local!stability
linear!stability
stability!linear
neighborhood
equation!of variations

Chapter 4

Local stability

(R. Mainieri and P. Cvitanović)

So far we have concentrated on description of the trajectory of a single initial point. Our next task is to define and determine the size of a *neighborhood* of $x(t)$. We shall do this by assuming that the flow is locally smooth, and describe the local geometry of the neighborood by studying the flow linearized around $x(t)$. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. As we shall demonstrate in chapter 14, in hyperbolic systems what matters are the expanding directions. The repercussion are far-reaching: As long as the number of unstable directions is finite, the same theory applies to finite-dimensional ODEs, phase-space volume preserving Hamiltonian flows, and dissipative, volume contracting infinite-dimensional PDEs.

4.1 Flows transport neighborhoods

As a swarm of representative points moves along, it carries along and distorts neighborhoods, as sketched in figure 2.1(b). The deformation of an infinitesimal neighborhood is best understood by considering a trajectory originating near $x_0 = x(0)$ with an initial infinitesimal displacement $\delta x(0)$, and letting the flow transport the displacement $\delta x(t)$ along the trajectory $x(x_0, t) = f^t(x_0)$. The system of linear *equations of variations* for the displacement of the infinitesimally close neighbor $x + \delta x$ follows from the flow equations (2.5) by Taylor expanding to linear order

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j \, .$$

The infinitesimal displacement δx is thus transported along the trajectory $x(x_0, t)$, with time variation given by

might want to cite Arnold [7.1]

tangent!space
tangent!bundle
matrix!of variations
initial!point
$“xInit$
Jacobian matrix
linearized!flow
flow!linearized
Lagrangian!coordinates
perhaps replace A_{ij} by
A_{ij}? and $\mathbf{J}_{ij}$?
Eulerian!coordinates
eigendirection

$$\frac{d}{dt}\delta x_i(x_0,t) = \sum_j \left.\frac{\partial v_i(x)}{\partial x_j}\right|_{x=x(x_0,t)} \delta x_j(x_0,t)\,. \tag{4.1}$$

As both the displacement and the trajectory depend on the initial point x_0 and the time t, we shall often abbreviate the notation to $x(x_0,t) \to x(t) \to x$, $\delta x_i(x_0,t) \to \delta x_i(t) \to \delta x$ in what follows. Taken together, the set of equations

$$\dot{x}_i = v_i(x)\,, \quad \dot{\delta x}_i = \sum_j A_{ij}(x)\delta x_j \tag{4.2}$$

fix notation: $(\mathbf{T})\mathcal{M}$?

governs the dynamics in the tangent bundle $(x,\delta x) \in \mathbf{T}\mathcal{M}$ obtained by adjoining the d-dimensional tangent space $\delta x \in \mathbf{T}_x\mathcal{M}$ to every point $x \in \mathcal{M}$ in the d-dimensional phase space $\mathcal{M} \subset \mathbb{R}^d$. The matrix of variations [1]

draw infinitesim. displc., explain what you mean by shearing

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} \tag{4.3}$$

describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow.

Taylor expanding a *finite time* flow to linear order,

$$f_i^t(x_0+\delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}}\delta x_j + \cdots\,, \tag{4.4}$$

one finds that the linearized neighborhood is transported by the *Jacobian* (or *fundamental*) *matrix*

draw finite displc.

$$\delta x(t) = \mathbf{J}^t(x_0)\delta x(0)\,, \qquad \mathbf{J}^t_{ij}(x_0) = \left.\frac{\partial x_i(t)}{\partial x_j}\right|_{x=x_0}\,. \tag{4.5}$$

add a section on eigenvalues of matrices

which describes the deformation of an infinitesimal neighborhood at finite time t in the co-moving frame of $x(t)$, that is, transformation of the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

As this is a deformation in the linear approximation, one can think of it as a linear deformation of an infinitesimal sphere enveloping x_0 into an ellipsoid around $x(t)$, described by the eigenvectors and eigenvalues of the Jacobian matrix of the linearized flow, figure 4.1 Nearby trajectories separate along the *unstable directions*, approach each other along the *stable directions*, and change their distance along the *marginal directions* at a rate slower than exponential. In the literature adjectives *neutral* or *indifferent* are often used instead of “marginal”, (attracting) stable directions

[1] Mason: say ”variational matrix”. PC: I follow Arnold’s ”matrix of variations”, will recheck

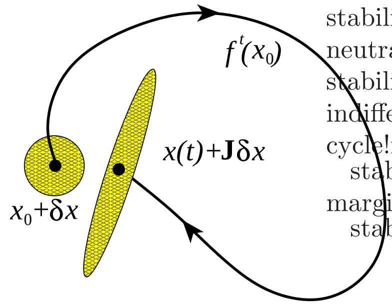


Figure 4.1: Jacobian matrix maps an infinitesimal spherical neighborhood of x_0 into an ellipsoidal neighborhood time t later.

stability!neutral
neutral!stability
stability!indifferent
indifferent!stability
cycle!marginal stability
marginal stability!cycle

are sometimes called "asymptotically stable", the (neutrally) stable directions "stable", and so on: but all one is saying is that the eigenvalues of the Jacobian matrix have maginitude smaller, equal, or larger than 1.

