Chapter 14

Trace formulas

> The trace formula is not a formula, it is an idea.
> Martin Gutzwiller

Dynamics is posed in terms of local equations, but the ergodic averages require global information. How can we use a local description of a flow to learn something about the global behavior? We have given a quick sketch of this program in sects. 1.5 and 1.6; now we redo the same material in greater depth. In chapter 10 we have related global averages to the eigenvalues of appropriate evolution operators. Traces of evolution operators can be evaluated as integrals over Dirac delta functions, and in this way the spectra of evolution operators become related to periodic orbits. If there is one idea that one should learn about chaotic dynamics, it happens in this chapter, and it is this: there is a fundamental local $\leftrightarrow$ global duality which says that

the spectrum of eigenvalues is dual to the spectrum of periodic orbits

For dynamics on the circle, this is called Fourier analysis; for dynamics on well-tiled manifolds, Selberg traces and zetas; and for generic nonlinear dynamical systems the duality is embodied in the trace formulas that we will now introduce. These objects are to dynamics what partition functions are to statistical mechanics.

14.1 Trace of an evolution operator

Our extraction of the spectrum of $\mathcal{L}$ commences with the evaluation of the trace. To compute an expectation value using (10.21) we have to integrate over all the values of the kernel $\mathcal{L}^t(x, y)$. If $\mathcal{L}^t$ were a matrix we would be computing a weighted sum of its eigenvalues which is dominated by the leading eigenvalue as $t \to \infty$. As the trace of $\mathcal{L}^t$ is also dominated by the leading eigenvalue as $t \to \infty$, we might just as well look at the trace

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monodromy matrix
stability!eigenvalues
expanding!stability eigenvalues
marginal!stability eigenvalues
contracting!stability eigenvalues

$$\mathrm{tr}\,\mathcal{L}^t = \int dx\,\mathcal{L}^t(x,x) = \int dx\,\delta\big(x - f^t(x)\big)\,e^{\beta\cdot A^t(x)}\,. \tag{14.1}$$

Assume that $\mathcal{L}$ has a spectrum of discrete eigenvalues $s_0, s_1, s_2, \cdots$ ordered so that $\mathrm{Re}\ s_\alpha \geq \mathrm{Re}\, s_{\alpha+1}$. We ignore for the time being the question of what function space the eigenfunctions belong to, as we shall compute the eigenvalue spectrum without constructing any explicit eigenfunctions.

By definition, the trace is the sum over eigenvalues (for the time being we choose not to worry about convergence of such sums),

$$\mathrm{tr}\,\mathcal{L}^t = \sum_{\alpha=0}^{\infty} e^{s_\alpha t}\,. \tag{14.2}$$

On the other hand, we have learned in sect. 9.2 how to evaluate the delta-function integral (14.1).

As the case of discrete time mappings is somewhat simpler, we first derive the trace formula for maps, and then for flows. The final formula (14.19) covers both cases.

14.1.1 Hyperbolicity assumption

According to (9.8) the trace (14.1) picks up a contribution whenever $x - f^n(x) = 0$, that is, whenever x belongs to a periodic orbit. For reasons which we will explain in sect. 14.3, it is wisest to start by focusing on discrete time systems. The contribution of an isolated prime cycle p of period n_p for a map f can be evaluated by restricting the integration to an infinitesimal open neighborhood $\mathcal{M}_p$ around the cycle,

$$\mathrm{tr}\,_p\mathcal{L}^{n_p} = \int_{\mathcal{M}_p} dx\,\delta(x - f^{n_p}(x)) = \frac{n_p}{\left|\det\left(\mathbf{1} - \mathbf{J}_p\right)\right|} = n_p \prod_{i=1}^{d} \frac{1}{|1-\Lambda_{p,i}|} \tag{14.3}$$

