dynamical!system

Chapter 2

Go with the flow

Poetry is what is lost in translation.
Robert Frost

(R. Mainieri, P. Cvitanović and E.A. Spiegel) [1]

this chapter essentially finished

We start out with a recapitulation of the basic notions of dynamics. Our aim is narrow; we keep the exposition focused on prerequisites to the applications to be developed in this text. We assume that the reader is familiar with dynamics on the level of the introductory texts mentioned in sect. 1.8, and concentrate here on developing intuition about what a dynamical system can do. It will be a coarse brush sketch - a full description of all possible behaviors of dynamical systems is beyond human ken. Anyway, for a novice there is no shortcut through this lengthy detour; a sophisticated traveler might prefer to skip this well-trodden territory and embark upon the journey at chapter 9.

fast track:
chapter 9, p. 125

2.1 Dynamical systems

to Benny: "mirror" aludes to the Leibnitz quote

In a dynamical system we observe the world as a function of time. We express our observations as numbers and record how they change with time; given sufficiently detailed information and understanding of the underlying natural laws, we see the future in the present as in a mirror. The motion of the planets against the celestial firmament provides an example. Against the daily motion of the stars from East to West, the planets distinguish themselves by moving among the fixed stars. Ancients discovered that by knowing a sequence of planet's positions - latitudes and longitudes - its future position could be predicted.

sect. 1.3

[1] Lautrup: improve language twrd end of chapter - too "kort for hovedet"

state space
phase space
representative point
deterministic!dynamics
dynamical!system!deterministic
phase space
dynamical!system
dynamical!system!smooth
smooth!dynamics
flow
iteration
map

For the solar system, tracking the latitude and longitude in the celestial sphere suffices to completely specify the planet's apparent motion. All possible values for positions and velocities of the planets form the *phase space* of the system. More generally, a state of a physical system, at a given instant in time, can be represented by a single point in an abstract space called *state space* or *phase space* $\mathcal{M}$. As the system changes, so does the *representative point* in phase space. We refer to the evolution of such points as *dynamics*, and the function f^t which specifies where the representative point is at time t as the *evolution rule*.

If there is a definite rule f that tells us how this representative point moves in $\mathcal{M}$, the system is said to be deterministic. For a deterministic dynamical system, the evolution rule takes one point of the phase space and maps it into exactly one point. However, this is not always possible. For example, knowing the temperature today is not enough to predict the temperature tomorrow; knowing the value of a stock today will not determine its value tomorrow. The phase space can be enlarged, in the hope that in a sufficiently large phase space it is possible to determine an evolution rule, so we imagine that knowing the state of the atmosphere, measured over many points over the entire planet should be sufficient to determine the temperature tomorrow. Even that is not quite true, and we are less hopeful when it comes to stocks.

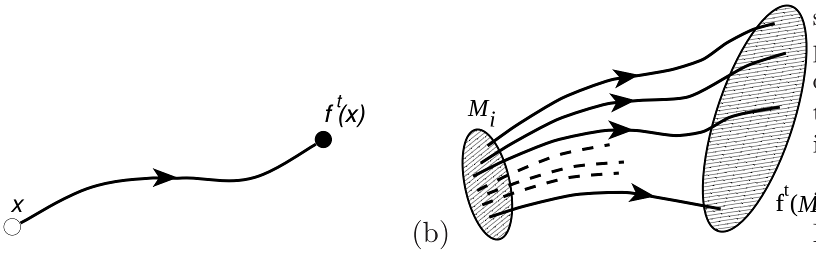
For a deterministic system almost every point has a unique future, so trajectories cannot intersect. We say "almost" because there might exist a set of measure zero (tips of wedges, cusps, *etc.*) for which a trajectory is not defined. We may think such sets a nuisance, but it is quite the contrary - they will enable us to partition phase space, so that the dynamics can be better understood.

