Chapter 16

Why does it work?

Bloch: "Space is the field of linear operators." Heisenberg: "Nonsense, space is blue and birds fly through it." give ref to Bloch
Felix Bloch, *Heisenberg and the early days of quantum mechanics*

[1]

(R. Artuso, H.H. Rugh and P. Cvitanović)

The trace formulas and spectral determinants work well, sometimes very well indeed. The question is: why? The heuristic manipulations of chapter 14 were naive and reckless, as we are facing infinite-dimensional vector spaces and singular integral kernels.

We now outline the key ingredients of proofs that put the trace and determinant formulas on solid mathematical footing. This requires taking a closer look at the evolution operators from a mathematical point of view, since up to now we have talked about eigenvalues without any reference to what kind of a function space the corresponding eigefunctions belong to. We shall restrict our considerations to the spectral properties of the Perron-Frobenius operator for maps, as proofs for more general evolution operators follow along the same lines. What we refer to as a "the set of eigenvalues" acquires meaning only within a precisely specified functional setting: this sets the stage for a discussion of analyticity properties of spectral determinants. In sect. 16.1 we show on a very simple example that the spectrum is quite sensitive to the regularity properties of the functions considered. In sect. 16.4 we review the basic facts of the classical Fredholm theory of integral equations. The program is sketched in sect. 16.5, motivated by an explicit study of eigenspectrum of the Bernoulli shift map, and in sect. 16.6 generalized to piecewise real-analytic hyperbolic maps acting on appropriate densities. For expanding and hyperbolic finite-subshift maps analyticity leads to a very strong result; not only do the determinants have

[1]Artuso: exercises 7.9 and 7.19 have to be removed (7.9 migrated to chapter 8, while 7.19 is actually solved in the text ..).

entire function
single fixed point!repeller
define entire function
repeller!single fixed point
remark 16.1
Perron-Frobenius!operator
eigenfunctions!Perron-Frobenius

better analyticity properties than the trace formulas, but the spectral determinants are singled out as being entire functions in the complex s plane.

The goal of this chapter is not to provide an exhaustive review of the rigorous theory of the Perron-Frobenius operators and their spectral determinants, but rather to give you a feeling for how our heuristic considerations can be put on a firm basis. The mathematics underpinning the theory is both hard and profound.

If you are primarily interested in applications of periodic orbit theory, you should skip this chapter on the first reading.

fast track:
chapter 17, p. 305

16.1 Linear maps: exact spectra

We start gently; in example 16.1 we work out the *exact* eigenvalues and eigenfunctions of the Perron-Frobenius operator for the simplest example of unstable (expanding) dynamics, a linear 1-d map with one unstable fixed point. Not only that, but in example 16.5 we compute the exact spectrum for the simplest example of a dynamical system with an infinity of unstable periodic orbits, the Bernoulli shift. In example 16.7 we show that these results carry over to any analytic single-branch 1-d repeller, ??, and motivate by these examples the steps that lead to a proof that spectral determinants for more general analytic 1-dimensional expanding maps, and - in sect. 16.6, for 2-dimensional hyperbolic mappings - are also entire functions.

Example 16.1 The simplest of eigenspectra - a single fixed point: *In order to get some feeling for the determinants defined so formally in sect. 15.2, let us work out a trivial example: a repeller with only one expanding linear branch*

$$f(x) = \Lambda x\,, \qquad |\Lambda| > 1\,,$$

and only one fixed point $x^* = 0$. The action of the Perron-Frobenius operator (9.10) is

$$\mathcal{L}\phi(y) = \int dx\, \delta(y - \Lambda x)\, \phi(x) = \frac{1}{|\Lambda|}\phi(y/\Lambda)\,. \qquad (16.1)$$

From this one immediately gets that the monomials y^k are eigenfunctions:

$$\mathcal{L}y^k = \frac{1}{|\Lambda|\Lambda^k} y^k\,, \qquad k = 0, 1, 2, \ldots \qquad (16.2)$$

What are these eigenfunctions? Think of eigenfunctions of the Schrödinger equation: k labels the kth eigenfunction x^k in the same spirit in which the number of nodes labels the kth quantum-mechanical eigenfunction. A quantum-mechanical amplitude with more nodes has more variability, hence a higher kinetic energy. Analogously, for a Perron-Frobenius operator, a higher k eigenvalue $1/|\Lambda|\Lambda^k$ is getting exponentially smaller because densities that vary more rapidly decay more rapidly under the expanding action of the map.

Example 16.2 The trace formula for a single fixed point: *The eigenvalues Λ^{-k-1} fall off exponentially with k, so the trace of $\mathcal{L}$ is a convergent sum*

$$\mathrm{tr}\,\mathcal{L} = \frac{1}{|\Lambda|}\sum_{k=0}^{\infty}\Lambda^{-k} = \frac{1}{|\Lambda|(1-\Lambda^{-1})} = \frac{1}{|f(0)'-1|}\,,$$

in agreement with (14.6). A similar result follows for powers of $\mathcal{L}$, yielding the single-fixed point version of the trace formula for maps (14.9):

$$\sum_{k=0}^{\infty}\frac{ze^{s_k}}{1-ze^{s_k}} = \sum_{r=1}^{\infty}\frac{z^r}{|1-\Lambda^r|}\,, \qquad e^{s_k} = \frac{1}{|\Lambda|\Lambda^k}\,. \tag{16.3}$$

The left hand side of (16.3) is a meromorphic function, with the leading zero at $z = |\Lambda|$.

Example 16.3 Meromorphic functions and exponential convergence: *As an illustration of how exponential convergence of a truncated series is related to analytic properties of functions, consider, as the simplest possible example of a meromorphic function, the ratio*

$$h(z) = \frac{z-a}{z-b}$$

with a, b real and positive and $a < b$. Within the spectral radius $|z| < b$ *the function h can be represented by the power series*

$$h(z) = \sum_{k=0}^{\infty}\sigma_k z^k\,,$$

where $\sigma_0 = a/b$, and the higher order coefficients are given by $\sigma_j = (a-b)/b^{j+1}$. Consider now the truncation of order N of the power series

$$h_N(z) = \sum_{k=0}^{N}\sigma_k z^k = \frac{a}{b} + \frac{z(a-b)(1-z^N/b^N)}{b^2(1-z/b)}\,.$$

Let $\hat{z}_N$ be the solution of the truncated series $h_N(\hat{z}_N) = 0$. To estimate the distance between a and $\hat{z}_N$ it is sufficient to calculate $h_N(a)$. It is of order $(a/b)^{N+1}$, so finite order estimates converge exponentially to the asymptotic value.

This example shows that: (1) an estimate of the leading pole (the leading eigenvalue of $\mathcal{L}$) from a finite truncation of a trace formula converges exponentially, and (2) the non-leading eigenvalues of $\mathcal{L}$ lie outside of the radius of convergence of the trace formula and cannot be computed by means of such cycle expansion. However, as we shall now see, the whole spectrum is reachable at no extra effort, by computing it from a determinant rather than a trace.

Euler!formula
Bernoulli!shift
shift!Bernoulli
Bernoulli!polynomials

Example 16.4 The spectral determinant for a single fixed point: *The spectral determinant (15.3) follows from the trace formulas of example 16.2:*

$$\det(1 - z\mathcal{L}) = \prod_{k=0}^{\infty} \left(1 - \frac{z}{|\Lambda|\Lambda^k}\right) = \sum_{n=0}^{\infty} (-t)^n Q_n\,, \quad t = \frac{z}{|\Lambda|}\,, \tag{16.4}$$

16.3
page 303

where the cummulants Q_n are given explicitly by the Euler formula

$$Q_n = \frac{1}{1-\Lambda^{-1}} \frac{\Lambda^{-1}}{1-\Lambda^{-2}} \cdots \frac{\Lambda^{-n+1}}{1-\Lambda^{-n}} \quad . \tag{16.5}$$

(If you cannot figure out how to derive this formula, the solutions on p. 1010 offer several proofs.)

The main lesson to glean from this simple example is that the cummulants Q_n decay asymptotically *faster* than exponentially, as $\Lambda^{-n(n-1)/2}$. For example, if we approximate series such as (16.4) by the first 10 terms, the error in the estimate of the leading zero is $\approx 1/\Lambda^{50}$!