One of the eigendirections is what one might expect, the direction of the flow itself. To see that, consider two points along a trajectory separated by infinitesimal flight time δt: $\delta x(0) = f^{\delta t}(x_0) - x_0 = v(x_0)\delta t$. At time t,

$$\delta x(t) = f^{t+\delta t}(x_0) - f^t(x_0) = f^{\delta t}(x(t)) - x(t) = v(x(t))\,\delta t\,,$$

while from (4.5) $\delta x(t) = \mathbf{J}^t(x_0)\delta x(0)$. Dividing both equations by δt we observe that $\mathbf{J}^t(x_0)$ transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

$$v(x(t)) = \mathbf{J}^t(x_0)\,v(x_0)\,. \tag{4.6}$$

As noted above, the Jacobian matrix (4.5) maps the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

The velocity at point $x(t)$ in general does not point in the same direction as the velocity at point x_0, so this is not an eigenvalue condition for $\mathbf{J}^t$; the Jacobian matrix computed for an arbitrary segment of an arbitrary trajectory has no invariant meaning.

As the eigenvalues of finite time $\mathbf{J}^t$ have invariant meaning only for periodic orbits, we postpone their interpretation to chapter 8. However, already at this stage we see that if the orbit is periodic, $x(T_p) = x(0)$, at any point along cycle p the velocity v is an eigenvector of the Jacobian matrix $\mathbf{J}_p = \mathbf{J}^{T_p}$ with a unit eigenvalue,

$$\mathbf{J}^{T_p}(x)\,v(x) = v(x)\,, \qquad x \in p\,. \tag{4.7}$$

Two successive points along the cycle separated by $\delta x(0)$ have exactly the same separation after a completed period $\delta x(T_p) = \delta x(0)$, hence eigenvalue 1.

As we started by assuming that we know the equations of motion, from (4.3) we also know $\mathbf{A}$, the instantaneous rate of shear of an infinitesimal

linear!flows
flow!linear
trajectory

neighborhood $\delta x_i(t)$ of the trajectory $x(t)$. What we do not know is the finite time deformation matrix $\mathbf{J}^t$. Our next task is to relate the instantaneous shear rate $\mathbf{A}$ to $\mathbf{J}^t$.

We are interested in smooth, differentiable flows. If a flow is smooth, in a sufficiently small neighborhood it is essentially linear. Hence the next section, which might seem an embarrassment (what is a section on *linear* flows doing in a book on *non*linear dynamics?), offers a firm stepping stone on the way to understanding nonlinear flows. If you know your eigenvalues and eigenvectors, you may prefer to fast forward here.

fast track:
sect. 4.3, p. 66

4.2 Linear flows

Linear fields are the simplest vector fields. Described by linear differential equations which can be solved explicitly, with solutions that are good for all times. The phase space for linear differential equations is $\mathcal{M} = \mathbb{R}^d$, and the equations of motion (2.5) are written in terms of a vector x and a constant matrix of variations $\mathbf{A}$ as

$$\dot{x} = v(x) = \mathbf{A}x\,. \tag{4.8}$$

Solving this equation means finding the phase space trajectory

$$x(t) = (x_1(t), x_2(t), \ldots, x_d(t))$$

passing through the point x_0.

If $x(t)$ is a solution with $x(0) = x_0$ and $y(t)$ another solution with $y(0) = y_0$, then the linear combination $ax(t) + by(t)$ with $a, b \in \mathbb{R}$ is also a solution, but now starting at the point $ax_0 + by_0$. At any instant in time, the space of solutions is a d-dimensional vector space, which means that one can find a basis of d linearly independent solutions. How do we solve the linear differential equation (4.8)? If instead of a matrix equation we have a scalar one, $\dot{x} = ax$, with a a real number, then the solution is

$$x(t) = e^{ta}x(0)\,. \tag{4.9}$$

In order to solve the matrix case, it is helpful to rederive the solution (4.9) by studying what happens for a short time step δt. If at time $t = 0$ the position is $x(0)$, then

$$\frac{x(\delta t) - x(0)}{\delta t} = ax(0)\,, \tag{4.10}$$

which we iterate m times to obtain the Euler's formula for compounding interest

exponential!of a matrix
matrix!exponential
stability!eigenvalue
stability!exponent
Lyapunov exponent