(in (9.9) and here we set the observable $e^{A_p} = 1$ for the time being). Periodic orbit Jacobian matrix $\mathbf{J}_p$ is also known as the *monodromy matrix* (from Greek *mono-* = alone, single, and *dromo* = run, racecourse), and its eigenvalues $\Lambda_{p,1}$, $\Lambda_{p,2}$, …, $\Lambda_{p,d}$ as the Floquet multipliers. We sort the eigenvalues $\Lambda_{p,1}$, $\Lambda_{p,2}$, …, $\Lambda_{p,d}$ of the p-cycle $[d\times d]$ Jacobian matrix $\mathbf{J}_p$ into expanding, marginal and contracting sets $\{e, m, c\}$, as in (8.2). As the integral (14.3) can be carried out only if $\mathbf{J}_p$ has no eigenvalue of unit magnitude, we assume that no eigenvalue is marginal (we shall show in sect. 14.3 that the longitudinal $\Lambda_{p,d+1} = 1$ eigenvalue for flows can be eliminated by restricting the consideration to the transverse Jacobian matrix $\mathbf{J}_p$), and factorize the trace (14.3) into a product over the expanding and the contracting eigenvalues

$$\left|\det\left(\mathbf{1} - \mathbf{J}_p\right)\right|^{-1} = \frac{1}{|\Lambda_p|} \prod_e \frac{1}{1 - 1/\Lambda_{p,e}} \prod_c \frac{1}{1 - \Lambda_{p,c}}\ , \tag{14.4}$$

where $\Lambda_p = \prod_e \Lambda_{p,e}$ is the product of expanding eigenvalues. Both $\Lambda_{p,c}$ and $1/\Lambda_{p,e}$ are smaller than 1 in absolute value, and as they are either real or come in complex conjugate pairs we are allowed to drop the absolute value brackets $|\cdots|$ in the above products.

hyperbolicity assumption
three-disk@3-disk!hyperbolicity
Dirac delta function
prime cycle
cycle!prime

The *hyperbolicity assumption* requires that the stabilities of all cycles included in the trace sums be exponentially bounded away from unity:

$$\begin{array}{lcll} |\Lambda_{p,e}| & > & e^{\lambda_e T_p} & \text{any } p, \text{ any expanding } |\Lambda_{p,e}| > 1 \\ |\Lambda_{p,c}| & < & e^{-\lambda_c T_p} & \text{any } p, \text{ any contracting } |\Lambda_{p,c}| < 1 \,, \end{array} \qquad (14.5)$$

where $\lambda_e, \lambda_c > 0$ are strictly positive bounds on the expanding, contracting cycle Lyapunov exponents. If a dynamical system satisfies the hyperbolicity assumption (for example, the well separated 3-disk system clearly does), the $\mathcal{L}^t$ spectrum will be relatively easy to control. If the expansion/contraction is slower than exponential, let us say $|\Lambda_{p,i}| \sim {T_p}^2$, the system may exhibit "phase transitions", and the analysis is much harder - we shall discuss this in chapter 21.

It follows from (14.4) that for long times, $t = rT_p \to \infty$, only the product of expanding eigenvalues matters, $\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right| \to |\Lambda_p|^r$. We shall use this fact to motivate the construction of dynamical zeta functions in sect. 15.3. However, for evaluation of the full spectrum the exact cycle weight (14.3) has to be kept.

14.2 A trace formula for maps

If the evolution is given by a discrete time mapping, and all periodic points have stability eigenvalues $|\Lambda_{p,i}| \neq 1$ strictly bounded away from unity, the trace $\mathcal{L}^n$ is given by the sum over all periodic points i of period n:

$$\mathrm{tr}\,\mathcal{L}^n = \int dx\,\mathcal{L}^n(x,x) = \sum_{x_i \in \mathrm{Fix} f^n} \frac{e^{\beta\cdot A_i}}{|\det\,(\mathbf{1} - \mathbf{J}^n(x_i))|}\,. \qquad (14.6)$$

Here Fix $f^n = \{x : f^n(x) = x\}$ is the set of all periodic points of period n, and A_i is the observable (10.5) evaluated over n discrete time steps along the cycle to which the periodic point x_i belongs. The weight follows from the properties of the Dirac delta function (9.8) by taking the determinant of $\partial_i(x_j - f^n(x)_j)$. If a trajectory retraces itself r times, its Jacobian matrix is $\mathbf{J}_p^r$, where $\mathbf{J}_p$ is the $[d\times d]$ Jacobian matrix (4.5) evaluated along a single traversal of the prime cycle p. As we saw in (10.5), the integrated observable A^n is additive along the cycle: If a prime cycle p trajectory retraces itself r times, $n = rn_p$, we obtain A_p repeated r times, $A_i = A^n(x_i) = rA_p$, $x_i \in p$.

