The Darboux-Bianchi-Bäcklund transformation and soliton surfaces

Jan Cieśliński

Uniwersytet w Białymstoku, Instytut Fizyki,
15-424 Białystok, ul. Lipowa 41, Poland
e-mail: janek@alpha.fuwb.edu.pl, janek@fuw.edu.pl

Abstract: In the first part of the paper we present the dressing method which generates multi-soliton solutions to integrable systems of nonlinear partial differential equations. We compare the approach of Neugebauer with that of Zakharov, Shabat and Mikhailov. In both cases we discuss the group reductions and reductions defined by some multilinear constraints on matrices of the linear problem. The second part of the paper describes the soliton surfaces approach. The so-called Sym–Tafel formula simplifies the explicit reconstruction of the surface from the knowledge of its fundamental forms, unifies various integrable nonlinearities and enables one to apply powerful methods of the theory of solitons to geometrical problems. The Darboux–Bianchi–Bäcklund transformation (i.e., the dressing method on the level of soliton surfaces) reconstructs explicitly many classical transformations of 19th century. We present examples of interesting classes of surfaces obtained from spectral problems. In particular, we consider spectral problems in Clifford algebras associated with orthogonal coordinates. Finally, compact formulas for multi-soliton surfaces are discussed and applied for the Localized Induction Equation with axial flow.

1 Introduction

An integrable system can be represented as integrability conditions for a *linear problem* (a system of linear partial differential equations containing the spectral parameter). In this paper we confine ourselves to linear problems of the form

$$\Phi_{,k} = U_k \Phi, \tag{1}$$

where x^1, x^2 are independent variables, $\Phi_{,k} := \partial\Phi/\partial x^k$ and U_1, U_2 are complex $n \times n$ matrices which depend in the prescribed way on the dependent variables (shortly, "soliton fields"), also through their derivatives or integrals, and on the so called *spectral parameter* λ. In practice U_k are meromorphic (usually rational) functions of λ.

Considering the system (1) as equations for an unknown function Φ one can easily see that a non-trivial solution exists if and only if the following integrability condition holds:

$$U_{1,2} - U_{2,1} + [U_1, U_2] = 0. \tag{2}$$

The "zero curvature condition" (2), considered as an identity with respect to λ, is equivalent to a system of nonlinear partial differential equations for soliton fields. The system (2) has many interesting properties which justify to call it *integrable* [2, 48, 59, 85]. One of the properties, the existence of the Darboux–Bäcklund transformation, is discussed thoroughly in this paper.

The system (1) has n independent vector solutions. It is convenient to consider them as columns of a non-degenerate $n \times n$ matrix called the fundamental solution of (1). In the sequel we always assume that Φ, sometimes called the *wave function*, is the matrix.

REMARK 1. *A linear combination of vector solutions is a solution as well. In other words, if Φ and Φ' are fundamental solutions of (1) then there exists a constant non-degenerate matrix C such that $\Phi' = \Phi C$.*

As an example consider the *nonlinear Schrödinger equation*:

$$iq_{,2} + fq_{,11} + 2q|\,q\,|^2 = 0, \tag{3}$$

where $q = q(x^1, x^2) \in \mathbf{C}$ is the soliton field. The associated linear problem reads [2, 85]:

$$\Phi_{,1} = (i\lambda\sigma_3 + Q)\Phi, \qquad \Phi_{,2} = (-2i\lambda^2\sigma_3 - 2\lambda Q + R)\Phi, \tag{4}$$

where $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\quad Q := \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$, $\quad R := \begin{pmatrix} i|\,q\,|^2 & iq_{,1} \\ i\bar{q}_{,1} & -i|\,q\,|^2 \end{pmatrix}$.

The *Darboux–Bianchi–Bäcklund (DBB)* transformation is a gauge transformation which preserves the form of the linear problem. On the level of soliton fields this transformation simply adds a soliton solution. Usually the name of Bianchi is not mentioned in this context. However, because of his enormous contribution to this field it seems justified to name after him at least the corresponding transformation on the level of *soliton surfaces* (see Section 4).