☞ chapter 12

Locally, the phase space $\mathcal{M}$ looks like $\mathbb{R}^d$, meaning that d numbers are sufficient to determine what will happen next. Globally, it may be a more complicated manifold formed by patching together several pieces of $\mathbb{R}^d$, forming a torus, a cylinder, or some other geometric object. When we need to stress that the dimension d of $\mathcal{M}$ is greater than one, we may refer to the point $x \in \mathcal{M}$ as x_i where $i = 1, 2, 3, \ldots, d$. The evolution rule $f^t : \mathcal{M} \to \mathcal{M}$ tells us where a point x is in $\mathcal{M}$ after a time interval t. The pair $(\mathcal{M}, f)$ constitute a *dynamical system*.

The dynamical systems we will be studying are smooth. This is expressed mathematically by saying that the evolution rule f^t can be differentiated as many times as needed. Its action on a point x is sometimes indicated by $f(x, t)$ to remind us that f is really a function of two variables: the time and a point in phase space. Note that time is relative rather than absolute, so only the time interval is necessary. This follows from the fact that a point in phase space completely determines all future evolution, and it is not necessary to know anything else. The time parameter can be a real variable ($t \in \mathbb{R}$), in which case the evolution is called a *flow*, or an integer ($t \in \mathbb{Z}$), in which case the evolution advances in discrete steps in time, given by *iteration* of a *map*. Actually, the evolution parameter need not be the physical time; for example, a time-stationary solution of a partial

(a) (b)



spatial!profile
profile!spatial
orbit
trajectory
initial!point
$“xInit$
initial!state $“xInit$
Lagrangian!coordinates
Eulerian!coordinates
orbit
wandering point
point!wandering

Figure 2.1: (a) A trajectory traced out by the evolution rule f^t. Starting from the phase space point x, after a time t, the point is at $f^t(x)$. (b) The evolution rule f^t can be used to map a region $\mathcal{M}_i$ of the phase space into the region $f^t(\mathcal{M}_i)$.

differential equation is parametrized by spatial variables. In such situations one talks of a “spatial profile” rather than a “flow”.

Nature provides us with innumerable dynamical systems. They manifest themselves through their trajectories: given an initial point x_0, the evolution rule traces out a sequence of points $x(t) = f^t(x_0)$, the *trajectory* through the point $x_0 = x(0)$. A trajectory is parameterized by the time t and thus belongs to $(f^t(x_0), t) \in \mathcal{M} \times \mathbb{R}$. By extension, we can also talk of the evolution of a region $\mathcal{M}_i$ of the phase space: just apply f^t to every point in $\mathcal{M}_i$ to obtain a new region $f^t(\mathcal{M}_i)$, as in figure 2.1.

2.1
page 43

Because f^t is a single-valued function, any point of the trajectory can be used to label the trajectory. If we mark the trajectory by its inital point x_0, we are describing it in the *Lagrangian coordinates*. We can regard the transport of the material point at $t = 0$ to its current point $x(t) = f^t(x_0)$ as a coordinate transformation from the Lagrangian coordinates to the *Eulerian coordinates*.

The subset of points in $\mathcal{M}$ that belong to the (possibly infinite) trajectory of a given point x_0 is called the *orbit* of x_0; we shall talk about forward orbits, backward orbits, periodic orbits, etc.. For a flow, an orbit is a continuous curve; for a map, it is a sequence of points.

What are the possible trajectories? This is a grand question, and there are many answers, chapters to follow offering some. Here we shall classify possible trajectories as:

$$\begin{array}{rll} \text{stationary:} & f^t(x) = x & \text{for all } t \\ \text{periodic:} & f^t(x) = f^{t+T_p}(x) & \text{for a given minimum period } T_p \\ \text{aperiodic:} & f^t(x) \neq f^{t'}(x) & \text{for all } t \neq t' \ . \end{array}$$

a little picture here of the three trajectory types.

define quasiperiodicity

The ancients tried to make sense of all dynamics in terms of periodic motions; epicycles, integrable systems. The embarassing truth is that for a generic dynamical systems most motions are aperiodic. We will classify aperiodic motions into two subtypes: those that wander off, and those that keep coming back.