A prime cycle is a single traversal of the orbit, and its label is a non-repeating symbol string. There is only one prime cycle for each cyclic

Laplace!transform, discrete
trace!formula!maps

permutation class. For example, the four cycle points $\overline{0011} = \overline{1001} = \overline{1100} = \overline{0110}$ belong to the same prime cycle $p = 0011$ of length 4. As both the stability of a cycle and the weight A_p are the same everywhere along the orbit, each prime cycle of length n_p contributes n_p terms to the sum, one for each cycle point. Hence (14.6) can be rewritten as a sum over all prime cycles and their repeats

chapter 11

$$\mathrm{tr}\,\mathcal{L}^n = \sum_p n_p \sum_{r=1}^{\infty} \frac{e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|}\delta_{n,n_p r}\,, \tag{14.7}$$

with the Kronecker delta $\delta_{n,n_p r}$ projecting out the periodic contributions of total period n. This constraint is awkward, and will be more awkward still for the continuous time flows, where it will yield a series of Dirac delta spikes (14.17). Such sums are familiar from the density-of-states sums of statistical mechanics, where they are dealt with in the same way as we shall do here: we smooth this distribution by taking a Laplace transform which rids us of the $\delta_{n,n_p r}$ constraint.

We *define* the trace formula for maps to be the Laplace transform of $\mathrm{tr}\,\mathcal{L}^n$ which, for discrete time mappings, is simply the generating function for the trace sums

$$\sum_{n=1}^{\infty} z^n \mathrm{tr}\,\mathcal{L}^n = \mathrm{tr}\,\frac{z\mathcal{L}}{1 - z\mathcal{L}} = \sum_p n_p \sum_{r=1}^{\infty} \frac{z^{n_p r} e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|}\,. \tag{14.8}$$

box this relation

Expressing the trace as in (14.2), in terms of the sum of the eigenvalues of $\mathcal{L}$, we obtain the *trace formula for maps*:

$$\sum_{\alpha=0}^{\infty} \frac{z e^{s_\alpha}}{1 - z e^{s_\alpha}} = \sum_p n_p \sum_{r=1}^{\infty} \frac{z^{n_p r}\, e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|}\,. \tag{14.9}$$

This is our second example of the duality between the spectrum of eigenvalues and the spectrum of periodic orbits, announced in the introduction to this chapter.

⇓PRELIMINARY

If the above derivation is too formal for your taste, you might want to read sect. **??** in parallel with this chapter.

Rondoni's section s-Still-another was killed by Vattay and needs resurrection as an appendix

⇑PRELIMINARY

fast track:
sect. 14.3, p. 245

14.2.1 A trace formula for transfer operators

For a piecewise-linear map (10.17), we can explicitly evaluate the trace formula. By the piecewise linearity and the chain rule $\Lambda_p = \Lambda_0^{n_0}\Lambda_1^{n_1}$, where the cycle p contains n_0 symbols 0 and n_1 symbols 1, the trace (14.6) reduces to

space!density functions

piecewise constant function

function!space!piecewise constant

$$\mathrm{tr}\,\mathcal{L}^n = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{|1-\Lambda_0^m \Lambda_1^{n-m}|} = \sum_{k=0}^{\infty} \left(\frac{1}{|\Lambda_0|\Lambda_0^k} + \frac{1}{|\Lambda_1|\Lambda_1^k} \right)^n . \tag{14.10}$$

The eigenvalues are simply

$$e^{s_k} = \frac{1}{|\Lambda_0|\Lambda_0^k} + \frac{1}{|\Lambda_1|\Lambda_1^k} \,. \tag{14.11}$$

restore the smooth 2-branch case here.

⇓PRELIMINARY

move this to symbolic.tex

The simplest example of spectrum for such dynamical system is the spectrum for the symmetric piecewise-linear 2-branch repeller (10.17) for which $\Lambda = \Lambda_1 = -\Lambda_0$. In this case all odd eigenvalues vanish, and the even eigenvalues are given by $e^{s_k} = 2/\Lambda^{k+1}$, k even.

