local!stability
linear!stability
stability!linear
cycle!stability
cycle!stability
topological!invariant
invariant!topological

# Chapter 8

# Cycle stability

Topological features of a dynamical system – singularities, periodic orbits, and the ways in which the orbits intertwine – are invariant under a general continuous change of coordinates. Surprisingly, there exist quantities that depend on the notion of metric distance between points, but nevertheless do not change value under a smooth change of coordinates. Local quantities such as the stability eigenvalues of equilibria and periodic orbits, and global quantities such as Lyapunov exponents, metric entropy, and fractal dimensions are examples of properties of dynamical systems independent of coordinate choice.

We now turn to the first, local class of such invariants, linear stability of periodic orbits of flows and maps. This will give us metric information about local dynamics. The material covered here is aimed at unbelieving Thomases - if you already know that the stability eigenvalues of periodic orbits are invariants of a flow, you can skip this chapter.

fast track:
chapter 9, p. 125

## 8.1 Stability of periodic orbits

As noted on page 34, a trajectory can be stationary, periodic or aperiodic. For chaotic systems almost all trajectories are aperiodic – nevertheless, stationary and periodic orbits will turn out to be the key to unraveling chaotic dynamics. Here we note a few of the properties that makes them so precious to a theorist. [1]

add figs

An obvious virtue of periodic orbits is that they are *topological* invariants: a fixed point remains a fixed point for any choice of coordinates, and similarly a periodic orbit remains periodic in any representation of the dynamics. Any re-parametrization of a dynamical system that preserves its

[1] Mason: does not jive with relative equilibira in a 3-body problem? This must be rephrased more carefully

metric!invariant
invariant!metric
stability!eigenvalues
expanding!stability eigenvalues
marginal!stability
contracting!stability eigenvalues
stability!exponents
exponent!stability

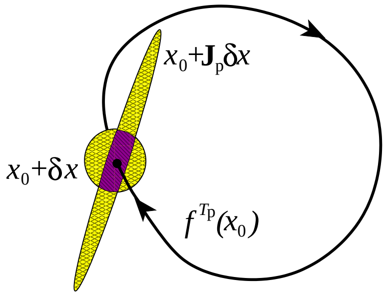


**Figure 8.1:** For a prime cycle $p$, Jacobian matrix $\mathbf{J}_p$ returns an infinitesimal spherical neighborhood of $x_0 \in p$ stretched into an ellipsoid, with overlap ratio along the eigenvector $\mathbf{e}_i$ of $\mathbf{J}_p(x)$ given by the eigenvalue $\Lambda_{p,i}$. These ratios are invariant under smooth nonlinear reparametrizations of phase space coordinates, and are intrinsic property of cycle $p$.

topology has to preserve topological relations between periodic orbits, such as their relative inter-windings and knots. So the mere existence of periodic orbits suffices to partially organize the spatial layout of a non–wandering set. No less important, as we shall now show, is the fact that cycle stability eigenvalues are *metric* invariants: they determine the relative sizes of neighborhoods in a non–wandering set.

give references to templates

To prove this, we start by noting that due to the multiplicative structure (4.31) of Jacobian matrices, the stability of the $r$th repeat of a prime cycle $p$ of period $T_p$ is

$$\mathbf{J}^{rT_p}(x) = \mathbf{J}^{T_p}(f^{(r-1)T_p}(x)) \cdots \mathbf{J}^{T_p}(f^{T_p}(x))\mathbf{J}^{T_p}(x) = (\mathbf{J}_p(x))^r \ , \quad (8.1)$$

emphasize - eigenvalues are NOT multiplicative unless on a periodic orbit

where $\mathbf{J}_p(x) = \mathbf{J}^{T_p}(x)$ is the stability matrix for a single traversal of the prime cycle $p$, $x \in p$ is any point on the cycle, and $f^{rT_p}(x) = x$ as $f^t(x)$ returns to $x$ every multiple of the period $T_p$. Hence, it suffices to restrict our considerations to the stability of prime cycles.