A point $x \in \mathcal{M}$ is called a *wandering point*, if there exists an open

transient
recurrence
point!non-wandering
non-wandering set
contracting!flow
flow!contracting
attractor
Lyapunov functional
functional!Lyapunov
basin of attraction
attractor!basin
attractor!strange
strange!attractor
repeller

neighborhood $\mathcal{M}_0$ of x to which the trajectory never returns

$$f^t(x) \notin \mathcal{M}_0 \qquad \text{for all} \quad t > t_{\text{min}} \,. \tag{2.1}$$

In physics literature, the dynamics of such state is often referred to as *transient.*

A *periodic orbit* corresponds to a trajectory that returns exactly to the initial point in a finite time. Periodic orbits form a very small subset of the phase space, in the same sense that rational numbers are a set of zero measure on the unit interval. For times much longer than a typical "turnover" time, it makes sense to relax the notion of exact periodicity, and replace it by the notion of *recurrence.* A point is *recurrent* or *non-wandering* if for any open neighborhood $\mathcal{M}_0$ of x and any time t_{min} there exists a later time t, such that

$$f^t(x) \in \mathcal{M}_0 \,. \tag{2.2}$$

In other words, the trajectory of a non-wandering point reenters the neighborhood $\mathcal{M}_0$ infinitely often. We shall denote by Ω the *non–wandering set* of f, that is, the union of all the non-wandering points of $\mathcal{M}$. The set Ω, the non–wandering set of f, is the key to understanding the long-time behavior of a dynamical system; all calculations undertaken here will be carried out on non–wandering sets.

So much about individual trajectories. What about clouds of initial points? If there exists a connected phase space volume that maps into itself under forward evolution (and you can prove that by the method of Lyapunov functionals, or several other methods available in the literature), the flow is globally contracting onto a subset of $\mathcal{M}$ which we shall refer to as the *attractor.* The attractor may be unique, or there can coexist any number of distinct attracting sets, each with its own *basin of attraction*, the set of all points that fall into the attractor under foward evolution. The attractor can be a fixed point, a periodic orbit, aperiodic, or any combination of the above. The most interesting case is that of an aperiodic recurrent attractor, to which we shall refer loosely as a *strange attractor.* We say 'loosely', as will soon become apparent that diagnosing and proving existence of a genuine, card-carrying strange attractor is a highly nontrivial undertaking. [2]

☞ sect. 2.2

Conversely, if we can enclose the non–wandering set Ω by a connected phase space volume $\mathcal{M}_0$ and then show that almost all points within $\mathcal{M}_0$, but not in Ω, eventually exit $\mathcal{M}_0$, we refer to the non–wandering set Ω as a *repeller.* An example of a repeller is not hard to come by - the pinball game of sect. 1.3 is a simple chaotic repeller.

It would seem, having said that the periodic points are so exceptional that almost all non-wandering points are aperiodic, that we have given up

[2] Mason: do you distinguish strange chaotic attractor strange non-chaotic attractor, in spirit of Merry Landia? PC: no.

the ancients' fixation on periodic motions. Nothing could be further from truth. As longer and longer cycles approximate more and more accurately finite segments of aperiodic trajectories, we shall establish control over non–wandering sets by defining them as the closures of the union of all periodic points.

flow
irreversible!dynamics
dynamics!irreversible
semigroup!dynamical
group!dynamical
Lyapunov time
reversible!dynamics
dynamics!reversible

Before we can work out an example of a non–wandering set and get a better grip on what chaotic motion might look like, we need to ponder flows in a little more depth.

