag{6.19}$$

where

$$\mathbf{J}_R(x_k^-) = \frac{\partial R(x_k^-)}{\partial x}\,, \qquad v_k^+ = v(x_k^+)\,, \quad v_k^- = v(x_k^-)\,.$$

volume contraction
billiards
baker's map
three-disk@3-disk!pinball

If you wish, you can compute the extra flight time δt by elementary geometry, from figure 4.4, as in (4.40). However, $\mathbf{J}_R(x^-)$ is a Jacobian matrix like any other; you can think of it as a limit of a smooth flow where the trajectory is strongly deflected over a time interval 2ϵ, $\epsilon \to 0$. According to (4.6), a Jacobian matrix transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

$$v(x(t)) = \mathbf{J}^t(x_0)\, v(x_0)\,,$$

hence the prefactor of δt in (6.19) *vanishes*. Another way to think of this is as in sect. 4.4.1, where we have proven that the spectrum of transverse eigenvalues of a Jacobian matrix for a continuous flow is independent of the choice of a transverse Poincaré sections, with different choices differing by such δt's contributions along the direction of the flow.

The cycle Jacobian matrix for n_p consecutive bounces describes a beam of trajectories defocused by $\mathbf{J}_T$ along the flight segments and defocused/refocused at kicks by $\mathbf{J}_R$

$$\mathbf{J}_p = \prod_{k=n_p}^{1} \mathbf{J}_R(x^-_{k+1})\mathbf{J}_T(x_k)\,. \tag{6.20}$$

To summarize: we have shown that the time delay contributes only along the direction of the flow, and by (4.6) the longitudinal part does not contribute computation of stability eigenvalues of a cycle.

⇓PRELIMINARY

Guys, what am I missing? If you do no believe me, why does the Hamiltonian linearized stability for particles bouncing in gravity not require this δt term? There are accelerations...

The generalized billiard dynamics is not phase space volume preserving; for impact normal to the billiard wall $\det \mathbf{J} = \gamma^2$.

fix this
⇑PRELIMINARY
⇑PRIVATE

Commentary

start with Sinai (convex), then Bunimovich (concave)

Remark 6.1 Billiards. The 3-disk game of pinball is to chaotic dynamics what a pendulum is to integrable systems; the simplest physical example that captures the essence of chaos. Another contender for the title of the "harmonic oscillator of chaos" is the baker's map which is used as the red thread through Ott's introduction to chaotic dynamics [1.7]. The baker's map is the simplest reversible dynamical system which is hyperbolic and has positive entropy. We will not have much use for the baker's map here, as due to its piecewise linearity it is so non-generic that it misses all of the subtleties of cycle expansions curvature corrections that will be central to this treatise.

☞ chapter 18

That the 3-disk game of pinball is a quintessential example of deterministic chaos appears to have been first noted by B. Eckhardt [6.1]. The model was studied in depth classically, semiclassically and quantum mechanically by P. Gaspard

intermittency
bifurcation!generic
renormalization
smooth!potential
three-disk@3-disk!pinball
symbolic!dynamics!at a bifurcation
three-dimensional@3-dimensional sphere!scattering
scattering!3-dimensional spheres

and S.A. Rice [6.2], and used by P. Cvitanović and B. Eckhardt [37.4] to demonstrate applicability of cycle expansions to quantum mechanical problems. It has been used to study the higher order $\hbar$ corrections to the Gutzwiller quantization by P. Gaspard and D. Alonso Ramirez [6.4], construct semiclassical evolution operators and entire spectral determinants by P. Cvitanović and G. Vattay [6.5], and incorporate the diffraction effects into the periodic orbit theory by G. Vattay, A. Wirzba and P.E. Rosenqvist [6.6]. The full quantum mechanics and semiclassics of scattering systems is developed here in the 3-disk scattering context in chapter 35. Gaspard's monograph [1.4], which we warmly recommend, utilizies the 3-disk system in much more depth than will be attained here. For further links check `ChaosBook.org`.

A pinball game does miss a number of important aspects of chaotic dynamics: generic bifurcations in smooth flows, the interplay between regions of stability and regions of chaos, intermittency phenomena, and the renormalization theory of the "border of order" between these regions. To study these we shall have to face up to much harder challenge, dynamics of smooth flows.

Nevertheless, pinball scattering is relevant to smooth potentials. The game of pinball may be thought of as the infinite potential wall limit of a smooth potential, and pinball symbolic dynamics can serve as a *covering* symbolic dynamics in smooth potentials. One may start with the infinite wall limit and adiabatically relax an unstable cycle onto the corresponding one for the potential under investigation. If things go well, the cycle will remain unstable and isolated, no new orbits (unaccounted for by the pinball symbolic dynamics) will be born, and the lost orbits will be accounted for by a set of pruning rules. The validity of this adiabatic approach has to be checked carefully in each application, as things can easily go wrong; for example, near a bifurcation the same naive symbol string assignments can refer to a whole island of distinct periodic orbits.

☞ sect. 34.1

Remark 6.2 Further reading. The chapter 1 of Gaspard monograph [1.4] is recommended reading if you are interested in Hamiltonian flows, and billiards in particular. A. Wirzba has generalized the stability analysis of sect. 6.2 to scattering off 3-dimensional spheres (follow the links in `ChaosBook.org/extras`). A clear discussion of linear stability for the general d-dimensional case is given in Gaspard [1.4], sect. 1.4.

⇓PRELIMINARY

find Knauff scattering reference

Knauff's scattering problem is surprising in this context, as it is Anosov flow that resembles a billiard.

⇑PRELIMINARY

⇓PRIVATE

Remark 6.3 Impact oscillators. Relate sect. 6.3 to the elastic impact oscillator literature, refs. [6.11, 6.12, 6.13, 6.14].

Remark 6.4 Restitution coefficients. (From Ref. [6.15]): In the normal direction, the restitution laws mostly used are those given by Newton and by Poisson. Newton's is a purely kinematical law, Poisson's is dynamical and Begin-Boulanger uses energy consideration.

The coefficient of restitution given by Newton considers the normal relative velocities pre and post-collision.

The coefficient of restitution given by Poisson is the ratio between the normal impulse in the expansion phase and the normal impulse in the compression phase.

This coefficient of restitution takes into consideration the dynamics of the system in the virtual process of collision.

The coefficient of restitution given by Beghin-Boulanger considers the relation between the kinetic energies in the expansion phase and compression phases. This coefficient of restitution considers the exchange of energy during the virtual process of collision.

When friction is not considered all three coefficients of restitution are equivalent (ref. [3] in Ref. [6.15]).

γ_t = coefficient of tangential restitution

⇑PRIVATE

Résumé

A particulary natural application of the Poincaré section method is the reduction of a billiard flow to a boundary-to-boundary return map.

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