do not like name "stability multipliers", as $\Lambda$'s are not multiplicative

### 8.1.1 Stability eigenvalues and exponents

We sort the *stability eigenvalues* $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ of the $[d\times d]$ Jacobian matrix $\mathbf{J}_p$ evaluated on the $p$-cycle into sets $\{e, m, c\}$

$$\begin{aligned} \text{expanding:} \quad & \{\Lambda_p\}_e &= \{\Lambda_{p,j} : |\Lambda_{p,j}| > 1\} \\ \text{marginal:} \quad & \{\Lambda_p\}_m &= \{\Lambda_{p,j} : |\Lambda_{p,j}| = 1\} \\ \text{contracting:} \quad & \{\Lambda_p\}_c &= \{\Lambda_{p,j} : |\Lambda_{p,j}| < 1\} . \end{aligned} \quad (8.2)$$

and denote by $\Lambda_p$ (no $j$th eigenvalue index) the product of *expanding* eigenvalues

$$\Lambda_p = \prod_e \Lambda_{p,e} . \quad (8.3)$$

As $\mathbf{J}_p$ is a real matrix, complex eigenvalues always come in complex conjugate pairs, $\Lambda_{p,i+1} = \Lambda^*_{p,i}$, so the product of expanding eigenvalues $\Lambda_p$ is always real.

Cycle *stability exponents* are the stretching/contraction rates per unit time

Lyapunov exponent!cycle
cycle!Lyapunov exponent
stability!window
hyperbolic!flow
flow!hyperbolic
nonhyperbolic!flow
flow!nonhyperbolic
flow!elliptic

$$\lambda_{p,i} = \frac{1}{T_p} \ln |\Lambda_{p,i}| \,. \tag{8.4}$$

This definition is motivated by the form of the stability exponents for the linear dynamical systems, for example (4.14), as well as the fact that exponents so defined can be interpreted as Lyapunov exponents (10.32) evaluated on the prime cycle $p$. As in the three cases of (8.2), we sort the stability exponents into three sets

☞ sect. 10.3

$$\begin{aligned} \text{expanding:} \quad & \{\lambda_p\}_e && = \{\lambda_{p,i}:\ \lambda_{p,i} > 0\} \\ \text{marginal:} \quad & \{\lambda_p\}_m && = \{\lambda_{p,i}:\ \lambda_{p,i} = 0\} \\ \text{contracting:} \quad & \{\lambda_p\}_c && = \{\lambda_{p,i}:\ \lambda_{p,i} < 0\}\,. \end{aligned} \tag{8.5}$$

A periodic orbit $p$ of a $d$-dimensional flow or a map is *stable* if all its stability exponents (other than the vanishing longitudinal stability exponent, to be explained in sect. 8.2.1 below) are strictly negative, $|\lambda_{p,i}| < 0$. The region of system parameter values for which a periodic orbit $p$ is stable is called the *stability window* of $p$. The set $\mathcal{M}_p$ of initial points that are asymptotically attracted to $p$ as $t \to +\infty$ (for a fixed set of system parameter values) is called the *basin of attraction* of $p$.

If *all* stability exponents (other than the vanishing longitudinal stability exponent) of *all* periodic orbits of a flow are strictly bounded away from zero, $|\lambda_i| \geq \lambda_{min} > 0$, the flow is said to be *hyperbolic*. Otherwise the flow is said to be *nonhyperbolic*. In particular, if all $|\lambda_i| = 0$, the orbit is said to be *elliptic*. Such orbits proliferate in Hamiltonian flows. [2]

☞ sect. 5.3

We often do care about the sign of $\Lambda_{p,i}$ and, if $\Lambda_{p,i}$ is complex, its phase

$$\Lambda_{p,j} = \pm e^{T_p(\lambda_{p,j} \pm i\theta_{p,j})} \,. \tag{8.6}$$

☞ sect. 5.2

Keeping track of this by case-by-case enumeration is a self-inflicted, unnecessary nuisance, followed in much of the literature. To avoid this, almost all of our formulas will be stated in terms of the stability eigenvalues $\Lambda_j$ rather than in the terms of the overall signs, stability exponents $\lambda_j$ and phases $\theta_j$.