2.2 Flows

There is no beauty without some strangeness.
William Blake

crosscheck Blake quote

A *flow* is a continuous-time dynamical system. The evolution rule f^t is a family of mappings of $\mathcal{M} \to \mathcal{M}$ parameterized by $t \in \mathbb{R}$. Because t represents a time interval, any family of mappings that forms an evolution rule must satisfy:

2.2
page 43

this excludes billiards?

(a) $f^0(x) = x$ (in 0 time there is no motion)

(b) $f^t(f^{t'}(x)) = f^{t+t'}(x)$ (the evolution law is the same at all times)

(c) the mapping $(x,t) \mapsto f^t(x)$ from $\mathcal{M} \times \mathbb{R}$ into $\mathcal{M}$ is continuous.

The family of mappings $f^t(x)$ thus forms a continuous (forward semi-) group. Why "semi-"group? It may fail to form a group if the dynamics is not reversible, and the rule $f^t(x)$ cannot be used to rerun the dynamics backwards in time, with negative t; with no reversibility, we cannot define the inverse $f^{-t}(f^t(x)) = f^0(x) = x$, in which case the family of mappings $f^t(x)$ does not form a group. In exceedingly many situations of interest - for times beyond the Lyapunov time, for asymptotic attractors, for dissipative partial differential equations, for systems with noise, for non-invertible maps - the dynamics cannot be run backwards in time, hence, the circumspect emphasis on *semi*groups. On the other hand, there are many settings of physical interest, where dynamics is reversible (such as finite-dimensional Hamiltonian flows), and where the family of evolution maps f^t does form a group.

sect. 24.1

For infinitesimal times, flows can be defined by differential equations. We write a trajectory as

$$x(t+\tau) = f^{t+\tau}(x_0) = f(f(x_0,t),\tau) \tag{2.3}$$

and express the time derivative of a trajectory at point $x(t)$,

$$\left.\frac{dx}{d\tau}\right|_{\tau=0} = \partial_\tau f(f(x_0,t),\tau)|_{\tau=0} = \dot{x}(t)\,. \tag{2.4}$$

vector field
tangent!bundle
Duffing oscillator

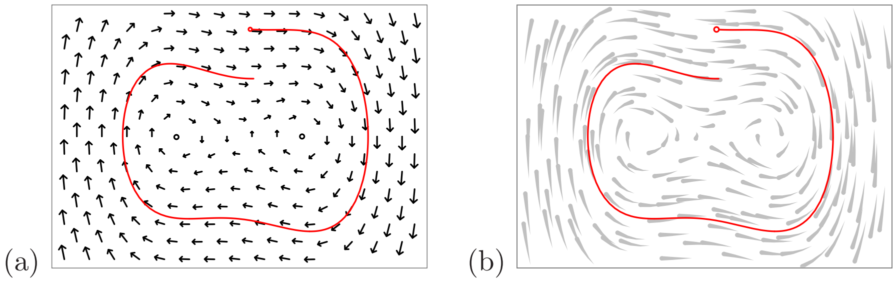


Figure 2.2: (a) The two-dimensional vector field for the Duffing system (2.6), together with a short trajectory segment. (b) The flow lines. Each "comet" represents the same time interval of a trajectory, starting at the tail and ending at the head. The longer the comet, the faster the flow in that region.

2.3
page 43

as the time derivative of the evolution rule, a vector evaluated at the same point. By considering all possible trajectories, we obtain the vector $\dot{x}(t)$ at any point $x \in \mathcal{M}$. This *vector field* is a (generalized) velocity field:

$$v(x) = \dot{x}(t) \,. \tag{2.5}$$

Newton's laws, Lagrange's method, or Hamilton's method are all familiar procedures for obtaining a set of differential equations for the vector field $v(x)$ that describes the evolution of a mechanical system. Equations of mechanics may appear different in form from (2.5), as they are often involve higher time derivatives, but an equation that is second or higher order in time can always be rewritten as a set of first order equations.

