Exercises

Check the IOP exercises format...

Exercise 13.1 **A transition matrix for 3-disk pinball.**

a) *Draw the Markov graph corresponding to the 3-disk ternary symbolic dynamics, and write down the corresponding transition matrix corresponding to the graph. Show that iteration of the transition matrix results in two coupled linear difference equations, - one for the diagonal and one for the off diagonal elements. (Hint: relate* $\mathrm{tr}\, T^n$ *to* $\mathrm{tr}\, T^{n-1} + \ldots$*.)*

b) *Solve the above difference equation and obtain the number of periodic orbits of length* n*. Compare with table 13.3.*

c) *Find the eigenvalues of the transition matrix* $\mathbf{T}$ *for the 3-disk system with ternary symbolic dynamics and calculate the topological entropy. Compare this to the topological entropy obtained from the binary symbolic dynamics* $\{0, 1\}$*.*

Exercise 13.2 **Sum of A_{ij} is like a trace.** *Let* A *be a matrix with eigenvalues* λ_k*. Show that*

$$\Gamma_n = \sum_{i,j} [A^n]_{ij} = \sum_k c_k \lambda_k^n .$$

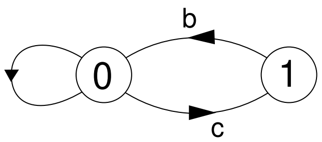
(a) *Use this to show that* $\ln |\mathrm{tr}\, A^n|$ *and* $\ln |\Gamma_n|$ *have the same asymptotic behavior as* $n \to \infty$*, that is, their ratio converges to one.*

(b) *Do eigenvalues* λ_k *need to be distinct,* $\lambda_k \neq \lambda_l$ *for* $k \neq l$*?*

Exercise 13.3 **Loop expansions.** *Prove by induction the sign rule in the determinant expansion (13.13):*

$$\det(1 - z\mathbf{T}) = \sum_{k \geq 0} \sum_{p_1 + \cdots + p_k} (-1)^k t_{p_1} t_{p_2} \cdots t_{p_k} .$$

Exercise 13.4 **Transition matrix and cycle counting.** *Suppose you are given the Markov graph*



This diagram can be encoded by a matrix T*, where the entry* T_{ij} *means that there is a link connecting node* i *to node* j*. The value of the entry is the weight of the link.*

a) Walks on the graph are given the weight that is the product of the weights of all links crossed by the walk. Convince yourself that the transition matrix for this graph is:

$$T = \left[\begin{array}{cc} a & b \\ c & 0 \end{array} \right] .$$

prime cycle
cycle!prime
three-disk@3-disk!prime cycles
prime cycle!3-disk

b) Enumerate all the walks of length three on the Markov graph. Now compute T^3 and look at the entries. Is there any relation between the terms in T^3 and all the walks?

c) Show that T^n_{ij} is the number of walks from point i to point j in n steps. (Hint: one might use the method of induction.)

d) Try to estimate the number $N(n)$ of walks of length n for this simple Markov graph.

e) The topological entropy h measures the rate of exponential growth of the total number of walks $N(n)$ as a function of n. What is the topological entropy for this Markov graph?

Exercise 13.5 3-disk prime cycle counting. *A* prime *cycle p of length n_p is a single traversal of the orbit; its label is a non-repeating symbol string of n_p symbols. For example, $\overline{12}$ is prime, but $\overline{2121}$ is not, since it is $\overline{21} = \overline{12}$ repeated.*

Verify that a 3-disk pinball has 3, 2, 3, 6, 9, $\cdots$ prime cycles of length 2, 3, 4, 5, 6, $\cdots$.

Exercise 13.6 "Golden mean" pruned map. *Continuation of exercise 11.8: Show that the total number of periodic orbits of length n for the "golden mean" tent map is*

$$\frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n} .$$

For continuation, see exercise 13.8. See also exercise 13.9.

Exercise 13.7 Alphabet {0,1}, prune $\_00\_$. *The Markov diagram figure 11.13(b) implements this pruning rule. The pruning rule implies that "0" must always be bracketed by "1"s; in terms of a new symbol $2 = 10$, the dynamics becomes unrestricted symbolic dynamics with with binary alphabet {1,2}. The cycle expansion (13.13) becomes*

$$\begin{aligned} 1/\zeta &= (1-t_1)(1-t_2)(1-t_{12})(1-t_{112})\ldots \\ &= 1 - t_1 - t_2 - (t_{12} - t_1 t_2) - (t_{112} - t_{12} t_1) - (t_{122} - t_{12} t_2)\ldots \end{aligned} \quad (13.35)$$

In the original binary alphabet this corresponds to:

$$\begin{aligned} 1/\zeta &= 1 - t_1 - t_{10} - (t_{110} - t_1 t_{10}) \\ &\quad - (t_{1110} - t_{110} t_1) - (t_{11010} - t_{110} t_{10}) \ldots \end{aligned} \quad (13.36)$$

This symbolic dynamics describes, for example, circle maps with the golden mean winding number, see chapter 25. For unimodal maps this symbolic dynamics is realized by the tent map of exercise 13.6.