$$x(t) \approx \left(1 + \frac{t}{m}a\right)^m x(0)\,. \tag{4.11}$$

⇓PRELIMINARY
??
page ??
⇑PRELIMINARY

The term in parentheses acts on the initial condition $x(0)$ and evolves it to $x(t)$ by taking m small time steps $\delta t = t/m$. As $m \to \infty$, the term in parentheses converges to e^{ta}. Consider now the matrix version of equation (4.10):

$$\frac{x(\delta t) - x(0)}{\delta t} = \mathbf{A} x(0)\,. \tag{4.12}$$

A representative point x is now a vector in $\mathbb{R}^d$ acted on by the matrix $\mathbf{A}$, as in (4.8). Denoting by $\mathbf{1}$ the identity matrix, and repeating the steps (4.10) and (4.11) we obtain Euler's formula for the exponential of a matrix

$$x(t) = \lim_{m\to\infty}\left(\mathbf{1} + \frac{t}{m}\mathbf{A}\right)^m x(0) = e^{t\mathbf{A}} x(0)\,. \tag{4.13}$$

We will use this expression as the *definition* of the exponential of a matrix.

in depth:
appendix M.2, p. 885

fast track:
sect. 4.3, p. 66

How do we compute the exponential (4.13)?

Example 4.1 Stability eigenvalues, diagonal case: *Should we be so lucky that* $\mathbf{A}$ *happens to be a diagonal matrix* $\mathbf{A}_D$ *with eigenvalues real* $(\lambda_1, \lambda_2, \ldots, \lambda_d)$*, the exponential is simply*

$$\mathbf{J}^t = e^{t\mathbf{A}_D} = \begin{pmatrix} e^{t\lambda_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{t\lambda_d} \end{pmatrix}. \tag{4.14}$$

Throughout this text the symbol Λ_k will always denote the kth *stability eigenvalue* (in literature sometimes referred to as the *stability multiplier*) of the finite time Jacobian matrix $\mathbf{J}^t$. Symbol λ_k will be reserved for the kth *stability exponent* (in cases of interest, often the same thing as the *Lyapunov exponent*)

$$|\Lambda_k| = e^{t\lambda_k}\,, \qquad \Lambda_k = \Lambda_k(x_0, t)\,,\ \lambda_k = \lambda_k(x_0, t)\,. \tag{4.15}$$

Example 4.2 Stability eigenvalues, diagonalizable case: *Suppose next that* $\mathbf{A}$ *is diagonalizable and that* $\mathbf{U}$ *is the matrix that brings it to its diagonal form* $\mathbf{A}_D = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$. *The transformation* $\mathbf{U}$ *is a linear coordinate transformation which rotates, skews, and possibly flips the coordinate axis of the vector space. Then* $\mathbf{J}$ *can also be brought to a diagonal form by inserting factors* $\mathbf{1} = \mathbf{U}\mathbf{U}^{-1}$ *between the steps of the product (4.13):*

4.2 page 73

$$\mathbf{J}^t = e^{t\mathbf{A}} = \mathbf{U}e^{t\mathbf{A}_D}\mathbf{U}^{-1} . \tag{4.16}$$

In either example, the action of both $\mathbf{A}$ and $\mathbf{J}$ is very simple; the axes of orthogonal coordinate system where $\mathbf{A}$ is diagonal are also the eigendirections of both $\mathbf{A}$ and $\mathbf{J}^t$, and under the flow the neighborhood is deformed by a multiplication by an eigenvalue factor for each coordinate axis.

As $\mathbf{A}$ has only real entries, it will in general have either real eigenvalues, or complex conjugate pairs of eigenvalues. That is not surprising, but also the corresponding eigenvectors can be either real or complex. All coordinates used in defining the flow are real numbers, so what is the meaning of a *complex* eigenvector?

Add explicit formulas for Jacobian matrixs, eigenvalues from the Notebooks, PC-KTH article.

Example 4.3 Complex stability eigenvalues: *To develop some intuition about that, let us work out the behavior for the simplest nontrivial case, the case where*

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} . \tag{4.17}$$

The eigenvalues λ_1, λ_2 *of* $\mathbf{A}$ *are the roots*

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}\,\mathbf{A} \pm \sqrt{(\mathrm{tr}\,\mathbf{A})^2 - 4\det\mathbf{A}}\right) \tag{4.18}$$

of the characteristic equation

$$\begin{aligned} \det(\mathbf{A} - z\mathbf{1}) &= (\lambda_1 - z)(\lambda_2 - z) = 0 , \qquad (4.19) \\ \left\| \begin{matrix} A_{11} - z & A_{12} \\ A_{21} & A_{22} - z \end{matrix} \right\| &= z^2 - (A_{11} + A_{22})\,z + (A_{11}A_{22} - A_{12}A_{21}) . \end{aligned}$$

The qualitative behavior of $e^{\mathbf{A}}$ *for real eigenvalues* $\lambda_1, \lambda_2 \in \mathbb{R}$ *will differ from the case that they form a complex conjugate pair,*

$$\lambda_1 = \lambda + i\theta , \quad \lambda_2 = \lambda_1^* = \lambda - i\theta .$$

give example

These two possibilities are refined further into sub-cases depending on the signs of the real part. The matrix might have only one eigenvector, or two linearly independent eigenvectors, which may or may not be orthogonal. Along each of these directions the motion is of the form $x_k \exp(t\lambda_k)$. *If the exponent* λ_k *is positive, then the component* x_k *will grow; if the exponent* λ_k *is negative, it will shrink.*

align figure 4.2

We sketch the full set of possibilities in figure 4.2(a), and work out in detail the case when $\mathbf{A}$ can be brought to the diagonal form, and the case of degenerate eigenvalues.