So far all is well for a rather boring example, a dynamical system with a single repelling fixed point. What about chaos? Systems where the number of unstable cycles increases exponentially with their length? We now turn to the simplest example of a dynamical system with an infinity of unstable periodic orbits, in order to motivate the steps that will eventually lead to a proof that spectral determinants for more general nonlinear 1-dimensional expanding maps are entire functions.

copy Driebe p. 19, draw figure

Example 16.5 Bernoulli shift: *Consider next the Bernoulli shift map*

$$x \mapsto 2x \;\; (mod\, 1)\,, \qquad\qquad x \in [0,1]\,. \tag{16.6}$$

The associated Perron-Frobenius operator (9.9) assambles $\rho(y)$ from its two preimages

$$\mathcal{L}\rho(y) = \frac{1}{2}\rho\left(\frac{y}{2}\right) + \frac{1}{2}\rho\left(\frac{y+1}{2}\right)\,. \tag{16.7}$$

For this simple example the eigenfunctions can be written down explicitly: they coincide, up to constant prefactors, with the Bernoulli polynomials $B_n(x)$. These polynomials are generated by the Taylor expansion of

$$\mathcal{G}_t(x) = \frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}\,, \quad B_0(x) = 1\,,\; B_1(x) = x - \frac{1}{2}\,, \ldots$$

The Perron-Frobenius operator (16.7) acts on the generating function $\mathcal{G}$ as

$$\mathcal{L}\mathcal{G}_t(x) = \frac{1}{2}\left(\frac{te^{xt/2}}{e^t-1} + \frac{te^{t/2}e^{xt/2}}{e^t-1}\right) = \frac{t/2e^{xt/2}}{e^{t/2}-1} = \sum_{k=1}^{\infty} B_k(x)\frac{(t/2)^k}{k!}\,,$$

hence each $B_k(x)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $1/2^k$.

The full operator has two components corresponding to the two branches. For the n times iterated operator we have a full binary shift, and for each of the 2^n branches the above calculations carry over, yielding the same trace $(2^n-1)^{-1}$ for every cycle on

I do not get your drift here: The persistence of a finite essential spectral radius would suggest that traces and determinants do not exist in this case either. The pleasant surprise is that they do, see remark 16.3.

length n. Without further ado we substitute everything back and obtain the determinant, escape!rate

$$\det(1 - z\mathcal{L}) = \exp\left(-\sum_{n=1} \frac{z^n}{n} \frac{2^n}{2^n - 1}\right) = \prod_{k=0}\left(1 - \frac{z}{2^k}\right), \tag{16.8}$$

verifying that the Bernoulli polynomials are eigenfunctions with eigenvalues 1, $1/2$, …, $1/2^n$, … .

The Bernoulli map spectrum looks reminiscent of the single fixed-point spectrum (16.2), with the difference that the leading eigenvalue here is 1, rather than $1/|\Lambda|$. The difference is significant: the single fixed-point map is a repeller, with escape rate (1.6) given by the $\mathcal{L}$ leading eigenvalue $\gamma = \ln|\Lambda|$, while there is no escape in the case of the Bernoulli map. As already noted in discussion of the relation (15.24), for bound systems the local expansion rate (here $\ln|\Lambda| = \ln 2$) is balanced by the entropy (here $\ln 2$, the log of the number of preimages F_s), yielding zero escape rate. link this to refsect??

So far we have demonstrated that our periodic orbit formulas are correct for two piecewise linear maps in 1 dimension, one with a single fixed point, and one with a full binary shift chaotic dynamics. What about higher dimensions? We check our formulas on a 2-d hyperbolic map next.

Example 16.6 The simplest of 2-d maps - a single hyperbolic fixed point: *We start by considering a very simple linear hyperbolic map with a single hyperbolic fixed point,*

$$f(x) = (f_1(x_1, x_2), f_2(x_1, x_2)) = (\Lambda_s x_1, \Lambda_u x_2), \quad 0 < |\Lambda_s| < 1, \;\; |\Lambda_u| > 1.$$

The Perron-Frobenius operator (9.10) acts on the 2-d density functions as

$$\mathcal{L}\rho(x_1, x_2) = \frac{1}{|\Lambda_s\Lambda_u|}\,\rho(x_1/\Lambda_s, x_2/\Lambda_u) \tag{16.9}$$

What are good eigenfunctions? Cribbing the 1-d eigenfunctions for the stable, contracting x_1 direction from example 16.1 is not a good idea, as under the iteration of $\mathcal{L}$ the high terms in a Taylor expansion of $\rho(x_1, x_2)$ in the x_1 variable would get multiplied by exponentially exploding eigenvalues $1/\Lambda_s^k$. This makes sense, as in the contracting directions hyperbolic dynamics crunches up initial densities, rather then smoothing them. So we guess instead that eigenfunctions are of form

$$\varphi_{k_1 k_2}(x_1, x_2) = x_2^{k_2}/x_1^{k_1+1}, \qquad k_1,\, k_2 = 0, 1, 2, \ldots, \tag{16.10}$$

a mixture of the Laurent series in the contraction direction, x_1 variable, and the Taylor series in the expanding direction, the x_2 variable. The action of Perron-Frobenius operator on this set of basis functions

$$\mathcal{L}\varphi_{k_1 k_2}(x_1, x_2) \;=\; \frac{1}{|\Lambda_u|}\frac{\Lambda_s^{k_1}}{\Lambda_u^{k_2}}\,\varphi_{k_1 k_2}(x_1, x_2)$$

is smoothing, with the higher k_1, k_2 eigenvectors decaying exponentially faster, by $\Lambda_s^{k_1}/\Lambda_u^{k_2+1}$ factor in the eigenvalue. One verifies by an explicit calculation (undoing the geometric series expansions to lead to (15.10)) that the trace of $\mathcal{L}$ indeed equals $1/|\det(\mathbf{1} - \mathbf{J})| = 1/|(1 - \Lambda_u)(1 - \Lambda_s)|$, from which it follows that all our trace and spectral determinant formulas apply.

So far we have checked the trace and spectral determinant formulas derived heuristically in chapters 14 and 15, but only for the case of 1- and 2-d linear maps. But for infinite-dimensional vector spaces this game is fraught with dangers, and we have already been mislead by piecewise linear examples into spectral confusions: contrast the spectra of example 9.1 and example 10.1 with the spectrum computed in sect. 14.2.1.

We show next that the above results do carry over to a sizable class of piecewise analytic expanding maps.

16.2 Evolution operator in a matrix representation

The standard, and for numerical purposes sometimes very effective way to look at operators is through their matrix representations. Evolution operators are moving density functions defined over some phase space, and as in general we can implement this only numerically, the temptation is to discretize the phase space as in sect. 10.4. The problem with such phase space discretization approaches that they sometimes yield plainly wrong spectra (compare example 10.1 with the result of sect. 14.2.1), so we have to think through carefully what is it that we *really* measure.

An expanding map $f(x)$ takes an initial smooth density $\phi_n(x)$, defined on a subinterval, stretches it out and overlays it over a larger interval, resulting in a new, smoother density $\phi_{n+1}(x)$. Repetition of this process smoothes the initial density, so it is natural to represent densities $\phi_n(x)$ by their Taylor series. By expanding both

$$\phi_n(y) = \sum_{k=0}^{\infty} \phi_n^{(k)}(0)\frac{y^k}{k!}\,, \qquad \phi_{n+1}(y)_k = \sum_{\ell=0}^{\infty} \phi_{n+1}^{(\ell)}(0)\frac{y^\ell}{\ell!}\,, \quad \text{and}$$

$$\phi_{n+1}^{(\ell)}(0) = \int dx\ \delta^{(\ell)}(y-f(x))\phi_n(x)\Big|_{y=0}\,, \qquad x = f^{-1}(0)\,,$$

and substituting the two Taylor series into (9.6)

$$\phi_{n+1}(y) = (\mathcal{L}\phi_n)(y) = \int_{\mathcal{M}} dx\, \delta(y-f(x))\, \phi_n(x)\,.$$

The matrix elements follow by evaluating the integral

$$\mathbf{L}_{\ell k} = \frac{\partial^\ell}{\partial y^\ell} \int dx\, \mathcal{L}(y,x)\frac{x^k}{k!}\Big|_{y=0}\,. \tag{16.11}$$

in a remark: credit Rugh [37.41] for this matrix rep

we obtain a matrix representation of the evolution operator

$$\int dx\, \mathcal{L}(y,x) \frac{x^k}{k!} = \sum_{k'} \frac{y^{k'}}{k'!} \mathbf{L}_{k'k}\,, \qquad k,k' = 0,1,2,\ldots$$

which maps the x^k component of the density of trajectories $\phi_n(x)$ into the $y^{k'}$ component of the density $\phi_{n+1}(y)$ one time step later, with $y = f(x)$.

