Exercises

Exercise 15.1 **Escape rate for a 1-d repeller, numerically.** *Consider the quadratic map*

$$f(x) = Ax(1-x) \tag{15.41}$$

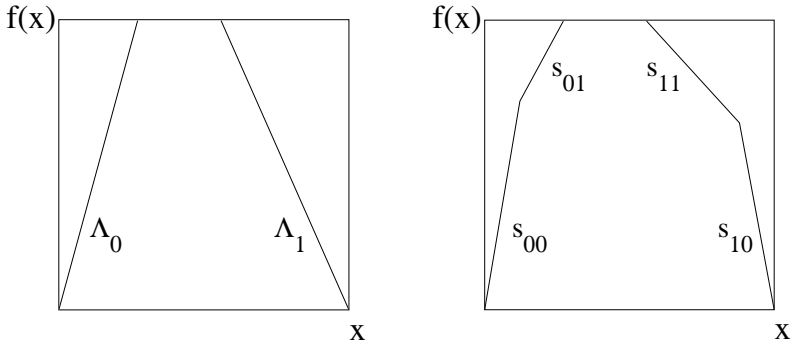
on the unit interval. The trajectory of a point starting in the unit interval either stays in the interval forever or after some iterate leaves the interval and diverges to minus infinity. Estimate numerically the escape rate (19.8), the rate of exponential decay of the measure of points remaining in the unit interval, for either $A = 9/2$ or $A = 6$. Remember to compare your numerical estimate with the solution of the continuation of this exercise, exercise 18.2.

Exercise 15.2 **Dynamical zeta functions.** *(easy)*

(a) *Evaluate in closed form the dynamical zeta function*

$$1/\zeta(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right) ,$$

for the piecewise-linear map (10.17) with the left branch slope Λ_0, the right branch slope Λ_1.



(b) *What if there are four different slopes s_{00}, s_{01}, s_{10}, and s_{11} instead of just two, with the preimages of the gap adjusted so that junctions of branches s_{00}, s_{01} and s_{11}, s_{10} map in the gap in one iteration? What would the dynamical zeta function be?*

Exercise 15.3 **Dynamical zeta functions from Markov graphs.** *Extend sect. 13.3 to evaluation of dynamical zeta functions for piecewise linear maps with finite Markov graphs. This generalizes the results of exercise 15.2.*

Exercise 15.4 **Zeros of infinite products.** *Determination of the quantities of interest by periodic orbits involves working with infinite product formulas.*

(a) *Consider the infinite product*

$$F(z) = \prod_{k=0}^{\infty}(1 + f_k(z))$$

where the functions f_k are "sufficiently nice." This infinite product can be converted into an infinite sum by the use of a logarithm. Use the properties of infinite sums to develop a sensible definition of infinite products.

(b) *If z^* is a root of the function F, show that the infinite product diverges when evaluated at z^*.*

(c) *How does one compute a root of a function represented as an infinite product?*

(d) *Let p be all prime cycles of the binary alphabet $\{0, 1\}$. Apply your definition of $F(z)$ to the infinite product*

$$F(z) = \prod_{p}(1 - \frac{z^{n_p}}{\Lambda^{n_p}})$$

(e) *Are the roots of the factors in the above product the zeros of $F(z)$?*

(Per Rosenqvist)

Exercise 15.5 Dynamical zeta functions as ratios of spectral determinants. *(medium) Show that the zeta function*

$$1/\zeta(z) = \exp\left(-\sum_{p}\sum_{r=1}\frac{1}{r}\frac{z^{n_p}}{|\Lambda_p|^r}\right)$$

can be written as the ratio $1/\zeta(z) = \det(1 - z\mathcal{L}_{(0)})/\det(1 - z\mathcal{L}_{(1)})$, *where* $\det(1 - z\mathcal{L}_{(s)}) = \prod_p \prod_{k=0}^{\infty}(1 - z^{n_p}/|\Lambda_p|\Lambda_p^{k+s})$.

Exercise 15.6 Contour integral for survival probability. *Perform explicitly the contour integral appearing in (15.20).*

Exercise 15.7 Dynamical zeta function for maps. *In this problem we will compare the dynamical zeta function and the spectral determinant. Compute the exact dynamical zeta function for the skew Ulam tent map (9.42)*

$$1/\zeta(z) = \prod_{p\in P}\left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right).$$

What are its roots? Do they agree with those computed in exercise 9.7?

Exercise 15.8 Dynamical zeta functions for Hamiltonian maps. *Starting from*

$$1/\zeta(s) = \exp\left(-\sum_{p}\sum_{r=1}^{\infty}\frac{1}{r}t_p^r\right)$$

for a 2-dimensional Hamiltonian map. Using the equality

$$1 = \frac{1}{(1-1/\Lambda)^2}(1-2/\Lambda+1/\Lambda^2)\,,$$

show that

$$1/\zeta = \det(1-\mathcal{L})\det(1-\mathcal{L}_{(2)})/\det(1-\mathcal{L}_{(1)})^2\,.$$

In this expression $\det(1-z\mathcal{L}_{(k)})$ *is the expansion one gets by replacing* $t_p \to t_p/\Lambda_p^k$ *in the spectral determinant.*

Exercise 15.9 Riemann ζ function. *The Riemann* ζ *function is defined as the sum*

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\,, \qquad s\in\mathbb{C}\,.$$

(a) *Use factorization into primes to derive the Euler product representation*

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}\,.$$

The dynamical zeta function exercise 15.16 is called a "zeta" function because it shares the form of the Euler product representation with the Riemann zeta function.