Example 4.4 Complex stability eigenvalues, diagonal: *If* $\mathbf{A}$ *can be brought to the diagonal form, the solution (4.13) to the differential equation (4.8) can be written either as*

in/out nodes
inward/outward spirals
center
saddle
singular value decomposition
stability!eigenvalue
Lagrangian!coordinates
Eulerian!coordinates
topological!equivalence
flow!stability
stability!flow

$$\left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right) = \left(\begin{array}{cc} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{array}\right) \left(\begin{array}{c} x_1(0) \\ x_2(0) \end{array}\right) , \qquad (4.20)$$

or

$$\left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right) = e^{t\lambda} \left(\begin{array}{cc} e^{it\theta} & 0 \\ 0 & e^{-it\theta} \end{array}\right) \left(\begin{array}{c} x_1(0) \\ x_2(0) \end{array}\right) . \qquad (4.21)$$

In the case $\lambda_1 > 0$, $\lambda_2 < 0$, x_1 *grows exponentially with time, and* x_2 *contracts exponentially. This behavior, called a* saddle, *is sketched in figure 4.2(b), as are the remaining possibilities: in/out nodes, inward/outward spirals, and the center. Spirals arise from taking a real part of the action of* $\mathbf{J}^t$ *on a complex eigenvector. The magnitude of* $|x(t)|$ *diverges exponentially when* $\lambda > 0$, *and contracts toward* 0 *when the* $\lambda < 0$, *whereas the imaginary phase* θ *controls its oscillations.*

spirals figure 4.2

refer to streamlines of figure 4.2(b)

⇓PRELIMINARY

finish example 4.5

Example 4.5 Degenerate stability eigenvalues: *The simplest nontrivial example of a matrix with a degenerate pair of eigenvalues is*

$$\mathbf{A} = \left(\begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array}\right) . \qquad (4.22)$$

The qualitative behavior of $e^{\mathbf{A}}$ *for eigenvalue* $\lambda \in \mathbb{R}$

The matrix might have only one eigenvector, or two linearly independent eigenvectors, which may or may not be orthogonal.

⇑PRELIMINARY

In general $\mathbf{J}^t$ is neither diagonal, nor diagonalizable, nor constant along the trajectory. Still, any matrix, including $\mathbf{J}^t$, can be expressed in the singular value decomposition form

⇓PRELIMINARY

☞ remark 4.2

⇑PRELIMINARY

$$\mathbf{J} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

where $\mathbf{D}$ is diagonal, and $\mathbf{U}$, $\mathbf{V}$ are orthogonal matrices. The diagonal elements $\Lambda_1, \Lambda_2, \ldots, \Lambda_d$ of $\mathbf{D}$ are called the *stability eigenvalues*.

give 2-d example of ellipsoid explicitelly

Under the action of the flow an infinitesimally small ball of initial points is deformed into an ellipsoid: Λ_i is the relative stretching of the ith principal axis of the ellipsoid, the columns of the matrix $\mathbf{V}$ are the principal axes $\mathbf{e}_i$ of stretching in the Lagrangian coordinate frame, and the orthogonal matrix $\mathbf{U}$ gives the orientation of the ellipse in the Eulerian coordinates.

rows of $\mathbf{V}$?

define Lagrangian=co-moving, Eulerian=external coordinate frames

Now that we have some feeling for the qualitative behavior of eigenvectors and eigenvalues, we are ready to return to the general case: nonlinear flows.

equilibrium!point
Rossler@Rössler system!equilibria
equilibrium!Rössler system

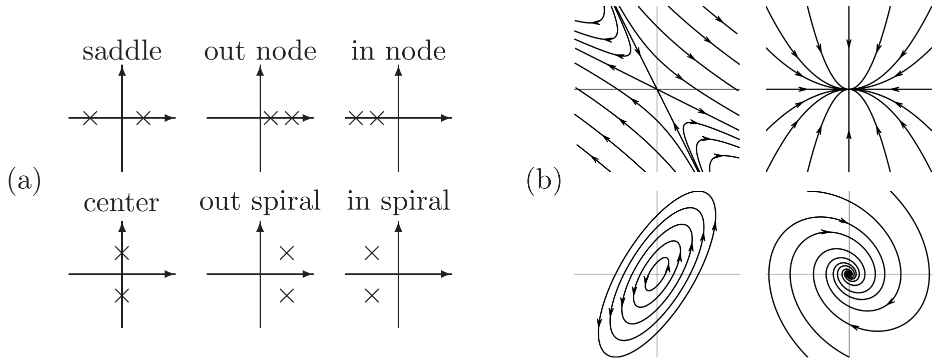


Figure 4.2: (a) Qualitatively distinct types of stability exponents of a $[2\times 2]$ stability matrix. (b) Streamlines for several typical 2-dimensional flows: saddle (hyperbolic), in node (attracting), center (elliptic), in spiral.