We already have some practice with evaluating derivatives $\delta^{(\ell)}(y) = \frac{\partial^\ell}{\partial y^\ell}\delta(y)$ from sect. 9.2. This yields a representation of the evolution operator centered on the fixed point, evaluated recursively in terms of derivatives of the map f:

$$\begin{aligned} (\mathbf{L})_{\ell k} &= \int dx\, \delta^{(\ell)}(x - f(x)) \frac{x^k}{k!}\bigg|_{x=f(x)} \\ &= \frac{1}{|f'|}\left(\frac{d}{dx}\frac{1}{f'(x)}\right)^{\ell} \frac{x^k}{k!}\bigg|_{x=f(x)} . \end{aligned} \tag{16.12}$$

The matrix elements vanish for $\ell < k$, so $\mathbf{L}$ is a lower triangular matrix. The diagonal and the successive off-diagonal matrix elements are easily evaluated iteratively by computer algebra

$$\mathbf{L}_{kk} = \frac{1}{|\Lambda|(\Lambda)^k}\,, \quad \mathbf{L}_{k+1,k} = -\frac{(k+2)!f''}{2k!|\Lambda|(\Lambda)^{k+2}}\,, \quad \cdots .$$

For chaotic systems the map is expanding, $|\Lambda| > 1$. Hence the diagonal terms drop off exponentially, as $1/|\Lambda|^{k+1}$, the terms below the diagonal fall off even faster, and truncating $\mathbf{L}$ to a finite matrix introduces only exponentially small errors.

The trace formula (16.3) takes now a matrix form

$$\mathrm{tr}\,\frac{z\mathcal{L}}{1 - z\mathcal{L}} = \mathrm{tr}\,\frac{\mathbf{L}}{1 - z\mathbf{L}}\,. \tag{16.13}$$

In order to illustrate how this works, we work out a simple example.

Example 16.7 Perron-Frobenius operator in a matrix representation: *As in example 16.1, we start with a map with a single fixed point, but this time with a nonlinear piecewise analytic map f with a nonlinear inverse $F = f^{-1}$, sign of the derivative $\sigma = \sigma(F') = F'/|F'|$, and the Perron-Frobenius operator acting on densities analytic in an open domain enclosing the fixed point $x = w^*$,*

compare to the Dettmann? papers

rewrite in Cauchy int form

$$\mathcal{L}\phi(y) = \int dx\, \delta(y - f(x))\, \phi(x) = \sigma\, F'(y)\, \phi(F(y))\,.$$

Assume that F is a contraction of the unit disk in the complex plane, that is,

$$|F(z)| < \theta < 1 \quad \textit{and} \quad |F'(z)| < C < \infty \quad \textit{for} \quad |z| < 1\,, \tag{16.14}$$

spectral!determinant!entire
exponential!convergence
convergence!super-exponential

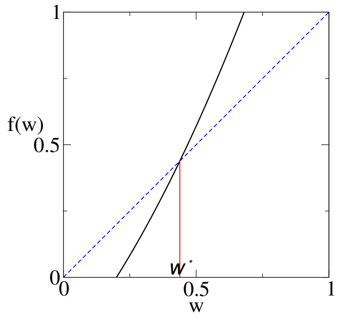

Figure 16.1: A nonlinear one-branch repeller with a single fixed point w^*.

and expand ϕ in a polynomial basis by means of the Cauchy formula

$$\phi(z) = \sum_{n\geq 0} z^n \phi_n = \oint \frac{dw}{2\pi i}\, \frac{\phi(w)}{w-z}\,, \quad \phi_n = \oint \frac{dw}{2\pi i}\, \frac{\phi(w)}{w^{n+1}}$$

forward reference (16.22)

Combining this with (16.22), we see that in this basis Perron-Frobenius operator $\mathcal{L}$ is represented by the matrix

$$\mathcal{L}\phi(w) = \sum_{m,n} w^m L_{mn} \phi_n\,, \quad L_{mn} = \oint \frac{dw}{2\pi i}\, \frac{\sigma\, F'(w)(F(w))^n}{w^{m+1}}\,. \qquad (16.15)$$

Taking the trace and summing we get:

$$\mathrm{tr}\ \mathcal{L} = \sum_{n\geq 0} L_{nn} = \oint \frac{dw}{2\pi i}\, \frac{\sigma\, F'(w)}{w - F(w)}\,.$$

do it for several poles

This integral has but one simple pole at the unique fixed point $w^* = F(w^*) = f(w^*)$. Hence

16.6
page 303

$$\mathrm{tr}\ \mathcal{L} = \frac{\sigma\, F'(w^*)}{1 - F'(w^*)} = \frac{1}{|f'(w^*) - 1|}\,.$$

This super-exponential decay of cummulants Q_k ensures that for a repeller consisting of a single repelling point the spectral determinant (16.4) is *entire* in the complex z plane.

In retrospect, the matrix representation method for solving the density evolution problems is eminently sensible — after all, that is the way one solves a close relative to classical density evolution equations, the Schrödinger equation. *When* available, matrix representations for $\mathcal{L}$ enable us to compute many more orders of cumulant expansions of spectral determinants and many more eigenvalues of evolution operators than the cycle expansions approach.

RM: The formula for product of zetas is missing

Now, if the spectral determinant is entire, formulas such as (15.26) imply that the dynamical zeta function is a meromorphic function. The practical import of this observation is that it guarantees that finite order estimates of zeroes of dynamical zeta functions and spectral determinants converge exponentially, or - in cases such as (16.4) - super-exponentially to the exact values, and so the cycle expansions to be discussed in chapter 18 represent a *true perturbative* approach to chaotic dynamics.

finite!matrix truncations
matrix!truncations

ELIMINARY

This result says that the single fixed-point piecewise-linear map example (16.2) applies to any analytic 1-d map with a single fixed point, with eigenfunctions $\phi_n = y^n$, and in this basis $\mathcal{L}$ diagonal, $L_{nm} = \delta_{nm}/|\Lambda|\Lambda^{-n}$. Thus we have rederived rigorously the trace formula (14.8) derived heuristically in sect. 14.2, so far only for the case of a 1-d expanding analytic map with a single fixed point. The requirement that map be analytic is needed to substitute bound (16.23) into the contour integral (16.24) in order to bound the $\mathcal{L}$ matrix elements

appendix K.4

$$|L_{mn}| \leq \sup_{|w|\leq 1} |F'(w)|\,|F(w)|^n \leq C\theta^n$$

which shows that finite $[N \times N]$ matrix truncations approximate the operator within an error exponentially small in N. It also follows that eigenvalues fall off as θ^n. In higher dimension similar considerations show that the entries in the matrix fall off as $1/\Lambda^{k^{1+1/d}}$, and eigenvalues as $1/\Lambda^{k^{1/d}}$.

When considering analytic maps

The size of such matrices is infinite, but if the matrix entries decay exponentially fast with the index size, calculations based on finite matrix truncations have exponentially small errors.

Furthermore, from bounds on the elements L_{mn} one calculates bounds on the cummulants (??) tr $\left(\wedge^k \mathcal{L}\right)$ and verifies that they fall off as $\Lambda^{-k^2/2}$, concluding that the $\mathcal{L}$ eigenvalues fall off exponentially for a general Axiom A 1-d map.

16.3 Eigenfunctions, an attractive fixed point

We have been hinting darkly all along that - in contrast to the more familiar quantum evolution operators acting on Hilbert spaces - evolution operators are in general *not* self-adjoint operators. For a finite-dimensional matrix lack of self-adjointness implies that the matrix cannot be brought to a diagonal form, and that its left and right eigenvectors are not (complex conjugate) transposes of each other.

For the infinite-dimensional evolution operators the implications are stark. Consider a single fixed point as an example. For a repelling fixed point the action of the Perron-Frobenius operator is to expand and smooth out the initial distribution, and hence the natural basis for densities an expanding maps evolves are smooth functions, such as polynomials: the *right* eigenfunctions are smooth.

Consider now the action of the Perron-Frobenius operator for the same map backward in time. Now the fixed point is attractive, densities get more

Fredholm theory
Fredholm!integral operator
⇑PRELIMINARY
Fredholm!integral equations
Hilbert-Schmidt! condition

contracted and kinkier with time, concentrating asymptotical on the fixed point itself; the *left* eigenfunctions are highly singular.

Before turning to specifics we summarize a few facts about classical theory of integral equations, something you might prefer to skip on first reading. The purpose of this exercise is to understand that the Fredholm theory, a theory that works so well for the Hilbert spaces of quantum mechanics does not necessarily work for deterministic dynamics - the ergodic theory is much harder.

fast track:
sect. 16.5, p. 288

16.4 Classical Fredholm theory

He who would valiant be
'Gainst all disaster
Let him in constancy
Follow the Master.
John Bunyan, *Pilgrim's Progress*

The Perron-Frobenius operator

$$\mathcal{L}\phi(x) = \int dy\, \delta(x - f(y))\, \phi(y)$$

has the same appearance as a classical Fredholm integral operator

$$\mathcal{K}\varphi(x) \;=\; \int_{\mathcal{M}} dy\, \mathcal{K}(x,y)\varphi(y)\,, \tag{16.16}$$

and one is tempted to resort to the classical Fredholm theory in order to establish analyticity properties of spectral determinants. This path to enlightenment is blocked by the singular nature of the kernel, which is a distribution, whereas the standard theory of integral equations usually concerns itself with regular kernels $\mathcal{K}(x,y) \in L^2(\mathcal{M}^2)$. Here we briefly recall some steps of the Fredholm theory, before working out the example of example 16.5.

