Laplace!transform

Exercises

Exercise 14.1 $t \to 0_+$ **regularization of eigenvalue sums****. *In taking the Laplace transform (14.19) we have ignored the $t \to 0_+$ divergence, as we do not know how to regularize the delta function kernel in this limit. In the quantum (or heat kernel) case this limit gives rise to the Weyl or Thomas-Fermi mean eigenvalue spacing (see sect. 33.1.1). Regularize the divergent sum in (14.19) following (for example) the prescription of appendix M.6 and assign to such volume term some interesting role in the theory of classical resonance spectra. E-mail the solution to the authors.*

Exercise 14.2 **General weights.** *(easy) Let f^t be a flow and $\mathcal{L}^t$ the operator*

recheck whether this is backwards

$$\mathcal{L}^t g(x) = \int dy\, \delta(x - f^t(y)) w(t, y) g(y)$$

where w is a weight function. In this problem we will try and determine some of the properties w must satisfy.

(a) *Compute $\mathcal{L}^s \mathcal{L}^t g(x)$ to show that*

$$w(s, f^t(x)) w(t, x) = w(t + s, x) .$$

(b) *Restrict t and s to be integers and show that the most general form of w is*

$$w(n, x) = g(x) g(f(x)) g(f^2(x)) \cdots g(f^{n-1}(x)) ,$$

for some g that can be multiplied. Could g be a function from $\mathbb{R}^{n_1} \mapsto \mathbb{R}^{n_2}$? ($n_i \in \mathbb{N}$.)

⇓PRELIMINARY

fix - done in the text?

(c) *(not yet tried out) Generalize the expression for w for continuous time. Use $g = e^p$ and convert the infinite product into an exponential of an integral.*

⇑PRELIMINARY

⇓PRELIMINARY

Exercise 14.3 **Higher order traces.** *One way to define the determinant of a matrix is to consider in a d-dimensional vector space the multilinear function Δ (this will turn out to be the determinant function where the matrix is specified by the column vectors, or if you like wedge products $\Delta(e_1, \ldots, e_d = e_1 \wedge \cdots \wedge e_d)$. It takes d vectors as arguments and is linear in each of them*

$$\Delta(e_1, \ldots, \alpha e_k + e'_k, \ldots, e_d) = \alpha \Delta(e_1, \ldots, e_k, \ldots, e_d) + \Delta(e_1, \ldots, e'_k, \ldots, e_d) .$$

This function is also anti-symmetric: if we swap the position of two of its arguments it changes sign

$$\Delta(e_1, \ldots, e_k, \ldots, e_j, \ldots, e_d) = -\Delta(e_1, \ldots, e_j, \ldots, e_k, \ldots, e_d) .$$

One can show that up to a multiplicative constant, this function is unique. Assume that it has been shown unique.

(a) *Show that* $\Delta(Ae_1, \ldots, Ae_d)$ *differs from* $\Delta(e_1, \ldots, e_d)$ *by a multiplicative constant (let us call it* $\det A$*), that is,*

$$\Delta(Ae_1, \ldots, Ae_d) = \det(A)\Delta(e_1, \ldots, e_d)$$

Take this as the definition of the determinant of a matrix.

(b) *Show that* $\det(AB) = \det(A)\det(B)$

(c) *Try and give a definition for trace of a matrix in terms of* Δ*. Think of* $\det(I + \epsilon A) = 1 + \epsilon \mathrm{tr}\, A$ *as* $\epsilon \to 0$ *and show that*

$$\mathrm{tr}\, A = \sum_k \Delta(e_1, \ldots, Ae_k, \ldots, e_d)$$

(d) *For a* 3×3 *matrix, expand* $\det(I - A)$*.*

(e) *Try the general case.*

$$\det(I - A) = \sum_{k \geq 0} (-1)^k \mathrm{tr}^{\,k}(A)\,,$$

where $\mathrm{tr}^{\,k}(A)$ *means the summation*

$$\sum_{j_1, \ldots, j_k} \Delta(\underbrace{\ldots Ae_{j_1}, \ldots, Ae_{j_k}}_{k\ A'\mathrm{s}})$$

(f) *If the* e_k *are a basis for the vector space, them*

$$\Delta(\underbrace{\ldots Ae_{j_1}, \ldots, Ae_{j_k}}_{k\ A'\mathrm{s}})$$

is really the determinant of the matrix A *with only* k *of the columns, that is, a* $k \times k$ *minor. So the expansion is really an expansion in minors.*

⇑PRELIMINARY