[2] Mason: for infinite dimensional flows one cannot have $\lambda_{min}$; in my thesis have a sequence $|\lambda_i| \to 0$, and all $|\lambda_i| \neq 0$.
PC: I think you are mixing up stability eigenvalues $\Lambda$ with stability exponents $\lambda$ in this and the next comment. My $\lambda_{min}$ is a lower bound on $|\lambda_i|$, not $\lambda_i$, so I'm OK unless I have eigenvalues arbitrarily close to 0; but then the flow is not hypebolic

critical point
cycle!superstable
superstable cycle
cycle!stable
stable!cycle
unstable!cycle
Henon@Hénon!map!stability
stability!billiards
billiards!stability

**Example 8.1 1-dimensional maps:** *The simplest example of cycle stability is afforded by 1-dimensional maps. The stability of a prime cycle $p$ follows from the chain rule (4.34) for stability of the $n_p$th iterate of the map*

$$\Lambda_p = \frac{d}{dx_0} f^{n_p}(x_0) = \prod_{m=0}^{n_p-1} f'(x_m)\,, \quad x_m = f^m(x_0)\,. \tag{8.7}$$

*$\Lambda_p$ is a property of the cycle, not the initial point, as taking any periodic point in the $p$ cycle as the initial point yields the same result.*

*A* critical point *$x_c$ is a value of $x$ for which the mapping $f(x)$ has vanishing derivative, $f'(x_c) = 0$. For future reference we note that a periodic orbit of a 1-dimensional map is* stable *if*

$$|\Lambda_p| = \left|f'(x_{n_p})f'(x_{n_p-1})\cdots f'(x_2)f'(x_1)\right| < 1\,,$$

*and* superstable *if the orbit includes a critical point, so that the above product vanishes. For a stable periodic orbit of period $n$ the slope of the $n$th iterate $f^n(x)$ evaluated on a periodic point $x$ (fixed point of the $n$th iterate) lies between $-1$ and $1$. If $|\Lambda_p| > 1$, $p$-cycle is* unstable.

**Example 8.2 Stability of cycles for maps:** *No matter what method we had used to determine the unstable cycles, the theory to be developed here requires that their stability eigenvalues be evaluated as well. For maps a Jacobian matrix is easily evaluated by picking any cycle point as a starting point, running once around a prime cycle, and multiplying the individual cycle point stability matrices according to (4.35). For example, the Jacobian matrix $\mathbf{J}_p$ for a Hénon map (3.12) prime cycle $p$ of length $n_p$ is given by (4.36),*

$$\mathbf{J}_p(x_0) = \prod_{m=n}^{1} \begin{pmatrix} -2ax_m & b \\ 1 & 0 \end{pmatrix}\,, \qquad x_m \in p\,,$$

*and the Jacobian matrix $\mathbf{J}_p$ for a 2-dimensional billiard prime cycle $p$ of length $n_p$*

$$\mathbf{J}_p = (-1)^{n_p} \prod_{k=n_p}^{1} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_k & 1 \end{pmatrix}$$

*follows from (6.11).*

## 8.2 Cycle stabilities are cycle invariants

The 1-dimensional map cycle stability $\Lambda_p$ is a product of derivatives over all points around the cycle, and is therefore independent of which periodic point is chosen as the initial one. In higher dimensions the form of the Jacobian matrix $\mathbf{J}_p(x_0)$ in (8.1) does depend on the choice of coordinates and the initial point $x_0 \in p$. Nevertheless, as we shall now show, the

invariance!of flows
nonhyperbolic!flow
*cycle!marginal stability*
*marginal stability!cycle*

cycle stability *eigenvalues* are intrinsic property of a cycle also for multi-dimensional flows. Consider the $i$th eigenvalue, eigenvector pair $(\Lambda_{p,i}, \mathbf{e}_i)$ computed from $\mathbf{J}_p$ evaluated at a cycle point,

$$\mathbf{J}_p(x)\mathbf{e}_i(x) = \Lambda_{p,i}\mathbf{e}_i(x)\,, \quad x \in p\,. \tag{8.8}$$

Consider another point on the cycle at time $t$ later, $x' = f^t(x)$ whose Jacobian matrix is $\mathbf{J}_p(x')$. By the group property (4.31), $\mathbf{J}^{T_p+t} = \mathbf{J}^{t+T_p}$, and the Jacobian matrix at $x'$ can be written either as

$$\mathbf{J}^{T_p+t}(x) = \mathbf{J}^{T_p}(x')\mathbf{J}^t(x) = \mathbf{J}_p(x')\mathbf{J}^t(x)\,, \quad \text{or} \quad \mathbf{J}^{t+T_p}(x) = \mathbf{J}^t(x)\mathbf{J}_p(x)\,.$$