The general form of Fredholm integral equations of the second kind is

$$\varphi(x) \;=\; \int_{\mathcal{M}} dy\, \mathcal{K}(x,y)\varphi(y) \;+\; \xi(x) \tag{16.17}$$

where $\xi(x)$ is a given function in $L^2(\mathcal{M})$ and the kernel $\mathcal{K}(x,y) \in L^2(\mathcal{M}^2)$ (Hilbert-Schmidt condition). The natural object to study is then the linear integral operator (16.16), acting on the Hilbert space $L^2(\mathcal{M})$: and the fundamental property that follows from the $L^2(Q)$ nature of the kernel is that such an operator is *compact*, that is close to a finite rank operator (see appendix M). A compact operator has the property that for every $\delta > 0$ only a *finite* number of linearly independent eigenvectors exist corresponding to eigenvalues whose absolute value exceeds δ, so we immediately realize (figure 16.4) that much work is needed to bring Perron-Frobenius operators into this picture.

resolvent!kernel
kernel!resolving

We rewrite (16.17) in the form

$$\mathcal{T}\varphi = \xi\,, \qquad \mathcal{T} = \mathbb{1} - \mathcal{K}\,. \tag{16.18}$$

The Fredholm alternative is now stated as follows: the equation $\mathcal{T}\varphi = \xi$ as a unique solution for every $\xi \in L^2(\mathcal{M})$ **or** there exists a non-zero solution of $\mathcal{T}\varphi_0 = 0$, with an eigenvector of $\mathcal{K}$ corresponding to the eigenvalue 1.

The theory remains the same if instead of $\mathcal{T}$ we consider the operator $\mathcal{T}_\lambda = \mathbb{1} - \lambda\mathcal{K}$ with $\lambda \neq 0$. As $\mathcal{K}$ is a compact operator there will be at most a denumerable set of λ for which the second part of Fredholm alternative holds: so apart from this set the inverse operator $(\mathbb{1} - \lambda\mathcal{T})^{-1}$ exists and is a bounded operator. When λ is sufficiently small we may look for a perturbative expression for such an inverse, as a geometric series

$$(\mathbb{1} - \lambda\mathcal{K})^{-1} = \mathbb{1} + \lambda\mathcal{K} + \lambda^2\mathcal{K}^2 + \cdots = \mathbb{1} + \lambda\mathcal{W}\,, \tag{16.19}$$

where each $\mathcal{K}^n$ is still a compact integral operator with kernel

$$\mathcal{K}^n(x,y) = \int_{\mathcal{M}^{n-1}} dz_1 \ldots dz_{n-1}\, \mathcal{K}(x,z_1)\cdots\mathcal{K}(z_{n-1},y)\,,$$

and $\mathcal{W}$ is also compact, as it is given by the convergent sum of compact operators. The problem with (16.19) is that the series has a finite radius of convergence, while apart from a denumerable set of λ's the inverse operator is well defined. A fundamental result in the theory of integral equations consists in rewriting the resolving kernel $\mathcal{W}$ as a ratio of two *analytic* functions of λ

$$\mathcal{W}(x,y) = \frac{\mathcal{D}(x,y;\lambda)}{D(\lambda)}.$$

If we introduce the notation

$$\mathcal{K}\begin{pmatrix} x_1 \ldots x_n \\ y_1 \ldots y_n \end{pmatrix} = \begin{vmatrix} \mathcal{K}(x_1,y_1) & \ldots & \mathcal{K}(x_1,y_n) \\ \ldots & \ldots & \ldots \\ \mathcal{K}(x_n,y_1) & \ldots & \mathcal{K}(x_n,y_n) \end{vmatrix}$$

essential!spectrum

we may write the explicit expressions

$$
\begin{aligned}
D(\lambda) &= 1+\sum_{n=1}^{\infty}(-1)^n\frac{\lambda^n}{n!}\int_{\mathcal{M}^n} dz_1\ldots dz_n\,\mathcal{K}\left(\begin{array}{c} z_1\ldots z_n \\ z_1\ldots z_n\end{array}\right) \\
&= \exp\left(-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\mathrm{tr}\,\mathcal{K}^m\right) \qquad (16.20) \\
\mathcal{D}(x,y;\lambda) &= \mathcal{K}\left(\begin{array}{c} x \\ y\end{array}\right)+\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int_{\mathcal{M}^n} dz_1\ldots dz_n\,\mathcal{K}\left(\begin{array}{cccc} x & z_1 & \ldots & z_n \\ y & z_1 & \ldots & z_n\end{array}\right)
\end{aligned}
$$

$D(\lambda)$ is known as the Fredholm determinant (see (15.25) and appendix M): it is an entire analytic function of λ, and $D(\lambda) = 0$ only if $1/\lambda$ is an eigenvalue of $\mathcal{K}$.

Worth emphasizing again: the Fredholm theory is based on the compactness of the integral operator, that is, on the functional properties (summability) of its kernel.

16.5 Analyticity of spectral determinants

> They savored the strange warm glow of being much more ignorant than ordinary people, who were only ignorant of ordinary things.
> Terry Pratchett

Spaces of functions integrable L^1, or square-integrable L^2 on interval $[0,1]$ are mapped into themselves by the Perron-Frobenius operator, and in both cases the constant function $\phi_0 \equiv 1$ is an eigenfunction with eigenvalue 1. If we focus our attention on L^1 we also have a family of L^1 eigenfunctions,

$$
\phi_\theta(y) = \sum_{k\neq 0}\exp(2\pi i k y)\frac{1}{|k|^\theta} \qquad (16.21)
$$

with complex eigenvalue $2^{-\theta}$, parametrized by complex θ with Re $\theta > 0$. By varying θ one realizes that such eigenvalues fill out the entire unit disk. Such *essential spectrum*, the case $k = 0$ of figure 16.4, hides all fine details of the spectrum.

What's going on? Spaces L^1 and L^2 contain arbitrarily ugly functions, allowing any singularity as long as it is (square) integrable - and there is no way that expanding dynamics can smooth a kinky function with a non-differentiable singularity, let's say a discontinuous step, and that is why the eigenspectrum is dense rather than discrete. Mathematicians love to wallow in this kind of muck, but there is no way for an experimental physicist to prepare a nowhere differentiable L^1 initial density. The only thing we can prepare and measure are piecewise smooth (real-analytic) density functions.

move this somewhere; define Banach space

For a bounded linear operator $\mathcal{A}$ on a Banach space $\mathbf{\Omega}$, the spectral radius is the smallest positive number ρ_{spec} such the spectrum is inside the disk of radius ρ_{spec}, while the essential spectral radius is the smallest positive number ρ_{ess} such that outside the disk of radius ρ_{ess} the spectrum consists only of isolated eigenvalues of finite multiplicity (see figure 16.4).

spectral!radius
essential!spectral radius

16.5
page 303

We may shrink the essential spectrum by letting the Perron-Frobenius operator act on a space of smoother functions, exactly as in the one-branch repeller case of sect. 16.1. We thus consider a smaller space, $\mathbb{C}^{k+\alpha}$, the space of k times differentiable functions whose k'th derivatives are Hölder continuous with an exponent $0 < \alpha \leq 1$: the expansion property guarantees that such a space is mapped into itself by the Perron-Frobenius operator. In the strip $0 < \mathrm{Re}\,\theta < k+\alpha$ most ϕ_θ will cease to be eigenfunctions in the space $\mathbb{C}^{k+\alpha}$; the function ϕ_n survives only for integer valued $\theta = n$. In this way we arrive at a finite set of *isolated* eigenvalues $1, 2^{-1}, \cdots, 2^{-k}$, and an essential spectral radius $\rho_{ess} = 2^{-(k+\alpha)}$.

this is not Bernuolli specific

We follow a simpler path and restrict the function space even further, namely to a space of analytic functions, that is, functions for which the Taylor expansion is convergent at each point of the interval $[0, 1]$. With this choice things turn out easy and elegant. To be more specific let ϕ be a holomorphic and bounded function on the disk $D = B(0, R)$ of radius $R > 0$ centered at the origin. Our Perron-Frobenius operator preserves the space of such functions provided $(1 + R)/2 < R$ so all we need is to choose $R > 1$. If F_s, $s \in \{0, 1\}$, denotes the s inverse branch of the Bernoulli shift (16.6), the corresponding action of the Perron-Frobenius operator is given by $\mathcal{L}_s h(y) = \sigma\ F_s'(y)\ h \circ F_s(y)$, using the Cauchy integral formula along the ∂D boundary contour:

copy Driebe p. 19, draw figure

$$\mathcal{L}_s h(y) = \sigma \oint \frac{dw}{2\pi i}\,_{\partial D} \frac{h(w)F_s'(y)}{w - F_s(y)} . \tag{16.22}$$

For reasons that will be made clear later we have introduced a sign $\sigma = \pm 1$ of the given real branch $|F'(y)| = \sigma\, F'(y)$. For both branches of the Bernoulli shift $s = 1$, but in general one is not allowed to take absolute values as this could destroy analyticity. In the above formula one may also replace the domain D by *any domain* containing $[0, 1]$ such that the inverse branches maps the closure of D into the interior of D. Why? simply because the kernel stays non-singular under this condition, that is, $w - F(y) \neq 0$ whenever $w \in \partial D$ and $y \in \mathrm{Cl}\, D$.