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Asymptotically the spectrum (14.11) is dominated by the lesser of the two fixed point slopes $\Lambda = \Lambda_0$ (if $|\Lambda_0| < |\Lambda_1|$, otherwise $\Lambda = \Lambda_1$), and the eigenvalues e^{s_k} fall off exponentially as $1/\Lambda^k$, just as in the single fixed-point examples.

⇑PRELIMINARY

For $k = 0$ this is in agreement with the explicit transfer matrix (10.19) eigenvalues (10.20).

The alert reader should experience anxiety at this point. Is it not true that we have already written down explicitly the transfer operator in (10.19), and that it is clear by inspection that it has only *one* eigenvalue $e^{s_0} = 1/|\Lambda_0| + 1/|\Lambda_1|$? The example at hand is one of the simplest illustrations of necessity of defining the space that the operator acts on in order to define the spectrum. The transfer operator (10.19) is the correct operator on the space of functions piecewise constant on the two defining intervals $\{\mathcal{M}_0, \mathcal{M}_1\}$; on this space the operator indeed has only the eigenvalue e^{s_0}. As we shall see in sect. 16.1, the full spectrum (14.11) corresponds to the action of the transfer operator on the space of real analytic functions.

The Perron-Frobenius operator trace formula for the piecewise-linear map (10.17) follows from (14.8)

$$\mathrm{tr}\, \frac{z\mathcal{L}}{1-z\mathcal{L}} = \frac{z\left(\frac{1}{|\Lambda_0-1|} + \frac{1}{|\Lambda_1-1|}\right)}{1-z\left(\frac{1}{|\Lambda_0-1|} + \frac{1}{|\Lambda_1-1|}\right)} \,, \tag{14.12}$$

verifying the trace formula (14.9).

14.3 A trace formula for flows

Amazing! I did not understand a single word.

Fritz Haake

(R. Artuso and P. Cvitanović)

Ole: clarify this — draw a picture.

As any pair of nearby points on a cycle returns to itself exactly at each cycle period, the eigenvalue of the Jacobian matrix corresponding to the eigenvector along the flow necessarily equals unity for all periodic orbits. Hence for flows the trace integral $\mathrm{tr}\,\mathcal{L}^t$ requires a separate treatment for the longitudinal direction. To evaluate the contribution of an isolated prime cycle p of period T_p, restrict the integration to an infinitesimally thin tube $\mathcal{M}_p$ enveloping the cycle (see figure 1.10), and choose a local coordinate system with a longitudinal coordinate $dx_\parallel$ along the direction of the flow, and d transverse coordinates $x_\perp$

$$\mathrm{tr}\,_p\mathcal{L}^t = \int_{\mathcal{M}_p} dx_\perp dx_\parallel\, \delta\big(x_\perp - f^t_\perp(x)\big)\, \delta\Big(x_\parallel - f^t_\parallel(x)\Big) \; . \qquad (14.13)$$

(here we again set the observable $\exp(\beta \cdot A^t) = 1$ for the time being). Let $v(x)$ be the magnitude of the velocity at the point x along the flow. $v(x)$ is strictly positive, as otherwise the orbit would stagnate for infinite time at $v(x) = 0$ points, and that would get us nowhere. Therefore we can parametrize the longitudinal coordinate $x_\parallel$ by the flight time

change $\sigma \to \tau'$ here

$$x_\parallel(\tau) \;=\; \int_0^\tau d\sigma\, v(\sigma)\Big|_{mod\, L_p}$$

where $v(\sigma) = v(x_\parallel(\sigma))$, and L_p is the length of the circuit on which the periodic orbit lies (for the time being the *mod* operation in the above definition is redundant, as $\tau \in [0, T_p]$). With this parametrization

$$\Big(f^t_\parallel(x) - x_\parallel\Big) \;=\; \int_\tau^{t+\tau} d\sigma\, v(\sigma)\Big|_{mod\, L_p}$$

the integral along the longitudinal coordinate can be written as

$$\int_0^{L_p} dx_\parallel\, \delta\Big(x_\parallel - f^t_\parallel(x)\Big) \;=\; \int_0^{T_p} d\tau\, v(\tau)\, \delta\left(\int_\tau^{t+\tau} d\sigma\, v(\sigma)\Big|_{mod\, L_p}\right) . \qquad (14.14)$$