Multiplying (8.8) by $\mathbf{J}^t(x)$, we find that the Jacobian matrix evaluated at $x'$ has the same eigenvalue,

$$\mathbf{J}_p(x')\mathbf{e}_i(x') = \Lambda_{p,i}\mathbf{e}_i(x')\,, \quad \mathbf{e}_i(x') = \mathbf{J}^t(x)\mathbf{e}_i(x)\,, \tag{8.9}$$

but with the eigenvector $\mathbf{e}_i$ transported along the flow $x \to x'$ to $\mathbf{e}_i(x') = \mathbf{J}^t(x)\mathbf{e}_i(x)$. Hence, $\mathbf{J}_p$ evaluated anywhere along the cycle has the same set of stability eigenvalues $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d}\}$. As quantities such as $\operatorname{tr}\mathbf{J}_p(x)$, $\det \mathbf{J}_p(x)$ depend only on the eigenvalues of $\mathbf{J}_p(x)$ and not on the starting point $x$, in expressions such as $\det\left(\mathbf{1} - \mathbf{J}_p^r(x)\right)$ we may omit reference to any particular cycle point $x$:

$$\det\left(\mathbf{1} - \mathbf{J}_p^r\right) = \det\left(\mathbf{1} - \mathbf{J}_p^r(x)\right) \quad \text{for any } x \in p\,. \tag{8.10}$$

We postpone the proof that the cycle stability eigenvalues are smooth conjugacy invariants of the flow to sect. 8.5.

### 8.2.1 Marginal eigenvalues

The presence of marginal eigenvalues signals either an invariance of the flow (which one should immediately exploit to simplify the problem), or a non-hyperbolicity of a flow (a source of much pain, hard to avoid).

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page 124

**Example 8.3 A periodic orbit of a continuous flow has a marginal eigenvalue:** *As $\mathbf{J}^t(x)$ transports the velocity field $v(x)$ by (4.6), after a complete period*

$$\mathbf{J}_p(x)v(x) = v(x)\,, \tag{8.11}$$

*so a periodic orbit of a* flow *always has an eigenvector $\mathbf{e}_\parallel(x) = v(x)$ parallel to the local velocity field with the unit eigenvalue*

$$\Lambda_{p,\parallel} = 1\,. \tag{8.12}$$

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page 112

Poincar“'e!return map!cycle stability!Poincar“'e map cycle

*The continuous invariance that gives rise to this marginal eigenvalues is the invariance of a cycle under a translation of its points along the cycle: two points on the cycle initially distance $\delta x$ apart, $x'(0) - x(0) = \delta x(0)$, are separated by the exactly same $\delta x$ after a full period $T_p$. As we shall see in sect. 8.3, this marginal stability direction can be eliminated by cutting the cycle by a Poincaré section and eliminating the continuous flow Jacobian matrix in favor of the Jacobian matrix of the Poincaré return map.*

explain “ odromy”?

If the flow is governed by a time-independent Hamiltonian, the energy is conserved, and that leads to an additional marginal eigenvalue (remember, by symplectic invariance (5.17) real eigenvalues come in pairs). For the collinear helium of sect. 7.3 this marginal eigenvalue is made explicit in appendix D.2.

⇓PRELIMINARY

⇑PRELIMINARY

## 8.3 Stability of Poincaré map cycles

(R. Paškauskas and P. Cvitanović)

If a continuous flow periodic orbit $p$ pierces the Poincaré section $\mathcal{P}$ once, the section point is a fixed point of the Poincaré return map $P$ with stability (4.40)

$$\hat{\mathbf{J}}_{ij} = \left(\delta_{ik} - \frac{v_i\, U_k}{(v \cdot U)}\right) \mathbf{J}_{kj}\,, \tag{8.13}$$

with all primes dropped, as the initial and the final points coincide, $x' = f^{T_p}(x) = x$. If the periodic orbit $p$ pierces the set of Poincaré sections $\mathcal{P}$ $n$ times, the same observation applies to the $n$th iterate of $P$.