The problem is now reduced to the standard theory for Fredholm determinants, sect. 16.4. The integral kernel is no longer singular, traces and determinants are well-defined, and we can evaluate the trace of $\mathcal{L}_F$ by means of the Cauchy contour integral formula:

$$\mathrm{tr}\,\mathcal{L}_F = \oint \frac{dw}{2\pi i}\, \frac{\sigma F'(w)}{w - F(w)} .$$

Elementary complex analysis shows that since F maps the closure of D into its own interior, F has a unique (real-valued) fixed point x^* with a multiplier strictly smaller than one in absolute value. Residue calculus therefore yields

16.6
page 303

$$\mathrm{tr}\,\mathcal{L}_F = \frac{\sigma F'(x^*)}{1-F'(x^*)} = \frac{1}{|f'(x^*)-1|}\,,$$

justifying our previous *ad hoc* calculations of traces by means of Dirac delta functions.

compare to the Dettmann? papers

rewrite in Cauchy int form

Example 16.8 Perron-Frobenius operator in a matrix representation: *As in example 16.1, we start with a map with a single fixed point, but this time with a nonlinear piecewise analytic map f with a nonlinear inverse $F = f^{-1}$, sign of the derivative $\sigma = \sigma(F') = F'/|F'|$*

$$\mathcal{L}\phi(z) = \int dx\, \delta(z - f(x))\, \phi(x) = \sigma\, F'(z)\, \phi(F(z))\,.$$

Assume that F is a contraction of the unit disk, that is,

$$|F(z)| < \theta < 1 \quad \textit{and} \quad |F'(z)| < C < \infty \quad \textit{for} \quad |z| < 1\,, \tag{16.23}$$

and expand ϕ in a polynomial basis by means of the Cauchy formula

$$\phi(z) = \sum_{n\geq 0} z^n \phi_n = \oint \frac{dw}{2\pi i}\,\frac{\phi(w)}{w-z}\,,\qquad \phi_n = \oint \frac{dw}{2\pi i}\,\frac{\phi(w)}{w^{n+1}}$$

Combining this with (16.22), we see that in this basis $\mathcal{L}$ is represented by the matrix

$$\mathcal{L}\phi(w) = \sum_{m,n} w^m L_{mn}\phi_n\,,\quad L_{mn} = \oint \frac{dw}{2\pi i}\,\frac{\sigma\, F'(w)(F(w))^n}{w^{m+1}}\,. \tag{16.24}$$

Taking the trace and summing we get:

$$\mathrm{tr}\;\mathcal{L} = \sum_{n\geq 0} L_{nn} = \oint \frac{dw}{2\pi i}\,\frac{\sigma\, F'(w)}{w - F(w)}\,.$$

do it for several poles

This integral has but one simple pole at the unique fixed point $w^* = F(w^*) = f(w^*)$. Hence

16.6
page 303

$$\mathrm{tr}\;\mathcal{L} = \frac{\sigma\, F'(w^*)}{1-F'(w^*)} = \frac{1}{|f'(w^*)-1|}\,.$$

the Bernoulli continues here

We worked out a very specific example, yet our conclusions can be generalized, provided a number of restrictive requirements are met by the dynamical system under investigation:

1) the evolution operator is *multiplicative* along the flow,
2) the symbolic dynamics is a *finite subshift*,
3) all cycle eigenvalues are *hyperbolic* (exponentially bounded away from 1),
4) the map (or the flow) is *real analytic*, that is, it has a piecewise analytic continuation to a complex extension of the phase space.

These assumptions are romantic projections not lived up to by the dynamical systems that we actually desire to understand. Still, they are not devoid of physical interest; for example, nice repellers like our 3-disk game of pinball do satisfy the above requirements.

three-disk@3-disk!convergence

Properties 1 and 2 enable us to represent the evolution operator as a matrix in an appropriate basis space; properties 3 and 4 enable us to bound the size of the matrix elements and control the eigenvalues. To see what can go wrong consider the following examples:

Property 1 is violated for flows in 3 or more dimensions by the following weighted evolution operator

$$\mathcal{L}^t(y,x) = |\Lambda^t(x)|^\beta \delta\big(y - f^t(x)\big) \ ,$$

where $\Lambda^t(x)$ is an eigenvalue of the Jacobian matrix transverse to the flow. Semiclassical quantum mechanics suggest operators of this form with $\beta = 1/2$, (see chapter 33). The problem with such operators is due to the fact that when considering the Jacobian matrices $\mathbf{J}_{ab} = \mathbf{J}_a \mathbf{J}_b$ for two successive trajectory segments a and b, the corresponding eigenvalues are in general *not* multiplicative, $\Lambda_{ab} \neq \Lambda_a \Lambda_b$ (unless a, b are repeats of the same prime cycle p, so $\mathbf{J}_a \mathbf{J}_b = \mathbf{J}_p^{r_a + r_b}$). Consequently, this evolution operator is not multiplicative along the trajectory. The theorems require that the evolution be represented as a matrix in an appropriate polynomial basis, and thus cannot be applied to non-multiplicative kernels, that is, kernels that do not satisfy the semi-group property $\mathcal{L}^{t'}\mathcal{L}^t = \mathcal{L}^{t'+t}$. Cure for this problem in this particular case is given in appendix I.1.

Property 2 is violated by the 1-d tent map (see figure 16.2 (a))

$$f(x) = \alpha(1 \ - \ |1 - 2x|) \ , \quad 1/2 < \alpha < 1 \ .$$

All cycle eigenvalues are hyperbolic, but in general the critical point $x_c = 1/2$ is not a pre-periodic point, there is no finite Markov partition and the symbolic dynamics does not have a finite grammar (see sect. 12.4 for definitions). In practice this means that while the leading eigenvalue of $\mathcal{L}$ might be computable, the rest of the spectrum is very hard to control; as the parameter α is varied, non-leading zeros of the spectral determinant move wildly about.

Property 3 is violated by the map (see figure 16.2 (b))

$$f(x) = \left\{ \begin{array}{lll} x + 2x^2 & , & x \in I_0 = [0, \frac{1}{2}] \\ 2 - 2x & , & x \in I_1 = [\frac{1}{2}, 1] \end{array} \right. .$$

Here the interval $[0,1]$ has a Markov partition into the two subintervals I_0 and I_1; f is monotone on each. However, the fixed point at $x = 0$ has marginal stability $\Lambda_0 = 1$, and violates the condition 3. This type of map

intermittency
marginal!stability
doubling map

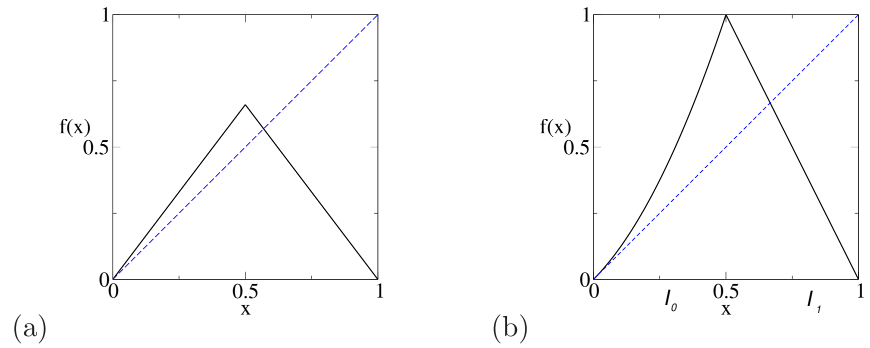

Figure 16.2: (a) A (hyperbolic) tent map without a finite Markov partition. (b) A Markov map with a marginal fixed point.

is called intermittent and necessitates much extra work. The problem is that the dynamics in the neighborhood of a marginal fixed point is very slow, with correlations decaying as power laws rather than exponentially. We will discuss such flows in chapter 21.

The property 4 is required as the heuristic approach of chapter 14 faces two major hurdles:

1. The trace (14.7) is not well defined since the integral kernel is singular.
2. The existence and properties of eigenvalues are by no means clear.

Actually this property is quite restrictive, but we need it in the present approach, in order that the Banach space of analytic functions in a disk is preserved by the Perron-Frobenius operator.

