Does the existence of a Lax pair imply integrability?

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April 30, 2002

Abstract

We give two examples of families of differential-difference equations which have Lax Pairs but are apparently non-integrable, contradicting the usual belief that the two properties are equivalent.

PACS: 63.10.+a, 63.20.Ry

Keywords: differential-difference equations, Lax pairs, modified KdV equation.

We consider some special cases of differential-difference equations that are continuous in time and discrete in a 1D-space variable:

\[ \dot{u}_n = F(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots), \] (1)

where \( u_n \equiv u_n(t), \dot{u}_n = \partial_t u_n \). A Lax pair (also know as a zero-curvature representation) for (1) is a set of two matrices \((M_n, N_n)\) which satisfies

\[ V_{n+1} = M_n V_n, \]
\[ \partial_t V_n = N_n V_n, \]
and also the compatibility condition

\[ \partial_t (SV_n) = S \partial_t V_n, \]  

(2)

where \( S \) is the shift operator \( SV_n = V_{n+1} \).

The required connection between (1) and the Lax pair is that the condition (2) is

\[ \partial_t M_n = N_{n+1} M_n - M_n N_n, \]  

(3)

which on substituting for \( (M_n, N_n) \) gives the dynamical equation (1) for \( u_n \).

Although it is known that the existence of a Lax Pair for a set of differential-difference equations does not automatically imply that these equations are integrable [1], it is usually assumed in the literature that the two are equivalent. We show here that there exists some simple systems which do have Lax pairs but which fail a commonly accepted test for integrability, the singularity confinement method, due to Ramani et al. [2].

The Volterra3 (V3) equation

\[ \dot{u}_n = u_n (u_{n+1} - u_{n-1}), \]  

(4)

is known to be integrable. A zero curvature representation (Lax pair) given by Faddeev and Takhtajan [1] for this equation is

\[ M_n = \begin{bmatrix} \lambda & u_n \\ -1 & 0 \end{bmatrix}, \quad N_n = \begin{bmatrix} u_n & \lambda u_n \\ -\lambda & -\lambda^2 + u_{n-1} \end{bmatrix} \]

Some more recent references on this equation are [3, 4].

A fifth order Volterra5 (V5) equation is

\[ \dot{u}_n = u_n (u_{n+1} - u_{n-1} + \alpha (u_{n+2} - u_{n-2})), \]  

(5)

which has the Lax pair

\[ M_n = \begin{bmatrix} \lambda & \lambda^2 + u_n \\ -1 & -\lambda \end{bmatrix}, \]

and

\[ N_n = \begin{bmatrix} u_n + \alpha (u_{n-1} + u_{n+1}) & \lambda (u_n - u_{n-1} + \alpha (u_{n+1} - u_n + u_{n-1} - u_{n-2})) \\ 0 & u_{n-1} + \alpha (u_n + u_{n-2}) \end{bmatrix} \]
Note that this $N_n$ can be written in the simpler form

$$N_n = \begin{bmatrix} A_n & \lambda (A_n - A_{n-1}) \\ 0 & A_{n-1} \end{bmatrix} \quad (6)$$

where $A_n = u_n + \alpha (u_{n-1} + u_{n+1})$. Taking $\alpha = 0$ in the V5 pair gives the following zero curvature representation for the V3 equation

$$M_n = \begin{bmatrix} \lambda & \lambda^2 + u_n \\ -1 & -\lambda \end{bmatrix}, \quad N_n = \begin{bmatrix} u_n & \lambda(u_n - u_{n-1}) \\ 0 & u_{n-1} \end{bmatrix}$$

For the general VolterraVm equation

$$\dot{u}_n = u_n \sum_{j=1}^{m} \alpha_j (u_{n+j} - u_{n-j}),$$

a zero curvature representation is (6) with

$$A_n = \sum_{j=1}^{m} \alpha_j \sum_{i=0}^{j-1} u_{n+j-1+2i}.$$

Narita [5] uses the double D operator method to show that the Vm equation is integrable providing $\alpha_i = 1, i = 1 \ldots m$, see also [6, 7, 8, 9]. However, tests using the singularity confinement method [2] appear to confirm that the $\alpha_i = 1$ case is integrable, but the more general $\alpha_i$ case is not.

Another integrable example [10] is the discrete mKdV equation

$$\dot{u}_n = (1 + u_n^2) (u_{n+1} - u_{n-1}) \quad (7)$$

(other integrable discretizations of the mKdV equation are discussed in [11]). An obvious generalisation of (9) is

$$\dot{u}_n = (1 + u_n^2) \sum_{j=1}^{m} \beta_j (u_{n+j} - u_{n-j}), \quad (8)$$

this has a Lax pair

$$M_n = \begin{bmatrix} \lambda & u_n \\ -\lambda^2 u_n & \lambda \end{bmatrix}, \quad N_n = \begin{bmatrix} A_n & B_n/\lambda \\ -\lambda B_n & A_n \end{bmatrix},$$
with
\[ A_n = \sum_{j=1}^{m} \beta_j \sum_{i=0}^{j-1} u_{n-j+i} u_{n+i}, \quad B_n = \sum_{j=1}^{m} (u_{n-j} + u_{n+j-1}) \sum_{i=j}^{m} \beta_i \]

but here the spectral parameter \( \lambda \) is entering in a trivial way. Tests with the singularity confinement method [2] suggests that the general case is not integrable for any choice of \( m > 1 \) and \( \beta_i \neq 0 \).

There is a transformation [12] connecting (7) to
\[ \dot{u}_n = (1 + u_n)^2 (u_{n+1} - u_{n-1}) \]

and we also have a zero curvature representation for this, so we suspect this one is integrable. Tests using the singularity confinement method suggests that a more general form of the \( m = 1 \) equation
\[ \dot{u}_n = (u_n - \alpha_1)(u_n - \alpha_2)(u_{n+1} - u_{n-1}) \] (9)
is also integrable (the case \( \alpha_2 \to \infty \) would give the Volterra equation). Moreover there is a simple transformation connecting all these equations: \( u_n = (u_{n-1} - \alpha_1)(u_n - \alpha_2) \) satisfies (4) whenever \( u_n \) satisfies (9).

Acknowledgements.
One of us (JCE) is grateful to the Alliance programme for support for a visit to Lyon where some of this work was carried out.

References