We are concerned here with a much larger world of general flows, mechanical or not, defined by a time-independent vector field (2.5). At each point of the phase space a vector indicates the local direction in which the orbit evolves. The length of the vector $|v(x)|$ is proportional to the speed at the point x, and the direction and length of $v(x)$ changes from point to point. When the phase space is a complicated manifold embedded in $\mathbb{R}^d$, one can no longer think of the vector field as being embedded in the phase space. Instead, we have to imagine that each point x of phase space has a different tangent plane $T\mathcal{M}_x$ attached to it. The vector field lives in the union of all these tangent planes, a space called the *tangent bundle* $\mathbf{T}\mathcal{M}$.

Example 2.1 A two-dimensional vector field $v(x)$: *A simple example of a flow is afforded by the unforced Duffing system*

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -0.15\, y(t) + x(t) - x(t)^3 \end{aligned} \tag{2.6}$$

plotted in figure 2.2. The velocity vectors are drawn superimposed over the configuration coordinates $(x(t), y(t))$ of phase space $\mathcal{M}$, but they belong to a different space, the tangent bundle $\mathbf{T}\mathcal{M}$.

equilibrium!point
flow!autonomous
flow!stationary
autonomous flow
stationary!flow

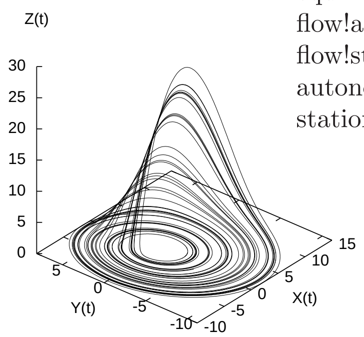


Figure 2.3: A trajectory of the Rössler flow at time $t = 250$. (G. Simon)

2.4
43

$$\text{If} \qquad v(x_q) = 0\,, \tag{2.7}$$

x_q is an *equilibrium point* (often referred to as a *stationary*, *fixed*, or *stagnation* point), and the trajectory remains forever stuck at x_q. Otherwise the trajectory passing through x_0 at time $t = 0$ can be obtained by integrating the equations (2.5):

$$x(t) = f^t(x_0) = x_0 + \int_0^t d\tau\, v(x(\tau))\,, \qquad x(0) = x_0\,. \tag{2.8}$$

We shall consider here only *autonomous* flows, that is, flows for which the velocity field v_i is *stationary*, not explicitly dependent on time. A non-autonomous system

$$\frac{dy}{d\tau} = w(y, \tau)\,, \tag{2.9}$$

can always be converted into a system where time does not appear explicitly. To do so, extend ("suspend") phase space to be $(d+1)$-dimensional by defining $x = \{y, \tau\}$, with a stationary vector field

$$v(x) = \left[\begin{array}{c} w(y,\tau) \\ 1 \end{array} \right]. \tag{2.10}$$

The new flow $\dot{x} = v(x)$ is autonomous, and the trajectory $y(\tau)$ can be read off $x(t)$ by ignoring the last component of x. 2.5 page 44

Example 2.2 A flow with a strange attractor: *The Duffing flow of figure 2.2 is bit of a bore - every trajectory ends up in one of the two attractive equilibrium points. Let's construct a flow that does not die out, but exhibits a recurrent dynamics. Start with a harmonic oscillator*

$$\dot{x} = -y\,, \qquad \dot{y} = x\,. \tag{2.11}$$

Rossler@Rössler system
contracting!flow
flow!contracting
attractor!strange
strange!attractor