(b) *(Not trivial:) For which complex values of* s *is the Riemann zeta sum convergent?*

(c) *Are the zeros of the terms in the product,* $s = -\ln p$*, also the zeros of the Riemann* ζ *function? If not, why not?*

Exercise 15.10 Finite truncations. *(easy) Suppose we have a one-dimensional system with complete binary dynamics, where the stability of each orbit is given by a simple multiplicative rule:*

$$\Lambda_p = \Lambda_0^{n_{p,0}}\Lambda_1^{n_{p,1}}\,, \qquad n_{p,0} = \#0 \text{ in } p\,,\ n_{p,1} = \#1 \text{ in } p\,,$$

so that, for example, $\Lambda_{00101} = \Lambda_0^3\Lambda_1^2$.

(a) *Compute the dynamical zeta function for this system; perhaps by creating a transfer matrix analogous to (10.19), with the right weights.*

(b) *Compute the finite* p *truncations of the cycle expansion, that is take the product only over the* p *up to given length with* $n_p \le N$*, and expand as a series in* z

$$\prod_p\left(1-\frac{z^{n_p}}{|\Lambda_p|}\right)\,.$$

Do they agree? If not, how does the disagreement depend on the truncation length N*?*

Exercise 15.11 Sequence of orbital measures for the Ulam map.

As an example, let us develop the sequence of orbital measures for the quadratic map, even if it is not precisely of the form we discussed above. Indeed, this illustrates how some of our assumptions could be relaxed, and still get correct results. Moreover, although for such a simple case it is possible to explicitly find all periodic points up to the desired period, we will only use those that are found by looking at close returns of a generic chaotic trajectory. Consider the Ulam map

$$x_{n+1} = 4x_n(1 - x_n) \ . \tag{15.42}$$

This is quadratic, and it has two fixed points, four period two periodic points (including the two repeats of the fixed points) and, in general, 2^n periodic points of period n, as explained in exercise 11.6. How these are structured is given in table 13.1.

make exercise, draw the figure

The natural measure of the Ulam map (19.25), figure C.2b is explicitly expressed by

need to derive this somewhere, refer to Ulam

$$d\mu(x) = \frac{dx}{\pi\sqrt{x - x^2}} \ . \tag{15.43}$$

this explicitely shows that periodic points do not have equal weights? But smooth conjugacy to tent map justifies it.

We can now calculate the average values of $\langle x \rangle$ and of $\langle x^2 \rangle$ exactly using the known analytic distribution function (19.26), and also numerically using the cycle expansion. This exercise illustrates well the use of orbital measures to calculate averages. The results are given in table 19.1, where we have used the periodic points of order 2 to 12 which are found by a generic chaotic trajectory, and we have neglected the cycles of a given period which are repeats of cycles of smaller period. This because the contributions of non prime (repeat) cycles become less and less important as the period grows, as one can see also from table 13.1. Notice also, that the number of cycles found numerically differs from the exact number (given in parenthesis for period 6 and larger). All this mimics what actually can be done, when sampling periodic points from the phase space of more realistic systems. Indeed, not being able to find all the existing cycles is a common numerical difficulty, linked to their instability. That can easily be remedied for the quadratic map, but in general one does not even know how many cycles of a certain period should be found. Notice the convergence of the method, despite these difficulties.

terrible; compare with the AACII calculation

yes, I note no convergence whatsoever

We have introduced the idea of periodic orbit expansions for simple maps, as an example of a theory of much wider generality. The reason why the method is expected to work in more general situations, is that the relevant attractors can be proven, or have been observed to be hierarchically approximated through sets of progressively longer and longer unstable periodic orbits. Indeed, the weak limit of these measures has been rigorously proved to be the natural measure for Axiom-A systems, independently of the (finite) dimensions of the relavant phase spaces.

plot $\log_{10}$ error from table 19.1, numbers are useless. Also plot the error of the correct cycle computation

n	$\langle x \rangle$	$\langle x^2 \rangle$	$\sum N_k$
2	0.62500	0.46875	2
3	0.54167	0.40625	6
4	0.50000	0.37500	12
5	0.50833	0.38125	30
6	0.50354	0.37766	52(54)
7	0.51240	0.38430	120(126)
8	0.51017	0.38263	230(240)
9	0.50914	0.38185	482(504)
10	0.50904	0.38178	940(990)
11	0.50795	0.38097	1962(2046)
12	0.50731	0.38049	3853(4020)
exact	0.5	0.375	

Table 15.1: An approximate cycle expansion using only periodic points of prime orbits, and only those found by a probabilistic exploration of the phase space. (In the parenthesis the correct total number of such periodic points, see table 13.2).

Remark 15.6 Nonhyperbolicity. In the above example we have succeeded in developing a sequence of orbital measures for the Ulam map, even though the Ulam map does not satisfy the hyperbolicity hypothesis (the critical point is superstable) invoked in the derivation. This illustrates how some of our assumptions could be relaxed, and still get sensible results. To solve this example *exactly* the cyclist way, consult refsect.?!

include the text from ref. [37.12]

(L. Rondoni)

⇑PRELIMINARY