4.3 Stability of flows

check Frøyla book.

How do you determine the eigenvalues of the finite time local deformation $\mathbf{J}^t$ for a general nonlinear smooth flow? The Jacobian matrix is computed by integrating the equations of variations (4.2)

$$x(t) = f^t(x_0)\,, \quad \delta x(x_0,t) = A^t(x_0)\delta x(x_0,0)\,. \tag{4.23}$$

The equations of variations are linear, so we should be able to integrate them - but in order to make sense of the answer, we derive it step by step.

4.3.1 Stability of equilibria

For a start, consider the case where x_q is an equilibrium point (2.7). Expanding around the equilibrium point x_q, using the fact that the matrix $\mathbf{A} = \mathbf{A}(x_q)$ in (4.2) is constant, and integrating,

$$f^t(x) = x_q + e^{\mathbf{A}t}(x - x_q) + \cdots\,, \tag{4.24}$$

we verify that the simple formula (4.13) applies also to the Jacobian matrix of an equilibrium point,

$$\mathbf{J}^t(x_q) = e^{\mathbf{A}t} \qquad \mathbf{A} = \mathbf{A}(x_q)\,. \tag{4.25}$$

Example 4.6 Equilibria of the Rössler flow. *The Rösler system (2.14) has two equilibrium points*

$$\begin{array}{l}(x^-, y^-, z^-) = (\,0.0070,\,-0.0351,\,0.0351\,)\\(x^+, y^+, z^+) = (\,5.6929,\,-28.4648,\,28.4648\,)\end{array}$$

2.9
page 44

Euler!limit

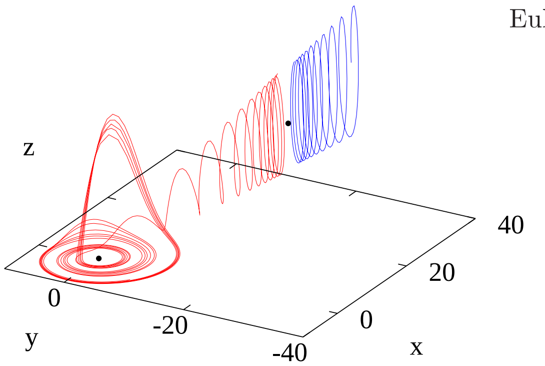


Figure 4.3: Two trajectories of the Rössler flow initiated in the neighborhood of the "+" equilibrium point (2.15). (R. Paškauskas)

The two equilibria together their stability exponents now yield quite detailed information about the flow. Figure 4.3 shows two trajectories which start in the neighborhood of the "+" equilibrium point. Trajectories to the right of the "+" equilibrium point escape, and those to the left spiral toward the "−" equilibrium point, where they seem to wander chaoticaly for all times. The stable manifold of "+" equilibrium point thus serves as a attraction basin boundary. Consider now the linearized stability exponents of the two equilibria

2.9
page 44

$$
\begin{array}{rll}
(\lambda_1^-, \lambda_2^- \pm i\,\theta_2^-) = & (-5.686, & 0.0970 \pm i\,0.9951) \\
(\lambda_1^+, \lambda_2^+ \pm i\,\theta_2^+) = & (0.1929, & -4.596\times 10^{-6} \pm i\,5.428)
\end{array}
\qquad (4.26)
$$

The $\lambda_2^+ \pm i\,\theta_2^+$ complex eigenvalue pair implies that that neighborhood of the outer equilibrium point rotates with angular period $T_+ \approx \left|2\pi/\theta_2^+\right| = 1.1575$. The stability multiplier by which a trajectory that starts near the "+" equilibrium point contracts in the stable manifold plane is the excrutiatingly slow $\Lambda_2^+ \approx \exp(\lambda_2^+ T_+) = 0.9999947$ per rotation. For each period the point of the stable manifold moves away along the unstable eigendirection by factor $\Lambda_1^+ \approx \exp(\lambda_1^+ T_+) = 1.2497$. Hence the slow spiraling on both sides of the "+" equilibrium point.