The solutions are Ae^{it}, Ae^{-it}*, and the whole* x*-*y *plane rotates with constant angular velocity* $\theta = 1$*, period* $T = 2\pi$*. Now make the system unstable by adding*

$$\dot{x} = -y\,, \qquad \dot{y} = x + ay\,, \qquad a > 0\,. \tag{2.12}$$

The plane is still rotating with constant angular velocity, but trajectories are now spiraling out. In general, any flow in the plane either escapes, falls into an attracting equilibrium point, or converges to a limit cycle - richer dynamics requires at least one more dimension. In order to prevent the trajectory from escaping to ∞*, kick it into 3rd dimension when* x *reaches some value* c *by adding*

$$\dot{z} = b + z(x - c)\,, \qquad c > 0\,. \tag{2.13}$$

Now z *shoots upwards exponentially,* $z \simeq e^{(x-c)t}$*. In order to bring it back, start decreasing* x *by modifing its evolution equation to*

$$\dot{x} = -y - z\,.$$

Large z *drives the trajectory toward* $x = 0$*; there the exponential contraction by* e^{-ct} *kicks in, and the trajectory drops back toward the* x*-*y *plane. This frequently studied example of an autonomous flow is called the* Rössler system *(for definitiveness we fix the parameters* a*,* b*,* c *in what follows):*

$$\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)\,, \qquad a = b = 0.2\,, \quad c = 5.7\,.
\end{aligned} \tag{2.14}$$

2.9
page 44

The system is as simple as they get - it would be linear, were it not for the sole bilinear term zx*. Even for so "simple" a system the nature of long-time solutions is far from obvious.*

There are two repelling equilibrium points:

$$\begin{aligned}
(x^-, y^-, z^-) &= (\,0.0070,\ -0.0351,\ 0.0351\,) \\
(x^+, y^+, z^+) &= (\,5.6929,\ -28.464,\ 28.464\,)
\end{aligned} \tag{2.15}$$

One is close to the origin by construction - the other, some distance away, must exist because the equilibrium has a 2nd-order nonlinearity.

refer also to figure 11.7
change to sect 3.2

3.5
page 58

To see what other solutions look like we need to resort to numerical integration. A typical numerically integrated long-time trajectory is sketched in figure 2.3. As we shall show in sect. 4.1, for this flow any finite volume of initial conditions shrinks with time, so the flow is contracting. *Trajectories that start out sufficiently close to the origin seem to converge to a* strange attractor. *We say "seem", as there exists no proof that such an attractor is asymptotically aperiodic - it might well be that what we see is but a long transient on a way to an attractive periodic orbit. For now, accept that figure 2.3 and similar figures in what follows are examples of "strange attractors". (continued in exercise 2.9)* *(Rytis Paškauskas)*

[3]

fast track:
chapter 3, p. 47

[3]Mason: Warwick Tucker has proven rigorously via interval arithmetic that the Lorentz attractor is strange for the original parameters, and a long periodic orbit for the slightly different parameters. PC: find the reference

2.3 Computing trajectories

Duffing system
Rossler@Rössler system

> On two occasions I have been asked [by members of Parliament], 'Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?' I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question.
> Charles Babbage

RM: expand

You have not learned dynamics unless you know how to integrate numerically whatever dynamical equations you face. Sooner or later, you need to implement some finite time-step prescription for integration of the equations of motion (2.5). The simplest is the Euler integrator which advances the trajectory by $\delta\tau \times$ velocity at each time step:

add figure

$$x_i \to x_i + v_i(x)\delta\tau \,. \tag{2.16}$$

This might suffice to get you started, but as soon as you need higher numerical accuracy, you will need something better. There are many excellent reference texts and computer programs that can help you learn how to solve differential equations numerically using sophisticated numerical tools, such as pseudo-spectral methods or implicit methods. If you are interested in Hamiltonian flows you might want to implement a symplectic integrator of type discussed in appendix E.2.1. If a "sophisticated" integration routine takes days and gobbles up terabits of memory, you are using brain-damaged high level software. Try writing a few lines of your own Runge-Kutta code in some mundane everyday language. While you absolutely need to master the requisite numerical methods, this is neither the time nor the place to expound upon them; how you learn them is your business. And if you have developed some nice routines for solving problems in this text or can point another student to some, let us know.