In attempting to generalize the results we encounter several problems. First, in higher dimensions life is not as simple. Multi-dimensional residue calculus is at our disposal but in general requires that we find poly-domains (direct product of domains in each coordinate) and this need not be the case. Second, and perhaps somewhat surprisingly, the 'counting of periodic orbits' presents a difficult problem. For example, instead of the Bernoulli shift consider the doubling map of the circle, $x \mapsto 2x \bmod 1$, $x \in R/Z$. Compared to the shift on the interval $[0,1]$ the only difference is that the endpoints 0 and 1 are now glued together. But since these endpoints are fixed points of the map the number of cycles of length n decreases by 1. The determinant becomes:

add a cat-map exercise

$$\det(1 - z\mathcal{L}) = \exp\left(-\sum_{n=1} \frac{z^n}{n}\,\frac{2^n - 1}{2^n - 1}\right) = 1 - z. \tag{16.25}$$

The value $z = 1$ still comes from the constant eigenfunction but the Bernoulli polynomials no longer contribute to the spectrum (they are not periodic). Proofs of these facts, however, are difficult if one sticks to the space of analytic functions.

Third, our Cauchy formulas *a priori* work only when considering purely expanding maps. When stable and unstable directions co-exist we have to resort to stranger function spaces, as shown in the next section.

16.6 Hyperbolic maps

(H.H. Rugh)

Moving onto hyperbolic systems, one faces the following paradox: If f is an area-preserving hyperbolic and real-analytic map of, for example, a 2-dimensional torus then the Perron-Frobenius operator is unitary on the space of L^2 functions, and its spectrum is confined to the unit circle. On the other hand, when we compute determinants we find eigenvalues scattered around inside the unit disk. Thinking back on our Bernoulli shift example one would like to imagine these eigenvalues as popping up from the L^2 spectrum by shrinking the function space. Shrinking the space, however, can only make the spectrum smaller so this is obviously not what happens. Instead one needs to introduce a 'mixed' function space where in the unstable direction one resort to analytic functions, as before, but in the stable direction one considers a 'dual space' of distributions on analytic functions. Such a space is neither included in nor does it include the L^2-space and we have thus resolved the paradox. But it still remains to be seen how traces and determinants are calculated.

The linear hyperbolic fixed point example example 16.6 is somewhat misleading, as we have made explicit use of a map that acts independently along the stable and unstable directions. For a more general hyperbolic map, there is no way to implement such direct product structure, and the whole argument falls apart. Her comes an idea; use the analyticity of the map to rewrite the Perron-Frobenius operator acting as follows (σ is the sign of the derivative in the unstable direction):

$$\mathcal{L}h(z_1, z_2) = \oint \oint \frac{\sigma\ h(w_1, w_2)}{(z_1 - f_1(w_1, w_2)(f_2(w_1, w_2) - z_2)} \frac{dw_1}{2\pi i} \frac{dw_2}{2\pi i}. \tag{16.26}$$

Here the function ϕ should belong to a space of functions analytic respectively *outside* a disk and *inside* a disk in the first and the second coordinate, with the additional property that the function decays to zero as the first coordinate tends to infinity. The contour integrals are along the boundaries of these disks. It is but an exercise in multi-dimensional residue calculus to verify that for the above linear example this expression reduces to (16.9). Such operators form the building bricks in the calculation of traces and determinants and one is able to prove the following:

Theorem: *The spectral determinant for hyperbolic analytic maps is entire.*

☞ remark 16.10

The proof, apart from the Markov property which is the same as for the purely expanding case, relies heavily on analyticity of the map in the

Axiom A!systems

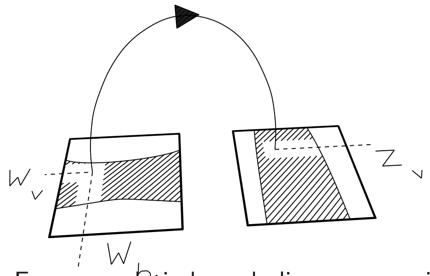

Figure 16.3: For an analytic hyperbolic map, specifying the contracting coordinate w_h at the initial rectangle and the expanding coordinate z_v at the image rectangle defines a unique trajectory between the two rectangles. In particular, w_v and z_h (not shown) are uniquely specified.

explicit construction of the function space. As we have also seen in the previous example the basic idea is to view the hyperbolicity as a cross product of a contracting map in the forward time and another contracting map in the backward time. In this case the Markov property introduced above has to be elaborated a bit. Instead of dividing the phase space into intervals, one divides it into rectangles. The rectangles should be viewed as a direct product of intervals (say horizontal and vertical), such that the forward map is contracting in, for example, the horizontal direction, while the inverse map is contracting in the vertical direction. For Axiom A systems (see remark 16.10) one may choose coordinate axes close to the stable/unstable manifolds of the map. With the phase space divided into N rectangles $\{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_N\}$, $\mathcal{M}_i = I_i^h \times I_i^v$ one needs complex extension $D_i^h \times D_i^v$, with which the hyperbolicity condition (which at the same time guarantees the Markov property) can be formulated as follows:

Analytic hyperbolic property: Either $f(\mathcal{M}_i) \cap \mathrm{Int}(\mathcal{M}_j) = \emptyset$, or for each pair $w_h \in \mathrm{Cl}(D_i^h)$, $z_v \in \mathrm{Cl}(D_j^v)$ there exist unique analytic functions of w_h, z_v: $w_v = w_v(w_h, z_v) \in \mathrm{Int}(D_i^v)$, $z_h = z_h(w_h, z_v) \in \mathrm{Int}(D_j^h)$, such that $f(w_h, w_v) = (z_h, z_v)$. Furthermore, if $w_h \in I_i^h$ and $z_v \in I_j^v$, then $w_v \in I_i^v$ and $z_h \in I_j^h$ (see figure 16.3).

What this means for the iterated map is that one replaces coordinates z_h, z_v at time n by the contracting pair z_h, w_v, where w_v is the contracting coordinate at time $n+1$ for the 'partial' inverse map.

In two dimensions the operator in (16.26) acts on functions analytic outside D_i^h in the horizontal direction (and tending to zero at infinity) and inside D_i^v in the vertical direction. The contour integrals are precisely along the boundaries of these domains.

A map f satisfying the above condition is called analytic hyperbolic and the theorem states that the associated spectral determinant is entire, and that the trace formula (14.7) is correct.

Examples of analytic hyperbolic maps are provided by small analytic perturbations of the cat map (where the Markov partitioning is non-trivial [?]), the 3-disk repeller, and the 2-d baker's map.

Perron-Frobenius!theorem
natural measure
check linear horseshoes in Nitecki book

16.7 Physics of eigenvalues and eigenfunctions

We appreciate by now that any honest attempt to look at spectral properties of the Perron-Frobenius operator involves hard mathematics: but the effort is rewarded by the fact that we are finally able to control analyticity properties of dynamical zeta functions and spectral determinants, and thus substantiate the claim that these objects provide a powerful and well founded perturbation theory.

Often (see chapter 10) the physically important part of the spectrum is just the leading eigenvalue, which gives us the escape rate from a repeller, or, for a general evolution operator, formulas for expectation values of observables and their higher moments. Also the eigenfunction associated to the leading eigenvalue has a physical interpretation (see chapter 9): it is the density of the natural measure, with singular measures ruled out by the proper choice of the function space. Such a conclusion is coherent with a the validity of a generalized Perron-Frobenius theorem for evolution operators. In the finite dimensional setting such theorem is formulated as follows:

replace "coherent"

☞ remark 16.8

- **Perron-Frobenius theorem:** let L_{nm} be a nonnegative matrix, such that some n exists for which $(L^n)_{ij} > 0 \ \forall i, j$: then
  1. the maximal modulus eigenvalue is non degenerate, real and positive
  2. the corresponding eigenvector (defined up to a constant) has nonnegative coordinates

We may ask what physical information is contained in eigenvalues beyond the leading one: suppose that we have a probability conserving system (so that the dominant eigenvalue is 1), for which the essential spectral radius is such that $0 < \rho_{ess} < \theta < 1$ on some Banach space $\mathcal{B}$ and denote by $\mathbf{P}$ the projection corresponding to the part of the spectrum inside a disk of radius θ. We denote by $\lambda_1, \lambda_2 \ldots \lambda_M$ the eigenvalues outside of this disk, ordered by the size of their absolute value (so that $\lambda_1 = 1$). Then we have the following decomposition

$$\mathcal{L}\varphi = \sum_{i=1}^{M} \lambda_i \psi_i L_i \psi_i^* \varphi + \mathbf{P}\mathcal{L}\varphi \tag{16.27}$$

when L_i are (finite) matrices in Jordan normal form ($L_0 = 0$ is a $[1 \times 1]$ matrix, as λ_0 is simple, due to Perron-Frobenius theorem), while ψ_i is a

correlation!function
correlation!spectrum

row vector whose elements are a basis on the eigenspace corresponding to λ_i, and ψ_i^* is a column vector of elements of $\mathcal{B}^*$ (the dual space, of linear functionals over $\mathcal{B}$) spanning the eigenspace of $\mathcal{L}^*$ corresponding to λ_i. For iterates of Perron-Frobenius operator (16.27) becomes

$$\mathcal{L}^n \varphi = \sum_{i=1}^{M} \lambda_i^n \psi_i L_i^n \psi_i^* \varphi + \mathbf{P}\mathcal{L}^n \varphi \tag{16.28}$$