The zeroes of the argument of the delta function do not depend on τ, as v is positive, so we may rewrite (14.14) as

$$\int_0^{L_p} dx_\parallel\, \delta\Big(x_\parallel - f^t_\parallel(x)\Big) \;=\; \sum_{r=1}^{\infty} \delta(t - rT_p) \int_0^{T_p} d\tau\, v(\tau) \frac{1}{v(\tau + t)},$$

having used (9.7). The r sum starts from r=1, as we are considering strictly positive times. Now we use another elementary property of delta functions,

$$h(x)\delta(x - x_0) \;=\; h(x_0)\delta(x - x_0).$$

Laplace!transform
trace!formula!flows

The velocities cancel, and we get

$$\oint_p dx_\parallel\, \delta\Big(x_\parallel - f^t_\parallel(x)\Big) \;=\; T_p \sum_{r=1}^{\infty} \delta(t - rT_p) \ . \qquad (14.15)$$

The prime period arises also for repeated orbits, because the space integration (14.14) sweeps the periodic orbit in phase space: This observation will also be important for the derivation of the semiclassical trace formula in chapter 33.

For the remaining transverse integration variables the Jacobian matrix is defined in a reduced Poincaré surface of section $\mathcal{P}$ of constant $x_\parallel$. Linearization of the periodic flow transverse to the orbit yields

$$\int_{\mathcal{P}} dx_\perp \delta\Big(x_\perp - f^{rT_p}_\perp(x)\Big) = \frac{1}{\big|\det\big(\mathbf{1} - \mathbf{J}^r_p\big)\big|} \ , \qquad (14.16)$$

where $\mathbf{J}_p$ is the p-cycle $[d\times d]$ *transverse* Jacobian matrix, and as in (14.5) we have to assume hyperbolicity, that is, that the magnitudes of all transverse eigenvalues are bounded away from unity.

Substituting (14.15), (14.16) into (14.13), we obtain an expression for $\mathrm{tr}\,\mathcal{L}^t$ as a sum over all prime cycles p and their repetitions

$$\mathrm{tr}\,\mathcal{L}^t = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r\beta\cdot A_p}}{\big|\det\big(\mathbf{1} - \mathbf{J}^r_p\big)\big|}\delta(t - rT_p) \ . \qquad (14.17)$$

A trace formula follows by taking a Laplace transform. This is a delicate step, since the transfer operator becomes the identity in the $t \to 0^+$ limit. In order to make sense of the trace we regularize the Laplace transform by a lower cutoff ϵ smaller than the period of any periodic orbit, and write

refer to sect. 9.4

$$\begin{aligned}\int_\epsilon^\infty dt\, e^{-st}\,\mathrm{tr}\,\mathcal{L}^t \;&=\; \mathrm{tr}\,\frac{e^{-(s-\mathcal{A})\epsilon}}{s-\mathcal{A}} = \sum_{\alpha=0}^{\infty} \frac{e^{-(s-s_\alpha)\epsilon}}{s-s_\alpha} \\ &=\; \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta\cdot A_p - sT_p)}}{\big|\det\big(\mathbf{1} - \mathbf{J}^r_p\big)\big|} \,, \qquad (14.18)\end{aligned}$$

where $\mathcal{A}$ is the generator of the semigroup of dynamical evolution, sect. 9.4. The *classical trace formula for flows* is the $\epsilon \to 0$ limit of the above expression:

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta\cdot A_p - sT_p)}}{\big|\det\big(\mathbf{1} - \mathbf{J}^r_p\big)\big|} \,. \qquad (14.19)$$

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This formula is still another example of the duality between the (local) cycles and (global) eigenvalues. If T_p takes only integer values, we can replace $e^{-s} \rightarrow z$ throughout, so the trace formula for maps (14.9) is a special case of the trace formula for flows. The relation between the continuous and discrete time cases can be summarized as follows:

make a learned comment about how this is a generalization of the Poisson resummation technique

$$\begin{array}{rcl} T_p & \leftrightarrow & n_p \\ e^{-s} & \leftrightarrow & z \\ e^{t\mathcal{A}} & \leftrightarrow & \mathcal{L}^n \,. \end{array} \tag{14.20}$$

We could now proceed to estimate the location of the leading singularity of $\mathrm{tr}\,(s-\mathcal{A})^{-1}$ by extrapolating finite cycle length truncations of (14.19) by methods such as Padé approximants. However, it pays to first perform a simple resummation which converts this divergence of a trace into a *zero* of a spectral determinant. We shall do this in sect. 15.2, but a brief discussion of how all this relates to the xxxx derivation offered in the introduction.