⇓PRELIMINARY

perhaps quote Numerical Recepies

⇑PRELIMINARY

2.7 page 44
2.8 page 44
2.10 page 45
2.11 page 46

In chapter 24 we shall dispose of the fear of "infinite-dimensional" dynamical systems - you might prefer to skip the section on first reading.

Commentary

Remark 2.1 Model ODE and PDE systems. The Duffing system (2.6) arises in the study of electronic circuits [2.2]. Rössler system was introduced in ref. [2.3] as a simplified set of equations describing no particular physical system, but capturing the essence of chaos in a simplest imaginable smooth flow. Otto Rössler, a man of classical education, was inspired in this quest by that rarely cited grandfather of chaos, Anaxagoras (456 B.C.). This, and references to earlier work can be found in refs. [2.4, 2.6]. We recommend in particular the inimitable Abraham and Shaw illustrated classic [2.5] for its beautiful sketches of the Rössler and many other flows.

⇓PRELIMINARY
⇑PRELIMINARY

move this to Hamiltonian chapter

chaos!diagnostics
frequency analysis

Remark 2.2 Diagnosing chaos. In sect. 1.3.1 we have stated that a deterministic system exhibits "chaos" if its dynamics is locally unstable (positive Lyapunov exponent) and globally mixing (positive entropy). In sect. 10.3 we shall define Lyapunov exponents, and discuss their evaluation, but already at this point it would be handy to have a few quick numerical methods to diagnose chaotic dynamics. Laskar's *frequency analysis* method [2.10] is useful for extracting quasi-periodic and weakly chaotic regions of phase space in Hamiltonian dynamics with many degrees of freedom. For references to several other numerical methods, see ref. [2.11].

⇓PRIVATE

Remark 2.3 Dynamical systems software.

J.D. Meiss [2.8] *Sci.nonlinear FAQ, Computational Resources*, partially superseded by the SIAM Dynamical Systems DSWeb site `www.dynamicalsystems.org`. There you will find a glossary that contains most of the answers in this FAQ plus new ones. There is also a growing software list [2.9]. You are encouraged to contribute to this list.

dig up Guckenheimer SIAM reference

[4]

⇑PRIVATE

Résumé

use in the maps.tex instead

Chaotic dynamics with a low-dimensional attractor can be visualized as a succession of nearly periodic but unstable motions. In the same spirit, turbulence in spatially extended systems can be described in terms of recurrent spatiotemporal patterns. Pictorially, dynamics drives a given spatially extended system through a repertoire of unstable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern. For any finite spatial resolution and finite time the system follows approximately a pattern belonging to a finite repertoire of possible patterns, and the long-term dynamics can be thought of as a walk through the space of such patterns. Recasting this image into mathematics is the subject of this book.

References

[2.1] E.N. Lorenz, *J. Atmospheric Phys.* **20**, 130 (1963).

[2.2] G. Duffing, *Erzwungene Schwingungen bei veränderlicher Eigenfrequenz* (Vieweg. Braunschweig 1918).

FIND good Duffing oscillator ref

[2.3] O. Rössler, Phys. Lett. **57A**, 397 (1976).

[2.4] J. Peinke, J. Parisi, O.E. Rössler, and R. Stoop, *Encounter with Chaos. Self-Organized Hierarchical Complexity in Semiconductor Experiments* (Springer, Berlin 1992).

[4] Mason: consider adding an appendix that lists available useful freeware, incl URLs: DSTool, xpp, AUTO, ... PC: links now included.

[2.5] R.H. Abraham, C.D. Shaw, *Dynamics - The Geometry of Behavior* (Addison-Wesley, Redwood, Ca, 1992).

[2.6] R. Gilmore and M. Lefranc, *The Topology of Chaos* (Wiley, New York, 2002).

[2.7] W.H. Press, B.P. Flannery, S.A. Teukolsky and W.T. Vetterling, *Numerical Recipes* (Cambridge University Press, 1986).