If we now consider expressions like

$$\langle \xi | \mathcal{L}^n | \varphi \rangle = \int_{\mathcal{M}} dy\, \xi(y) \left(\mathcal{L}^n \varphi\right)(y) = \int_{\mathcal{M}} dw\, (\xi \circ f^n)(w) \varphi(w) \tag{16.29}$$

we have

$$C_{\xi,\varphi}^{(n)} = \lambda_1^n \omega_1(\xi,\varphi) + \sum_{i=2}^{L} \lambda_i^n \omega_i^{(n)}(\xi,\varphi) + \mathcal{O}(\theta^n) \tag{16.30}$$

where

$$\omega_i^{(n)}(\xi,\varphi) = \int_{\mathcal{M}} dy\, \xi(y) \psi_i L_i^n \psi_i^* \varphi$$

In this way we see how eigenvalues beyond the leading one provide a twofold piece of information: they rule the convergence of expressions containing high powers of evolution operator to the leading order (the λ_1 contribution). Moreover if $\omega_1(\xi,\varphi) = 0$ then (16.29) defines a correlation function: as each term in (16.30) vanishes exponentially in the $n \to \infty$ limit, the eigenvalues $\lambda_2, \ldots \lambda_M$ rule the exponential decay of correlations for our dynamical system. We observe that prefactors ω depend on the choice of functions, while the exponential decay rates (logarithms of λ_i) do not: the correlation spectrum is thus an *universal* property of the dynamics (once we fix the overall functional space our Perron-Frobenius operator acts on).

16.7
page 303

Example 16.9 Bernoulli shift eigenfunctions: *So let us come back the Bernoulli shift example (16.6), on the space of analytic functions on a disk: apart from the origin we have only simple eigenvalues $\lambda_k = 2^{-k}$ $k = 0, 1, \ldots$. The eigenvalue $\lambda_0 = 1$ corresponds to probability conservation: the corresponding eigenfunction $B_0(x) = 1$ indicates that the natural measure has a constant density over the unit interval. If we now take any analytic function $\eta(x)$ with zero average (with respect to the Lebesgue measure), we have that $\omega_1(\eta,\eta) = 0$, and from (16.30) we have that the asymptotic decay of correlation function is (unless also $\omega_1(\eta,\eta) = 0$)*

$$C_{\eta,\eta}(n) \sim \exp(-n \log 2) \tag{16.31}$$

thus $-\log \lambda_1$ gives the exponential decay rate of correlations (with a prefactor that depends on the choice of the function). Actually the Bernoulli shift case may be treated *exactly, as for analytic functions we can employ the Euler-MacLaurin summation formula*

exponential!decay rate of correlations
decay!rate of correlations
Euler-MacLaurin!formula
space!analytic functions
functions!analytic

$$\eta(z) = \int_0^1 dw\, \eta(w) + \sum_{m=1}^{\infty} \frac{\eta^{(m-1)}(1) - \eta^{(m-1)}(0)}{m!} B_m(z)\,. \tag{16.32}$$

As we are considering zero–average functions, we have from (16.29), and the fact that Bernoulli polynomials are eigenvectors of the Perron-Frobenius operator

$$C_{\eta,\eta}(n) = \sum_{m=1}^{\infty} \frac{(2^{-m})^n(\eta^{(m)}(1) - \eta^{(m)}(0))}{m!} \int_0^1 dz\, \eta(z) B_m(z)\,.$$

The decomposition (16.32) is also useful to make us realize that the linear functionals ψ_i^* are quite singular objects: if we write it as

$$\eta(z) = \sum_{m=0}^{\infty} B_m(z)\, \psi_m^*[\eta]$$

we see that these functionals are of the form

might use Driebe sect 3.5.2 here; also say it as a left eigenstate $\hat{B}_j(x) = \frac{1}{j!}\frac{d^j}{dx^j}$

$$\psi_i^*[\varepsilon] = \int_0^1 dw\, \Psi_i(w)\varepsilon(w)$$

where

$$\Psi_i(w) = \frac{(-1)^{i-1}}{i!} \left(\delta^{(i-1)}(w-1) - \delta^{(i-1)}(w)\right) \tag{16.33}$$

when $i \geq 1$, while $\Psi_0(w) = 1$. Such a representation is only meaningful when the function ε is analytic in $w, w-1$ neighborhoods.

16.8 Troubles ahead

The above discussion confirms that for this simple example formal manipulations with traces and determinants are justified: the Perron-Frobenius operator has isolated eigenvalues, the trace formulas are explicitly verified, and the spectral determinant is an entire function whose zeroes yield the eigenvalues. Real life is harder, as we may appreciate through the following considerations:

- Our discussion tacitly assumed something that is physically entirely reasonable: our evolution operator is acting on the space of analytic functions, that is, we are allowed to represent the initial density $\rho(x)$ by its Taylor expansions in the neighborhoods of periodic points. This is however far from being the only possible choice: mathematicians often work with the function space $\mathbb{C}^{k+\alpha}$, that is, the space of k times differentiable functions whose k'th derivatives are Hölder continuous with an exponent $0 < \alpha \leq 1$: then every y^η with $\mathrm{Re}\,\eta > k$ is an eigenfunction of Perron-Frobenius operator and we have

16.1 page 303

$$\mathcal{L}y^\eta = \frac{1}{|\Lambda|\Lambda^\eta} y^\eta\,, \qquad \eta \in \mathbb{C}\,.$$

essential spectral radius
spectral!radius!essential
functions!L^2 square-integrable
L2@L^2 function space

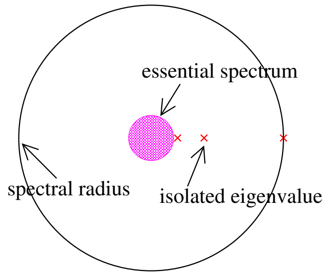

Figure 16.4: Spectrum of the Perron-Frobenius operator acting on the space of $\mathbb{C}^{k+\alpha}$ Hölder-continuous functions: only k isolated eigenvalues remain between the spectral radius, and the essential spectral radius which bounds the "essential", continuous spectrum.

define the essential spectral radius?

This spectrum is quite different from the analytic case: only a small number of isolated eigenvalues remain, enclosed between the spectral radius and a smaller disk of radius $1/|\Lambda|^{k+1}$, see figure 16.4. In literature the radius of this disk is called the *essential spectral radius.*

rewrite!

In sect. 16.5 we will discuss this point further, with the aid of a less trivial 1-dimensional example. The physical point of view is complementary to the standard setting of ergodic theory, where many chaotic properties of a dynamical system are encoded by the presence of a *continuous* spectrum, used to prove asymptotic decay of correlations in the space of L^2 square-integrable functions.

OK in QM, dumb in classical dynamics

16.2
page 303

- A deceptively innocent assumption hides behind many features discussed so far: that (16.1) maps a given function space into itself. This is strictly related to the *expanding* property of the map: if $f(x)$ is smooth in a domain D then $f(x/\Lambda)$ is smooth on a *larger* domain, provided $|\Lambda| > 1$. This is not obviously the case for hyperbolic systems in higher dimensions, and, as we shall see in sect. 16.6, extensions of the results obtained for expanding maps will be highly nontrivial,
- It is not at all clear that the above analysis of a simple one-branch, one fixed point repeller can be extended to dynamical systems with a Cantor set infinity of periodic points: we show that in sect. 16.5.

Commentary

Remark 16.1 Surveys of rigorous theory. We recommend references listed in sect. 1.8 for an introduction into the mathematical literature on this subject. For a physicist, Driebe's monograph [1.26] might be the most accessible introduction into mathematics touched upon in this chapter. There are a number of reviews of the mathematical approach to dynamical zeta functions and spectral determinants, with pointers to the original references, such as refs. [16.1, 16.2]. An alternative approach to spectral properties of the Perron-Frobenius operator is illustrated in ref. [16.3].

The ergodic theory, as presented by Sinai [16.14] and others, tempts one to describe the densities that the evolution operator acts on in terms of either integrable or square-integrable functions. As we have already seen, for our purposes, this space is not suitable. An introduction to ergodic theory is given by Sinai, Kornfeld and Fomin [16.15]; more advanced and more old fashioned presentations are Walters [16.16] and Denker, Grillenberger and Sigmund [16.17]; and a more formal Peterson [16.18].

smooth!dynamics!Spectral determinant
ceiling function

give credit to Prigopeople + ...(/)

Remark 16.2 Fredholm theory. Our brief summary of Fredholm theory is based on the exposition in ref. [16.4]. A technical introduction of the theory from an operatorial point of view is contained in ref. [16.5]. The theory has been generalized in ref. [16.6].