The $\lambda_2^- \pm i\,\theta_2^-$ complex eigenvalue pair tells us that neighborhood of the "−" equilibrium point rotates with angular period $T_- \approx \left|2\pi/\theta_2^-\right| = 6.313$, slightly faster than the harmonic oscillator estimate in (2.11). The stability multiplier by which a trajectory that starts near the "−" equilibrium point spirals away per one rotation is $\Lambda_{radial} \approx \exp(\lambda_2^- T_-) = 1.84$. The λ^- eigenvalue is essentially the z expansion correcting parameter c introduced in (2.13). For each Poincaré section return, the trajectory is contracted into the stable manifold by the amazing factor of $\Lambda_1 \approx \exp(\lambda_1^- T_-) = 10^{-15.6}$ (!).

Suppose you start with a 1 mm interval pointing in the Λ_1 eigendirection. After one Poincaré return the interval is of order of 10^{-4} fermi, the furthest we will get into subnuclear structure in this book. Of course, from the mathematical point of view, the flow is reversible, and the Poincaré return map is invertible. (Rytis Paškauskas)

4.3.2 Stability of trajectories

Next, consider the case of a general, non-stationary trajectory $x(t)$. The exponential of a constant matrix can be defined either by its Taylor series expansion, or in terms of the Euler limit (4.13):

appendix M.1

$$
e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k \qquad (4.27)
$$

time!ordered integration

$$= \lim_{m\to\infty}\left(\mathbf{1}+\frac{t}{m}\mathbf{A}\right)^m . \tag{4.28}$$

Taylor expanding is fine if $\mathbf{A}$ is a constant matrix. However, only the second, tax-accountant's discrete step definition of an exponential is appropriate for the task at hand, as for a dynamical system the local rate of neighborhood distortion $\mathbf{A}(x)$ depends on where we are along the trajectory. The linearized neighborhood is multiplicatively deformed along the flow, and the m discrete time step approximation to $\mathbf{J}^t$ is therefore given by a generalization of the Euler product (4.28):

$$\begin{aligned}\mathbf{J}^t &= \lim_{m\to\infty}\prod_{n=m}^{1}(\mathbf{1}+\delta t\mathbf{A}(x_n)) = \lim_{m\to\infty}\prod_{n=m}^{1}e^{\delta t\,\mathbf{A}(x_n)} \\ &= \lim_{m\to\infty}e^{\delta t\,\mathbf{A}(x_n)}e^{\delta t\,\mathbf{A}(x_{m-1})}\cdots e^{\delta t\,\mathbf{A}(x_2)}e^{\delta t\,\mathbf{A}(x_1)} ,\end{aligned} \tag{4.29}$$

where $\delta t = (t-t_0)/m$, and $x_n = x(t_0+n\delta t)$. Slightly perverse indexing of the products indicates that in our convention the successive infinitesimal deformation are applied by multiplying from the left. The two formulas for $\mathbf{J}^t$ agree to leading order in δt, and the $m\to\infty$ limit of this procedure is the integral

need integration exercises for (4.30)

☞ appendix E

$$\mathbf{J}^t_{ij}(x_0) = \left[\mathbf{T}e^{\int_0^t d\tau\mathbf{A}(x(\tau))}\right]_{ij} , \tag{4.30}$$

where $\mathbf{T}$ stands for time-ordered integration, *defined* as the continuum limit of the successive left multiplications (4.29). This formula for $\mathbf{J}$ is the main result of this chapter.

☞ appendix I.1

It is evident from the time-ordered product structure (4.29) that the Jacobian matrices are multiplicative along the flow,

$$\mathbf{J}^{t+t'}(x) = \mathbf{J}^{t'}(x')\mathbf{J}^t(x), \qquad \text{where } x' = f^t(x) . \tag{4.31}$$

The formula (4.29) is a matrix generalization of the crude Euler integrator (2.16) and is neither smart not accurate. Much better numerical accuracy is obtained by the following observation: To linear order in δt,

$$\mathbf{J}^{t+\delta t} - \mathbf{J}^t = \delta t\,\mathbf{A}\mathbf{J}^t + O\left((\delta t)^2\right) ,$$

so the Jacobian matrix itself satisfies the linearized equation (4.1)

$$\frac{d}{dt}\mathbf{J}^t(x) = \mathbf{A}(x)\,\mathbf{J}^t(x) , \quad \text{initial condition } \mathbf{J}^0(x) = \mathbf{1} . \tag{4.32}$$

left, right eigenvectors

Given a numerical routine for integrating the equations of motion, evaluation of the Jacobian matrix requires minimal additional programming effort; one simply extends the d-dimensional integration routine and integrates concurrently with $f^t(x)$ the d^2 elements of $\mathbf{J}^t(x)$.

map!stability
stability!maps
Jacobian matrix!mapping

The qualifier "simply" is perhaps too glib. Integration will work for short finite times, but for exponentially unstable flows one quickly runs into numerical over- and/or underflow problems, so further thought will have to go into implementating this calculation.