14.4 An asymptotic trace formula

In order to illuminate the manipulations of sect. 14.2 and relate them to something we already possess intuition about, we now rederive the heuristic sum of sect. 1.5.1 from the exact trace formula (14.9). The Laplace transforms (14.9) or (14.19) are designed to capture the time $\rightarrow \infty$ asymptotic behavior of the trace sums. By the hyperbolicity assumption (14.5), for $t = T_p r$ large the cycle weight approaches

$$\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right| \rightarrow |\Lambda_p|^r \,, \tag{14.21}$$

where Λ_p is the product of the expanding eigenvalues of $\mathbf{J}_p$. Denote the corresponding approximation to the nth trace (14.6) by

$$\Gamma_n = \sum_i^{(n)} \frac{1}{|\Lambda_i|} \,, \tag{14.22}$$

and denote the approximate trace formula obtained by replacing the cycle weights $\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|$ by $|\Lambda_p|^r$ in (14.9) by $\Gamma(z)$. Equivalently, think of this as a replacement of the evolution operator (10.23) by a transfer operator (as in sect. 14.2.1). For concreteness consider a dynamical system whose symbolic dynamics is complete binary, for example the 3-disk system figure 1.4. In this case distinct periodic points that contribute to the nth periodic points sum (14.7) are labeled by all admissible itineraries composed of sequences of letters $s_i \in \{0,1\}$:

ceiling function

$$\begin{aligned}
\Gamma(z) &= \sum_{n=1}^{\infty} z^n \Gamma_n = \sum_{n=1}^{\infty} z^n \sum_{x_i \in \mathrm{Fix} f^n} \frac{e^{\beta \cdot A^n(x_i)}}{|\Lambda_i|} \\
&= z \left\{ \frac{e^{\beta \cdot A_0}}{|\Lambda_0|} + \frac{e^{\beta \cdot A_1}}{|\Lambda_1|} \right\} + z^2 \left\{ \frac{e^{2\beta \cdot A_0}}{|\Lambda_0|^2} + \frac{e^{\beta \cdot A_{01}}}{|\Lambda_{01}|} + \frac{e^{\beta \cdot A_{10}}}{|\Lambda_{10}|} + \frac{e^{2\beta \cdot A_1}}{|\Lambda_1|^2} \right\} \\
&\quad + z^3 \left\{ \frac{e^{3\beta \cdot A_0}}{|\Lambda_0|^3} + \frac{e^{\beta \cdot A_{001}}}{|\Lambda_{001}|} + \frac{e^{\beta \cdot A_{010}}}{|\Lambda_{010}|} + \frac{e^{\beta \cdot A_{100}}}{|\Lambda_{100}|} + \ldots \right\} \qquad (14.23)
\end{aligned}$$

Both the cycle averages A_i and the stabilities Λ_i are the same for all points $x_i \in p$ in a cycle p. Summing over repeats of all prime cycles we obtain

$$\Gamma(z) = \sum_p \frac{n_p t_p}{1 - t_p}, \qquad t_p = z^{n_p} e^{\beta \cdot A_p} / |\Lambda_p| . \qquad (14.24)$$

This is precisely our initial heuristic estimate (1.8). Note that we could not perform such sum over r in the exact trace formula (14.9) as $\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right| \neq \left|\det\left(\mathbf{1} - \mathbf{J}_p\right)\right|^r$; the correct way to resum the exact trace formulas is to first expand the factors $1/|1 - \Lambda_{p,i}|$, as we shall do in (15.10).