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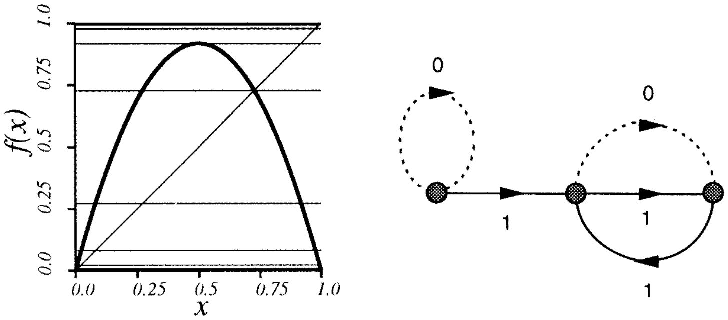

Figure 13.5: (a) A unimodal map for which the critical point maps into the right hand fixed point in three iterations, and (b) the corresponding Markov graph (Kai T. Hansen).

Exercise 13.8 **Spectrum of the "golden mean" pruned map.** *(medium - Exercise 13.6 continued)*

(a) *Determine an expression for* $\mathrm{tr}\,\mathcal{L}^n$*, the trace of powers of the Perron-Frobenius operator (9.10) for the tent map of exercise 13.6.*

(b) *Show that the spectral determinant for the Perron-Frobenius operator is*

$$\det(1-z\mathcal{L}) = \prod_{k \text{ even}} \left(1 - \frac{z}{\Lambda^{k+1}} - \frac{z^2}{\Lambda^{2k+2}}\right) \prod_{k \text{ odd}} \left(1 + \frac{z}{\Lambda^{k+1}} + \frac{z^2}{\Lambda^{2k+2}}\right). \tag{13.37}$$

Exercise 13.9 **A unimodal map example.** *Consider a unimodal map of figure 13.5(a) for which the critical point maps into the right hand fixed point in three iterations,* $S^+ = 100\overline{1}$*. Show that the admissible itineraries are generated by the Markov graph figure 13.5(b).*

(Kai T. Hansen)

⇓PRELIMINARY

Exercise 13.10 **Heavy pruning.** *(continuation of exercise 11.10.) Implement the grammar (12.11) by verifying all steps in the construction outlined in figure 13.3. Verify the entropy estimate (13.18). Perhaps count admissible trajectories up to some length of 5-10 symbols by your own method (generate all binary sequences, throw away the bad ones?), check whether this converges to the* h *value claimed in the text.*

⇑PRELIMINARY

Exercise 13.11 **Glitches in shadowing.**** *Note that the combination* t_{00011} *minus the "shadow"* $t_0 t_{0011}$ *in (13.17) cancels exactly, and does not contribute to the topological polynomial (13.18). Are you able to construct a smaller Markov graph than figure 13.3(e)?*

Exercise 13.12 **Whence Möbius function?** *To understand where the Möbius function comes from consider the function*

$$f(n) = \sum_{d|n} g(d) \tag{13.38}$$

where $d|n$ *stands for sum over all divisors* d *of* n*. Invert recursively this infinite tower of equations and derive the* Möbius inversion formula

$$g(n) = \sum_{d|n} \mu(n/d) f(d) \tag{13.39}$$

Exercise 13.13 <u>**Counting prime binary cycles.**</u> *In order to get comfortable with Möbius inversion reproduce the results of the second column of table 13.2.*

Write a program that determines the number of prime cycles of length n. You might want to have this program later on to be sure that you have missed no 3-pinball prime cycles.

Exercise 13.14 **Counting subsets of cycles.** *The techniques developed above can be generalized to counting subsets of cycles. Consider the simplest example of a dynamical system with a complete binary tree, a repeller map (11.8) with two straight branches, which we label 0 and 1. Every cycle weight for such map factorizes, with a factor t_0 for each 0, and factor t_1 for each 1 in its symbol string. Prove that the transition matrix traces (13.5) collapse to $tr(T^k) = (t_0 + t_1)^k$, and $1/\zeta$ is simply*

$$\prod_p (1 - t_p) = 1 - t_0 - t_1 \tag{13.40}$$

Substituting (13.40) into the identity

$$\prod_p (1 + t_p) = \prod_p \frac{1 - {t_p}^2}{1 - t_p}$$

we obtain

$$\begin{aligned}
\prod_p (1 + t_p) &= \frac{1 - t_0^2 - t_1^2}{1 - t_0 - t_1} = 1 + t_0 + t_1 + \frac{2t_0 t_1}{1 - t_0 - t_1} \\
&= 1 + t_0 + t_1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} 2 \binom{n-2}{k-1} t_0^k t_1^{n-k} . \qquad (13.41)
\end{aligned}$$

Hence for $n \geq 2$ the number of terms in the cumulant expansion with k 0's and $n-k$ 1's in their symbol sequences is $2\binom{n-2}{k-1}$.