in depth:
sect. 10.3, p. 152

4.4 Stability of maps

The transformation of an infinitesimal neighborhood of a trajectory under the iteration of a map follows from Taylor expanding the iterated mapping at *discrete* time n to linear order, as in (4.4). The linearized neighborhood is transported by the Jacobian matrix evaluated at a discrete set of times $n = 1, 2, \ldots,$

$$\mathbf{J}^n_{ij}(x_0) = \left.\frac{\partial f^n_i(x)}{\partial x_j}\right|_{x=x_0} . \tag{4.33}$$

This matrix is in the literature sometimes called the *fundamental* matrix. As the simplest example, consider a 1-dimensional map. The chain rule yields the stability of the nth iterate

$$\Lambda(x_0)_n = \frac{d}{dx} f^n(x_0) = \prod_{m=0}^{n-1} f'(x_m) , \quad x_m = f^m(x_0) . \tag{4.34}$$

The 1-step product formula for the stability of the nth iterate of a d-dimensional map

$$\begin{aligned} \mathbf{J}^n(x_0) &= \mathbf{J}(x_{n-1}) \cdots \mathbf{J}(x_1)\mathbf{J}(x_{x_0}) , \\ \mathbf{J}(x)_{kl} &= \frac{\partial}{\partial x_l} f_k(x) , \quad x_m = f^m(x_0) \end{aligned} \tag{4.35}$$

follows from the chain rule for matrix derivatives

$$\frac{\partial}{\partial x_i} f_j(f(x)) = \sum_{k=1}^{d} \left.\frac{\partial}{\partial y_k} f_j(y)\right|_{y=f(x)} \frac{\partial}{\partial x_i} f_k(x) .$$

If you prefer to think of a discrete time dynamics as a sequence of Poincaré section returns, then (4.35) follows from (4.31), Jacobian matrices are multiplicative along the flow.

10.1
page 161

Henon@Hénon!map!stability
Poincar"'e!return map!stability
stability!Poincar"'e return map

Example 4.7 Hénon map Jacobian matrix: *For the Hénon map (3.12) the Jacobian matrix for the nth iterate of the map is*

$$\mathbf{J}^n(x_0) = \prod_{m=n}^{1} \begin{pmatrix} -2ax_m & b \\ 1 & 0 \end{pmatrix}, \qquad x_m = f_1^m(x_0, y_0)\,. \tag{4.36}$$

The determinant of the Hénon one time step Jacobian matrix (4.36) is constant,

$$\det \mathbf{J} = \Lambda_1 \Lambda_2 = -b \tag{4.37}$$

so in this case only one eigenvalue $\Lambda_1 = -b/\Lambda_2$ needs to be determined. This is not an accident; a constant Jacobian was one of desiderata that led Hénon to construct a map of this particular form.

explain Henon = normal form in conjug.tex insert Rossler discussion here

fast track:
chapter 5, p. 75

4.4.1 Stability of Poincaré return maps

(R. Paškauskas and P. Cvitanović)

We now relate the linear stability of the Poincaré return map $P : \mathcal{P} \to \mathcal{P}$ defined in sect. 3.1 to the stability of the continuous time flow in the full phase space.

The hypersurface $\mathcal{P}$ can be specified implicitly through a function $U(x)$ that is zero whenever a point x is on the Poincaré section. A nearby point $x + \delta x$ is in the hypersurface $\mathcal{P}$ if $U(x + \delta x) = 0$, and the same is true for variations around the first return point $x' = x(\tau)$, so expanding $U(x')$ to linear order in δx leads to the condition

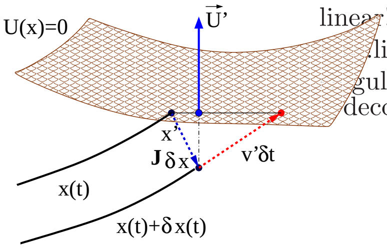
$$U_i(x') \left. \frac{dx'_i}{dx_j} \right|_{\mathcal{P}} = 0\,. \tag{4.38}$$

In what follows U_i is the gradient of U defined in (3.2), unprimed quantities refer to the starting point $x = x_0 \in \mathcal{P}$, $v = v(x_0)$, and the primed quantities to the first return: $x' = x(\tau)$, $v' = v(x')$, $U' = U(x')$. For brevity we shall also denote the full phase space Jacobian matrix at the first return by $\mathbf{J} = \mathbf{J}^\tau(x_0)$. Both the first return x' and the time of flight to the next Poincaré section $\tau(x)$ depend on the starting point x, so the Jacobian matrix

$$\hat{\mathbf{J}}(x)_{ij} = \left. \frac{dx'_i}{dx_j} \right|_{\mathcal{P}} \tag{4.39}$$

with both initial and the final variation constrained to the Poincaré section hypersurface $\mathcal{P}$ is related to the continuous flow Jacobian matrix by