[2.8] J.D. Meiss, *Sci.nonlinear FAQ, Computational Resources*, `amath.colorado.edu/faculty/jdm/faq.html` .

[2.9] DSWeb Dynamical Systems Software, `www.dynamicalsystems.org`.

[2.10] J. Laskar, *Icarus* **88**, 257 (1990).

[2.11] Ch. Skokos, *J. Phys* **A 34**, 10029 (2001).

[2.12] P. Cvitanović, "Periodic orbits as the skeleton of classical and quantum chaos", *Physica* **D 51**, 138 (1991).

[2.13] M.W. Hirsch, "The dynamical systems approach to differential equations", *Bull. Amer. Math. Soc.* **11**, 1 (1984)

⇓PRELIMINARY

move to another chapter

[2.14] J. Milnor, *Experimental Math.*, **1**, 5 (1992).

[2.15] S. Simo, *J. Stat. Phys.* **21**, 465 (1979).

[2.16] W. Parry, *Comm. Math. Phys.* **106**, 267 (1986).

[2.17] Hopf E 1942 *Abzweigung einer periodischen Lösung Bereich. Sächs. Acad. Wiss. Leipzig, Math. Phys. Kl.* **94** 19.

This is presumably not the reference

[2.18] R.H.G. Helleman, in S. Jorna, ed., *"Variational solutions of non-integrable systems", Topics in nonlinear dynamics AIP Conf. Proc.,* **46** , 264 (1978).

[2.19] R.H.G. Helleman and T. Bountis, in G. Casati and J. Ford, eds., *Stochastic behavior in classical and quantum Hamiltonian systems*, p. 353 (Springer, Berlin 1979).

remainder of refs. is probably of no Das-Buch interest

[2.20] T. Bountis and R.H.G. Helleman, *"On the stability of periodic orbits of two-dimensional mappings", J. Math. Phys.* **22**, 1867 (1981).

[2.21] E.W. Montroll and R.H.G. Helleman, *"On a nonlinear perturbation theory without secular terms. II. Carleman embedding of nonlinear equations in an infinite set of linear ones", AIP Conf. Proc.,* **27** , 75 (1976).

[2.22] R.F.S. Andrade and A. Rauh, *"The Lorenz model and the method of Carleman embedding", Phys. Lett.* **A 82**, 276 (1981).
The Lorenz model is investigated by means of the Carleman embedding. Attention is focused on the analytical properties of the Laplace transform of a trajectory. It is shown that beyond the turbulent threshold, finite linear embedding violates basic analytical requirements

[2.23] W.H. Steeb, *"Embedding of nonlinear finite dimensional systems in linear infinite dimensional systems and Bose operators", Hadronic Journal* **6** 68, (1983).
Nonlinear autonomous systems x=f(x), where f is a polynomial in x, can be embedded into an infinite linear system with a technique described by Carleman (1932). The author demonstrates that the infinite linear system can be written with the help of Bose annihilation and creation operators

[2.24] K. Kowalski, *"Hilbert space description of classical dynamical systems"*, *Physica* **A 145**, 408 (1987).
Classical dynamical systems can be described by a Hilbert space, Schrödinger-like equation with the Hamiltonian (a non-Hermitian one) expressed in terms of Bose creation and annihilation operators. The presented formalism includes the Carleman embedding as the special case.

[2.25] K. Kowalski, *"Nonlinear dynamical systems and classical orthogonal polynomials"*, *J. Math. Phys.* **38**, 2483 (1997).
Nonlinear dynamical systems with analytic nonlinearities can be brought down to the abstract Schrödinger equation a in Hilbert space with a boson Hamiltonian. The Fourier coefficients of the expansion of solutions to the Schrödinger equation in the particular occupation number representation are expressed by means of the classical orthogonal polynomials. The introduced formalism amounts to a generalization of the classical methods for linearization of nonlinear differential equations such as the Carleman embedding technique and Koopman approach.

⇑PRELIMINARY