In order to count the number of prime cycles in each such subset we denote with $M_{n,k}$ $(n = 1, 2, \dots ; k = \{0, 1\}$ for $n = 1;\ k = 1, \dots, n-1$ for $n \geq 2)$ the number of prime n-cycles whose labels contain k zeros. Show that

$$\begin{aligned}
M_{1,0} &= M_{1,1} = 1 \\
nM_{n,k} &= \sum_{m \mid \frac{n}{k}} \mu(m) \binom{n/m}{k/m} , \quad n \geq 2 , k = 1, \dots, n-1
\end{aligned}$$

where the sum is over all m which divide both n and k.

Exercise 13.15 **Logarithmic periodicity of $\ln {N_n}^*$.** *Plot $\ln N_n - nh$ for a system with a nontrivial finite Markov graph. Do you see any periodicity? If yes, why?*

Exercise 13.16 4-disk pinball topological polynomial. *Show that the 4-disk pinball topological polynomial (the pruning affects only the fixed points and the 2-cycles) is given by*

$$\begin{aligned} 1/\zeta_{4-disk} &= (1-3z)\frac{(1-z^2)^6}{(1-z)^3(1-z^2)^3} \\ &= (1-3z)(1+z)^3 = 1-6z^2-8z^3-3z^4\ . \end{aligned} \tag{13.42}$$

Exercise 13.17 N-disk pinball topological polynominal. *Show that for an N-disk pinball, the topological polynominal is given by*

$$\begin{aligned} 1/\zeta_{N-disk} &= (1-(N-1)z)\,\frac{(1-z^2)^{N(N-1)/2}}{(1-z)^{N-1}(1-z^2)^{(N-1)(N-2)/2}} \\ &= (1-(N-1)z)\,(1+z)^{N-1}\ . \end{aligned} \tag{13.43}$$

The topological polynomial has a root $z^{-1}=N-1$, as we already know it should from (13.29) or (13.15). We shall see in sect. 22.4 that the other roots reflect the symmetry factorizations of zeta functions.

Exercise 13.18 Alphabet $\{a,b,c\}$, prune $\_ab\_$. *The pruning rule implies that any string of "b"s must be preceeded by a "c"; so one possible alphabet is $\{a, cb^k;\ \overline{b}\}$, k=0,1,2.... As the rule does not prune the fixed point $\overline{b}$, it is explicitly included in the list. The cycle expansion (13.13) becomes*

$$\begin{aligned} 1/\zeta &= (1-t_a)(1-t_b)(1-t_c)(1-t_{cb})(1-t_{ac})(1-t_{cbb})\ldots \\ &= 1-t_a-t_b-t_c+t_at_b-(t_{cb}-t_ct_b)-(t_{ac}-t_at_c)-(t_{cbb}-t_{cb}t_b)\ldots \end{aligned}$$

The effect of the $\_ab\_$ pruning is essentially to unbalance the 2 cycle curvature $t_{ab}-t_at_b$; the remainder of the cycle expansion retains the curvature form.

Exercise 13.19 Alphabet $\{0,1\}$, prune n repeats. *of "0" $\_000\ldots00\_$.*

This is equivalent to the n symbol alphabet $\{1, 2, \ldots, n\}$ unrestricted symbolic dynamics, with symbols corresponding to the possible 10...00 block lengths: 2=10, 3=100, ..., n=100...00. The cycle expansion (13.13) becomes

$$1/\zeta = 1-t_1-t_2\ldots-t_n-(t_{12}-t_1t_2)\ldots-(t_{1n}-t_1t_n)\ldots\ . \tag{13.44}$$

⇓PRIVATE

Exercise 13.20 Alphabet $\{0,1\}$, prune $\_1000\_$, $\_00100\_$, $\_01100\_$. *This example is motivated by the pruning front description of the symbolic dynamics for the Hénon-type maps, sect. 12.4.*

⇑PRIVATE

Show that the topological zeta function is given by

$$1/\zeta = (1-t_0)(1-t_1-t_2-t_{23}-t_{113}) \tag{13.45}$$

with the unrestricted 4-letter alphabet $\{1,\ 2,\ \underline{23},\ \underline{113}\}$. Here 2, 3, refer to 10, 100 respectively, as in exercise 13.19.

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Exercise 13.21 Alphabet $\{0,1\}$, prune $\_1000\_$, $\_00100\_$, $\_01100\_$, $\_10011\_$. *This example of pruning we shall use in sect. 12.7. The first three pruning rules were incorporated in the preceeding exercise.*

(a) Show that the last pruning rule $\_10011\_$ leads (in a way similar to exercise 13.20) to the alphabet $\{\underline{21^k}, \underline{23}, \underline{21^k113}; \overline{1}, \overline{0}\}$, and the cycle expansion

$$1/\zeta = (1-t_0)(1-t_1-t_2-t_{23}+t_1t_{23}-t_{2113}) \tag{13.46}$$

Note that this says that 1, 23, 2, 2113 are the fundamental cycles; not all cycles up to length 7 are needed, only 2113.

(b) Show that the topological polynomial is

$$1/\zeta_{top} = (1-z)(1-z-z^2-z^5+z^6-z^7) \tag{13.47}$$

and check that it yields the exact value of the entropy $h = 0.522737642\ldots$.

Redo exercises by Markov graphs.