$$\left. \frac{dx'_i}{dx_j} \right|_{\mathcal{P}} = \frac{\partial x'_i}{\partial x_j} + \frac{dx'_i}{dt}\frac{dt}{dx_j} = \mathbf{J}_{ij} + v'_i \frac{dt}{dx_j}\,.$$



linear!flows
linear
singular value decomposition

Figure 4.4: If $x(t)$ intersects the Poincaré section $\mathcal{P}$ at time τ, the nearby $x(t) + \delta x(t)$ trajectory intersects it time $\tau + \delta t$ later. As $(U' \cdot v' \delta t) = -(U' \cdot \mathbf{J}\, \delta x)$, the difference in arrival times is given by $\delta t = -(U' \cdot \mathbf{J}\, \delta x)/(U' \cdot v')$.

The return time variation dt/dx, figure 4.4, is eliminated by substituting this expression into the constraint (4.38),

$$0 = U'_i\, \mathbf{J}_{ij} + (v' \cdot U') \frac{dt}{dx_j} \,,$$

redraw figure

yielding the projection of the full space $(d+1)$-dimensional Jacobian matrix to the Poincaré map d-dimensional Jacobian matrix:

$$\hat{\mathbf{J}}_{ij} = \left(\delta_{ik} - \frac{v'_i\, U'_k}{(v' \cdot U')} \right) \mathbf{J}_{kj} \,. \tag{4.40}$$

Substituting (4.6) we verify that the initial velocity $v(x)$ is a zero-eigenvector of $\hat{\mathbf{J}}$

$$\hat{\mathbf{J}} v = 0 \,, \tag{4.41}$$

so the Poincaré section eliminates variations parallel to v, and $\hat{\mathbf{J}}$ is a rank d matrix, that is, one less than the dimension of the continuus time flow.

Commentary

Remark 4.1 Linear flows. The theory of linear flows and their stability is only sketched in sect. 4.2. They are presented at length in many textbooks. We liked the discussion in chapter 1 of Perko [4.1] and chapters 3 and 5 of Glendinning [4.2].

⇓PRELIMINARY

Remark 4.2 Singular value decomposition of Jacobian matrix. For a discussion of the singular value decomposition (SVD) of Jacobian matrix, see for example Thiffeault [4.3].

⇑PRELIMINARY

time!ordered integration

Résumé

A neighborhood of a trajectory deforms as it is transported by a flow. In the linear approximation, the matrix of variations $\mathbf{A}$ describes the shearing/compression/expansion of an infinitesimal neighborhood in an infinitesimal time step. The deformation after a finite time t is described by the Jacobian matrix

$$\mathbf{J}^t(x_0) = \mathbf{T}e^{\int_0^t d\tau \mathbf{A}(x(\tau))} \,,$$

where $\mathbf{T}$ stands for the time-ordered integration, or, for discrete time maps, by multiplication along the n points x_0, x_1, $x_2, \ldots, x_{n-1}$ on the trajectory of x_0,

$$\mathbf{J}^n(x_0) = \prod_{m=n-1}^{0} \mathbf{J}(x_n)\mathbf{J}(x_{n-1})\cdots\mathbf{J}(x_2)\mathbf{J}(x_1) \,,$$

with $\mathbf{J}(x)$ the single time step Jacobian matrix. Here symbol Λ_k will always denote the kth stability *eigenvalue* of the finite time Jacobian matrix $\mathbf{J}^t$, and symbol λ_k the the kth stability *exponent*

$$|\Lambda| = e^{t\lambda} \,, \qquad \Lambda_\pm = e^{t(\lambda \pm i\theta)} \,.$$

For a complex stability eigenvalue the conjugate eigenvalue pair describes rotational motion in the plane defined by the corresponding pair of eigenvectors, with period $T = 2\pi/\theta$.

The eigenvalues and eigendirections of the Jacobian matrix describe the deformation of an initial infinitesimal sphere of neighboring trajectories into an ellipsoid a finite time t later. Nearby trajectories separate exponentially along unstable directions, approach each other along stable directions, and change slowly (algebraically) their distance along marginal directions. The Jacobian matrix $\mathbf{J}^t$ is in general neither symmetric, nor diagonalizable by a rotation, nor do its (left or right) eigenvectors define an orthonormal coordinate frame. Furthermore, although the stability matrices are multiplicative along the flow, in dimensions higher than one their eigenvalues in general are not. This lack of multiplicativity has important repercussions for both classical and quantum dynamics.

☞ appendix I.1

⇓PRIVATE

☞ chapter 37

⇑PRIVATE

add useful refs

References

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[4.2] P. Glendinning, *Stability, Instability, and Chaos* (Cambridge Univ. Press, Cambridge 1994).

[4.3] J.-L. Thiffeault, *Physica* **D 172**, 139 (2002); `nlin.CD/0101012`