We have already established in (4.41) that the velocity $v(x)$ is a zero-eigenvector of the Poincaré section Jacobian matrix, $\hat{\mathbf{J}} v = 0$. Consider next $(\Lambda_{p,\alpha}, \mathbf{e}_\alpha)$, the full phase space $\alpha$th (eigenvalue, eigenvector) pair (8.8), evaluated at a cycle point on a Poincaré section,

$$\mathbf{J}(x)\mathbf{e}_\alpha(x) = \Lambda_\alpha \mathbf{e}_\alpha(x)\,, \quad x \in \mathcal{P}\,. \tag{8.14}$$

Multiplying (8.13) by $\mathbf{e}_\alpha$ and inserting (8.14), we find that the Poincaré section Jacobian matrix $\hat{\mathbf{J}}$ has the same eigenvalue as the full phase space Jacobian matrix,

$$\hat{\mathbf{J}}(x)\hat{\mathbf{e}}_\alpha(x) = \Lambda_\alpha \hat{\mathbf{e}}_\alpha(x)\,, \quad x \in \mathcal{P}\,. \tag{8.15}$$

where $\hat{\mathbf{e}}_\alpha$ is a projection of the full phase space eigenvector onto the Poincaré section:

note: normalization not preserved

$$(\hat{\mathbf{e}}_\alpha)_i = \left(\delta_{ik} - \frac{v_i\, U_k}{(v \cdot U)}\right) (\mathbf{e}_\alpha)_k\,. \tag{8.16}$$

figure

Hence, $\hat{\mathbf{J}}_p$ evaluated on any Poincaré section point along the cycle $p$ has the same set of stability eigenvalues $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d}\}$ as the full phase space Jacobian matrix $\mathbf{J}_p$.

## 8.4 Rectification of a 1-dimensional periodic orbit

In sect. 7.4.1 we have constructed the conjugation function for a fixed point. Here we turn to the problem of constructing it for periodic orbits. Each point around the cycle has a differently distorted neighborhood, with differing second and higher order derivatives, so we need to compute a different conjugation function $h_a$ at each cycle point $x_a$. We expand the map $f$ around each cycle point along the cycle,

$$y_a(\phi) = f_a(\phi) - x_{a+1} = \phi f_{a,1} + \phi^2 f_{a,2} + \ldots$$

where $x_a$ is a point on the cycle, $f_a(\phi) = f(x_a+\phi)$ is centered on the periodic orbit, and the index $k$ in $f_{a,k}$ refers to the $k$th order in the expansion (7.20).

For a periodic orbit the conjugation formula (7.19) generalizes to

$$f_a(\phi) = h_{a+1}^{-1}(f_a'(0) h_a(\phi)), \qquad a = 1, 2, \cdots, n,$$

point by point. The conjugationg functions $h_a$ are obtained in the same way as before, by equating coefficients of the expansion (7.20), and assuming that the cycle stability $\Lambda = \prod_{a=0}^{n-1} f'(x_a)$ is not marginal, $|\Lambda| \neq 1$. The explicit expressions for $h_a$ in terms of $f$ are obtained by iterating around the whole cycle,

compute only onece, then propagate around the cycle by iteration?

$$f^n(x_a + \phi) = h_a^{-1}(\Lambda h_a(\phi)) + x_a \,. \tag{8.17}$$

evaluated at each cycle point $a$. Again we have the freedom to set $h_a'(0) = 1$ for all $a$.

☞ remark 7.1

### 8.4.1 Repeats of cycles

We have traded in our initial nonlinear map $f$ for a (locally) linear map $\Lambda y$ and an equally complicated conjugation function $h$. What is gained by rewriting the map $f$ in terms of the conjugacy function $h$? Once the neighborhood of a fixed point is linearized, the repeats of it are trivialized; from the conjugation formula (7.20) one can compute the derivatives of a function composed with itself $r$ times:

$$f^r(x) = h^{-1}(\Lambda^r h(x)) \,.$$

stability!eigenvalues, metric invariants
metric!invariant!stability eigenvalues
smooth!conjugacy
conjugacy!smooth
metric!invariant
invariant!metric
conjugacy!invariant

One can already discern the form of the expansion for arbitrary repeats; the answer will depend on the conjugacy function $h(x)$ computed for a *single* repeat, and all the dependence on the repeat number will be carried by factors polynomial in $\Lambda^r$, a considerable simplification. The beauty of the idea is difficult to gauge at this stage - an appreciation only sets in when one starts computing perturbative corrections, be it in celestial mechanics (where the method was born), be it the quantum or stochastic corrections to "semiclassical" approximations.

## 8.5 Smooth conjugacies and cycle stability

In sect. 8.2 we have established that for a given flow the cycle stability eigenvalues are intrinsic to a given cycle, independent of the staring point along the cycle. Now we can prove a much stronger statement; cycle stability eigenvalues are *metric invariants* of the flow, the same in *any* representation of the dynamical system.