☞ sect. 15.2

If the weights $e^{\beta A^n(x)}$ are multiplicative along the flow, and the flow is hyperbolic, for given β the magnitude of each $|e^{\beta A^n(x_i)}/\Lambda_i|$ term is bounded by some constant M^n. The total number of cycles grows as 2^n (or as e^{hn}, $h =$ topological entropy, in general), and the sum is convergent for z sufficiently small, $|z| < 1/2M$. For large n the nth level sum (14.6) tends to the leading $\mathcal{L}^n$ eigenvalue e^{ns_0}. Summing this asymptotic estimate level by level

$$\Gamma(z) \approx \sum_{n=1}^{\infty} (z e^{s_0})^n = \frac{z e^{s_0}}{1 - z e^{s_0}} \qquad (14.25)$$

we see that we should be able to determine s_0 by determining the smallest value of $z = e^{-s_0}$ for which the cycle expansion (14.24) diverges.

If one is interested only in the leading eigenvalue of $\mathcal{L}$, it suffices to consider the approximate trace $\Gamma(z)$. We will use this fact in section xxxx to motivate the introduction of dynamical zeta functions (15.15), and in sect. 15.5 we shall give the exact relation between the exact and the approximate trace formulas.

Commentary

Remark 14.1 Who's dunne it? Continuous time flow traces weighted by cycle periods were introduced by Bowen [14.1] who treated them as Poincaré section suspensions weighted by the "time ceiling" function (3.4). They were used by Parry and Pollicott [14.2]. The derivation presented here [14.3] paralleles closely to the derivation of the Gutzwiller semiclassical trace formula, chapters 31 and 33.

⇓PRELIMINARY

Tanner says he has fixed this.

Remark 14.2 $t \to 0_+$ regularization. We remark again that in taking the Laplace transform that led to (14.19), we have ignored a possible $t \to 0_+$ diverging term, as we do not know how to profitably regularize the delta function kernel in this limit. In the quantum (or heat kernel) case such volume term gives rise to the Weyl mean density of states (see sect. 33.1.1). A more careful treatment would regularize the divergent sum in (14.19), and possibly assign to the volume term some interesting role in the theory of classical resonance spectra. [1] [2]

⇑PRELIMINARY

Remark 14.3 Flat and sharp traces. In the above formal derivation of trace formulas we cared very little whether our sums were well posed. In the Fredholm theory traces like (14.1) require compact operators with continuous function kernels. This is not the case for our Dirac delta evolution operators: nevertheless, there is a large class of dynamical systems for which our results may be shown to be perfectly legal. In the mathematical literature expressions like (14.6) are called *flat* traces (see the review [14.4] and chapter 16). Other names for traces appear as well: for instance, in the context of $1-d$ mappings, *sharp* traces refer to generalizations of (14.6) where contributions of periodic points are weighted by the Lefschetz sign ± 1, reflecting whether the periodic point sits on a branch of nth iterate of the map which crosses the diagonal starting from below or starting from above [15.12]. Such traces are connected to the theory of kneading invariants (see ref. [14.4] and references therein). Traces weighted by ± 1 sign of the derivative of the fixed point have been used to study the period doubling repeller, leading to high precision estimates of the Feigenbaum constant δ, refs. [14.5, 37.5, 14.6].

Résumé

The description of a chaotic dynamical system in terms of cycles can be visualized as a tessellation of the dynamical system, figure 1.9, with a smooth flow approximated by its *periodic orbit skeleton*, each region $\mathcal{M}_i$ centered on a periodic point x_i of the topological length n, and the size of the region determined by the linearization of the flow around the periodic point. The integral over such topologically partitioned phase space yields the *classical trace formula*

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - s T_p)}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} .$$

[1] Artuso: the remark and the exercise almost coincide

[2] Baladi: (14.19) is nonzero only if $t > 0$ is an integer multiple of T. So the limit as $t \to 0_\pm$ is just zero, no? so what or why you should regularize? For $t = 0$ you get the identity operator whose trace is divergent so there should not be great hope to give a meaning to the l.h.s. of (14.19).

Now that we have a trace formula, we might ask for what is it good? As itstands, it is little more than a scary divergent formula which relates the unspeakable infinity of global eigenvalues to the unthinkable infinity of local unstable cycles. However, it is a good stepping stone on the way to construction of spectral determinants (to which we turn next), and a first hint that when the going is good, the theory might turn out to be convergent beyond our wildest dreams (chapter 16). In order to implement such formulas, we will have to determine "all" prime cycles. This task we postpone to chapters 12 and 17.

References

[14.1] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer Lecture Notes in Math. **470** (1975). replace by ref. [1.17]

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