Remark 16.3 Bernoulli shift. For a more detailed discussion, consult chaper 3 of ref. [1.26]. The extension of Fredholm theory to the case or Bernoulli shift on $\mathbb{C}^{k+\alpha}$ (in which the Perron-Frobenius operator is *not* compact – technically it is only *quasi-compact*, that is the essential spectral radius is strictly smaller than the spectral radius) has been given by Ruelle [16.7]: a concise and readable statement of the results is contained in ref. [16.8].

Remark 16.4 Higher dimensions and generalized Fredholm theory. When extending Bernoulli shift to higher dimensions. Extensions of Fredholm theory [16.6] which avoid problems with multi-dimensional residue calculus may be used: see ref. [16.9].

Remark 16.5 Hyperbolic dynamics. When dealing with hyperbolic systems one might try to reduce back to the expanding case by projecting the dynamics along the unstable directions. As mentioned in the text this might be technically quite involved, as usually such the unstable foliation is not characterized by very strong smoothness properties. For such an approach, see ref. [16.3].

Remark 16.6 Spectral determinants for smooth flows. The theorem on page 293 applies also to hyperbolic analytic maps in d dimensions and smooth hyperbolic analytic flows in $(d+1)$ dimensions, provided that the flow can be reduced to a piecewise analytic map by suspension on a Poincaré section complemented by an analytic "ceiling" function (3.4) which accounts for a variation in the section return times. For example, if we take as the ceiling function $g(x) = e^{sT(x)}$, where $T(x)$ is the time of the next Poincaré section for a trajectory staring at x, we reproduce the flow spectral determinant (15.14). Proofs are getting too hard for the purposes of this chapter; details are discussed in ref.(?).

⇓PRIVATE
⇑PRIVATE

Remark 16.7 Explicit diagonalization. For 1-d repellers a diagonalization of an explicit truncated L_{mn} matrix evaluated in a judiciously chosen basis may yield many more eigenvalues than a cycle expansion (see refs. [16.10, 16.11]). The reasons why one persists anyway in using the periodic orbit theory are partially aesthetic, and partially pragmatic. Explicit L_{mn} demands explicit choice of a basis and is thus non-invariant, in contrast to cycle expansions which utilize only the invariant information about the flow. In addition, we usually do not know how to construct L_{mn} for a realistic flow, such as the hyperbolic 3-disk game of pinball flow of sect. 1.3, whereas the periodic orbit formulas are general and straightforward to apply.

Perron-Frobenius!theorem
Axiom A

Remark 16.8 Perron-Frobenius theorem. A proof of the Perron-Frobenius theorem may be found in ref. [16.12]. For positive transfer operators such theorem has been generalized by Ruelle [16.13].

⇓PRELIMINARY
move into the Predrag appendix

Remark 16.9 Fried estimates. The form of the fall-off of the coefficients in the $F(z)$ expansion, as $u^{n^{1+1/d}}$, is in agreement with the estimates of Fried [16.19] for the spectral determinants of d-dimensional expanding flows.

⇑PRELIMINARY

Remark 16.10 Axiom A systems. Proofs outlined in sect. 16.6 follow the thesis work of H.H. Rugh [16.9, 16.19, 37.41]. For mathematical introduction to the subject, consult the excellent review by V. Baladi [16.1]. Rigorous treatment is given in refs. [16.9, 16.19, 37.41]. It would take us too far to give and explain the definition of the Axiom A systems (see refs. [**?**, **?**]). Axiom A implies, however, the existence of a Markov partition of the phase space from which the properties 2 and 3 assumed on page 281 follow.

Remark 16.11 Exponential mixing speed of the Bernoulli shift. We see from (16.31) that for the Bernoulli shift the exponential decay rate of correlations coincides with the Lyapunov exponent: while such an identity holds for a number of systems, it is by no means a general result, and there exist explicit counterexamples.

give references!

Remark 16.12 Left eigenfunctions. We shall never use explicit form of left eigenfunctions, corresponding to highly singular kernels like (16.33). Many details have been elaborated in a number of papers, like ref. [16.21], with a daring physical interpretation.

refer to Dettmann-Vattay

Remark 16.13 Ulam's idea. The approximation of Perron-Frobenius operator defined by (10.37) has been shown to reproduce correctly the spectrum for expanding maps, once finer and finer Markov partitions are used [16.22]. The subtle point of choosing a phase space partitioning for a "generic case" is discussed in ref. [16.23].

Résumé

A theory of evaluation of dynamical averages by means of trace formulas and spectral determinants requires a deeper understanding of their analyticity and convergence. In the case of especially well-behaved "Axiom *A*" systems where both the symbolic dynamics and hyperbolicity are under control, it is possible to treat traces and determinants in a rigorous fashion, and strong results about analyticity properties of dynamical zeta functions and spectral determinants outlined above follow.

⇓PRELIMINARY
this paragraph is already used elsewhere?

Most systems of interest are *not* of the "axiom A" category; they are neither purely hyperbolic nor (as we have seen in chapters 11 and 12 do they have finite grammar. Importance of symbolic dynamics is sometime grossly unappreciated; the crucial ingredient for nice analyticity properties of zeta functions is existence of finite grammar (coupled with uniform hyperbolicity). The dynamical systems that we are *really* interested in - for example, smooth bound Hamiltonian potentials - are presumably never really chaotic, and the central question remains: how to attack the problem in systematic and controllable fashion?

⇑PRELIMINARY

References

[16.1] V. Baladi, *A brief introduction to dynamical zeta functions*, in: DMV-Seminar 27, *Classical Nonintegrability, Quantum Chaos*, A. Knauf and Ya.G. Sinai (eds), (Birkhäuser, 1997).

add Baladi book

[16.2] M. Pollicott, *Periodic orbits and zeta functions*, 1999 AMS Summer Institute on *Smooth ergodic theory and applications*, Seattle (1999), To appear *Proc. Symposia Pure Applied Math., AMS*.

update

[16.3] M. Viana, *Stochastic dynamics of deterministic systems*, (Col. Bras. de Matemática, Rio de Janeiro,1997)

[16.4] A.N. Kolmogorov and S.V. Fomin, *Elements of the theory of functions and functional analysis* (Dover,1999).

[16.5] R.G. Douglas, *Banach algebra techniques in operator theory* (Springer, New York,1998).

[16.6] A. Grothendieck, *La théorie de Fredholm*, *Bull. Soc. Math. France* **84**, 319 (1956).

[16.7] D. Ruelle, *Inst. Hautes Études Sci. Publ. Math.* **72**, 175-193 (1990).

[16.8] V. Baladi, *Dynamical zeta functions*, in B. Branner and P. Hjorth, eds., Proceedings of the NATO ASI *Real and Complex Dynamical Systems* (1993), (Kluwer Academic Publishers, Dordrecht, 1995)

[16.9] D. Ruelle, *Inv. Math.* **34**, 231-242 (1976).

[16.10] F. Christiansen, P. Cvitanović and H.H. Rugh, *J. Phys* **A 23**, L713 (1990).

add Dettmann's 3rd

[16.11] D. Alonso, D. MacKernan, P. Gaspard and G. Nicolis, *Phys. Rev.* **E54**, 2474 (1996).

[16.12] P. Walters, *An introduction to ergodic theory* (Springer, New York 1982).

[16.13] D. Ruelle, *Commun. Math. Phys.* **9**, 267 (1968).

[16.14] Ya.G. Sinai, *Topics in ergodic theory* (Princeton Univ. Press, Princeton 1994).

[16.15] I. Kornfeld, S. Fomin and Ya. Sinai, *Ergodic Theory* (Springer, New York 1982).

[16.16] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Math. Vol **79** (Springer, New York 1982).

[16.17] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces* (Springer Lecture Notes in Math. **470**, 1975).

[16.18] K. Peterson, *Ergodic theory* (Cambridge Univ. Press, Cambridge 1983).

[16.19] D. Fried, *Ann. Scient. Éc. Norm. Sup.* **19**, 491 (1986).

[16.20] H.H. Rugh, *Nonlinearity* **5**, 1237 (1992).

[16.21] H.H. Hasegawa and W.C. Saphir, *Phys. Rev.* **A46**, 7401 (1992).

[16.22] G. Froyland, *Commun. Math. Phys.* **189**, 237 (1997)

[16.23] G. Froyland, *Extracting dynamical behaviour via markov models*, in A. Mees (ed.) *Nonlinear dynamics and statistics: Proceedings Newton Institute, Cambridge 1998* (Birkhäuser, 2000).

update ref. [16.24]

[16.24] V. Baladi, A. Kitaev, D. Ruelle, and S. Semmes, *"Sharp determinants and kneading operators for holomorphic maps"*, IHES preprint (1995).

[16.25] A. Zygmund, *Trigonometric series* (Cambridge Univ. Press, Cambridge 1959).