That the cycle stability eigenvalues are an invariant property of the given dynamical system follows from elementary considerations of sect. 7.1: If the same dynamics is given by a map $f$ in $x$ coordinates, and a map $g$ in the $y = h(x)$ coordinates, then $f$ and $g$ (or any other good representation) are related by (7.2), a reparametrization and a coordinate transformation $g = h \circ f \circ h^{-1}$. As both $f$ and $g$ are arbitrary representations of the dynamical system, the explicit form of the conjugacy $h$ is of no interest, only the properties invariant under any transformation $h$ are of general import. Furthermore, a good representation should not mutilate the data; $h$ must be a *smooth conjugacy* which maps nearby cycle points of $f$ into nearby cycle points of $g$. This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metrically invariant. For a fixed point $f(x) = x$ of a 1-dimensional map this follows from the chain rule for derivatives,

$$\begin{aligned} g'(y) &= h'(f \circ h^{-1}(y)) f'(h^{-1}(y)) \frac{1}{h'(x)} \\ &= h'(x) f'(x) \frac{1}{h'(x)} = f'(x)\,, \end{aligned} \tag{8.18}$$

and the generalization to the stability eigenvalues of periodic orbits of $d$-dimensional flows is immediate.

Is it? add $d$-dimensional case

As stability of a flow can always be rewritten as stability of a Poincaré section return map, we find that the stability eigenvalues of any cycle, for a flow or a map in arbitrary dimension, is a metric invariant of the dynamical system.

argue $det \to tr$ can be waste of time

add $d$-dimensional case

7.2
page 112

in depth:
appendix D.1, p. 797

## 8.6 Neighborhood of a cycle

Jacobian
Lyapunov exponent
contracting!flow
flow!contracting
incompressible flow
flow!incompressible
Roux!Henri

sect. 10.3

remark 10.3

explain that eigenvalue 1 along the flow requires "gauge fixing", choice of arbitrary starting point

4.1 page 73

The Jacobian of the flow (or the *sum* of stability exponents) is easily evaluated.

Consider $\det \mathbf{J}^t(x_0) = \prod_{i=1}^d \Lambda_i(x_0, t)$, the product of the stability eigenvalues. We shall refer to this determinant as the *Jacobian* of the flow. By means of the time-ordered product (4.29) and the identity $\ln \det M = \operatorname{tr} \ln M$ the Jacobian is given by

$$\det \mathbf{J}^t(x_0) = \exp \left[ \int_0^t d\tau \, \operatorname{tr} \mathbf{A}(x(\tau)) \right] = \exp \left[ \int_0^t d\tau \, \partial_i v_i(x(\tau)) \right] . \tag{8.19}$$

(Here, as elsewhere in this book, a repeated index implies summation.) As the divergence $\partial_i v_i$ is a scalar quantity, the integral in the exponent needs no time ordering. All we need to do is evaluate the time average

$$\begin{aligned} \langle \partial_i v_i \rangle_t &= \frac{1}{t} \int_0^t d\tau \sum_{i=1}^d A_{ii}(x(\tau)) \\ &= \frac{1}{t} \ln \left| \prod_{i=1}^d \Lambda_i(x_0, t) \right| = \sum_{i=1}^d \lambda_i(x_0, t) \end{aligned} \tag{8.20}$$

along the trajectory. If the flow is not singular (for example, the trajectory does not run head-on into the Coulomb $1/r$ singularity), the matrix of variations elements are bounded everywhere, $|A_{ij}| < M$ , and so is the trace $\sum_i A_{ii}$. The time integral in (8.20) grows at most linearly with $t$, hence $\langle \partial_i v_i \rangle_t$ is bounded for all times, and numerical estimates of the $t \to \infty$ limit of $\langle \partial_i v_i \rangle$ are not marred by any blowups.

Even if we were to insist on extracting $\langle \partial_i v_i \rangle$ from (4.29) by first multiplying stability matrices along the flow, and then taking the logarithm, we can avoid exponential blowups in $\mathbf{J}^t$ by using the multiplicative structure (4.31), $\det \mathbf{J}^{t'+t}(x_0) = \det \mathbf{J}^{t'}(x') \det \mathbf{J}^t(x_0)$ to restart with $\mathbf{J}^0(x') = \mathbf{1}$ whenever the eigenvalues of $\mathbf{J}^t(x_0)$ start getting out of hand. In numerical evaluations of Lyapunov exponents, $\lambda_i = \lim_{t\to\infty} \lambda_i(x_0, t)$, the sum rule (8.20) can serve as a helpful check on the accuracy of the computation. [3]

sect. 10.3

The divergence $\partial_i v_i$ is an important characterization of the flow - it describes the behavior of a phase space volume in the infinitesimal neighborhood of the trajectory. If $\partial_i v_i < 0$, the flow is *locally contracting*, and the trajectory might be falling into an attractor. If $\partial_i v_i = 0$, the flow preserves phase space volume and $\det \mathbf{J}^t = \mathbf{1}$. A flow with this property is called *incompressible*. An important class of such flows are the Hamiltonian flows considered in sect. 5.2.

[3] Mason: there is a good Matlab module that computes a Lyapunov exp - will provide a reference

neighborhood

But before we can get to that, Henri Roux, the perfect student always on alert, pipes up. He does not like our definition of the Jacobian matrix in terms of the time-ordered exponential (4.30). Depending on the signs of stability eigenvalues, the left hand side of (8.19) can be either positive or negative. But the right hand side is an exponential of a real number, and that can only be positive. What gives? As we shall see much later on in this text, in discussion of topological indices arising in semiclassical quantization, this is not at all a dumb question.

☞ sect. 30.2

⇓PRELIMINARY

☞ sect. 31.3.2

⇑PRELIMINARY

⇓PRELIMINARY

⇑PRELIMINARY

in depth:
appendix M.1, p. 883

in depth:
appendix D.1, p. 797

### 8.6.1 There goes the neighborhood

In what follows, our task will be to determine the size of a *neighborhood* of $x(t)$, and that is why we care about the stability eigenvalues, and especially the unstable (expanding) ones. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. The sub-volume $|\mathcal{M}_i| = \prod_i^e \Delta x_i$ of the set of points which get no further away from $f^t(x_0)$ than $L$, the typical size of the system, is fixed by the condition that $\Delta x_i \Lambda_i = O(L)$ in each expanding direction $i$. Hence the neighborhood size scales as $\propto 1/|\Lambda_p|$ where $\Lambda_p$ is the product of expanding eigenvalues (8.3) only; contracting ones play a secondary role. So secondary that even infinitely many of them (for example, the infinity of contracting eigendirections of the spatiotemporal dynamics of sect. 24.1) will not matter.

add a figure of hyperbolic flow

fix

So the physically important information is carried by the expanding sub-volume, not the total volume computed so easily in (8.20). That is also the reason why the dissipative and the Hamiltonian chaotic flows are much more alike than one would have naively expected for “compressible” *vs.* “incompressible” flows. In hyperbolic systems what matters are the expanding directions. Whether the contracting eigenvalues are inverses of the expanding ones or not is of secondary importance. As long as the number of unstable directions is finite, the same theory applies both to the finite-dimensional ODEs and infinite-dimensional PDEs.

⇓PRELIMINARY

## Commentary

**Remark 8.1** A resonance condition. In the hyperbolic case there is a resonance condition that must be satisfied: none of the stability exponents may be related by ratios of integers. That is, if $\Lambda_{p,1}, \Lambda_{p,2}, \ldots, \Lambda_{p,d}$ are the eigenvalues of the stability matrix, then they are in resonance if there exist integers $n_1, \ldots, n_d$ such that

$$(\Lambda_{p,1})^{n_1} (\Lambda_{p,2})^{n_2} \cdots (\Lambda_{p,d})^{n_d} = 1 \,.$$

If there is resonance, then one may get corrections to the basic conjugation formulas in the form of monomials in the variables of the map. give reference

(R. Mainieri)

⇑PRELIMINARY

## Résumé

Periodic orbits play a central role in any invariant characterization of the dynamics, because (a) their existence and inter-relations are a *topological*, coordinate-independent property of the dynamics, and (b) their stability eigenvalues form an infinite set of *metric invariants*: The stability eigenvalues of a periodic orbit remain invariant under any smooth nonlinear change of coordinates $f \to h \circ f \circ h^{-1}$ .

We shall show in chapter 11 that extending their local stability eigendirections into stable and unstable manifolds yields important global information about the topological organization of phase space.

The physically important information is carried by the unstable manifold, and the expanding sub-volume characterized by the product of expanding eigenvalues of $\mathbf{J}_p$. In hyperbolic systems what matters are the expanding directions. As long as the number of unstable directions is finite, the theory can be applied to flows of arbitrarily high dimension. create refsInvariant.tex