There are several approaches to the construction of multi-soliton solutions by gauge transformations. Let us mention the Zakharov–Shabat dressing method [51, 85, 89], the Neugebauer approach [61, 65], Darboux transformation for linear differential operators [58, 59], the method of Its based on axiomatics of the wave functions [43] and the approach of Gu [40]. These methods have been discovered to some extent independently. The fundamental concept of the "dressing" of linear operators proposed by Zakharov and Shabat [88, 89] was probably the starting point of all these researches. The details however are different and very often authors do not seem to be aware of the results of the other groups. In this paper we focus on the Neugebauer and Zakharov–Shabat methods.

The dressing method of Zakharov and Shabat [85, 86, 88, 89] is a general scheme to solve integrable nonlinear equations. Consider a gauge transformation

$$\tilde{\Phi} = D\Phi, \tag{5}$$

defined by a matrix $D = D(x^1, x^2; \lambda)$. Then, obviously, $\tilde{\Phi}$ satisfies a linear system (1) with matrices $\tilde{U}_k$

$$\tilde{U}_k = D_{,k} D^{-1} + DU_k D^{-1}, \qquad k = 1, 2. \tag{6}$$

If we are able to construct a matrix D such that $\tilde{U}_k$ given by (6) are of exactly the same form as U_k then D is called the *Darboux matrix* [18, 52, 59]. In other words, the dependence of $\tilde{U}_k$ on λ and on soliton fields should be exactly the same as that of U_k (of course, the soliton fields entering $\tilde{U}_k$ are in general different than those entering U_k). The existence of the Darboux matrix is one of the criterions of integrability.

It turns out that the structure of matrices U_k can be characterized completely in terms of some simple properties. The most important information is contained in the singularities of U_k. The matrices U_k are assumed to be meromorphic in λ and their poles are given.

In this paper we show that the construction of the Darboux matrix is practically algorithmic (see also [18]). The crucial point is to notice all relevant algebraic and group properties of the associated linear problem (especially to identify the reduction group) and then to apply appropriate general theorems. In particular, we present multilinear constraints which are usually invariant with respect to DBB transformation. Taking them into account we can avoid some cumbersome calculations and our construction assumes a more elegant form.

It is reasonable to consider as *equivalent* the Darboux matrices which generate the same DBB transformation on the level of soliton fields. In other words, D_1 and D_2 are equivalent iff the corresponding transformations (6) are identical.

Remark 2. *The Darboux matrix D is equivalent to fD where f is any scalar function independent on x^1, x^2, i.e. $f = f(\lambda)$.*

The methods of constructing the Darboux matrix are based on various approaches to matrices with rational coefficients. The following example shows that problems of that kind can be interesting even in the well-known scalar case. Namely, we proceed to the decomposition into partial fractions. The procedure (used in the integration of rational functions) is standard and algorithmic but usually is considered as cumbersome and boring. However, using local parameters in the neighbourhood of singularities one obtains a very effective method.

Let us show the main idea of the approach on the example of the function $F(x) = (2x+1)^{-10}x^{-3}$. Taking into account that $(1+\varepsilon)^\alpha = 1 + \alpha\varepsilon + \frac{1}{2!}\alpha(\alpha-1)\varepsilon^2 + \frac{1}{3!}\alpha(\alpha-1)(\alpha-2)\varepsilon^3 + \ldots$ we expand F around $x = -1/2$ and $x = 0$, choosing as local parameters respectively $\xi = 2x+1$ and $\eta = x$. Then

$$\frac{1}{\xi^{10}x^3} = \frac{1}{x^3}\,(1 - 20x + 220x^2) + \frac{8}{\xi^{10}}\,(1 + 3\xi + 6\xi^2 + 10\xi^3 + \ldots + 45\xi^8 + 55\xi^9) + h(x).$$

The crucial point is that $h(x) \equiv 0$. Indeed, by construction h is bounded and regular both in $x = 0$ and $x = -1/2$. Hence, by the Liouville theorem we have $h(x) \equiv \text{const}$. The constant is zero by the obvious boundary condition: $h \to 0$ for $|x| \to \infty$. The presented approach simplifies computations in a striking way. We can integrate very quickly any rational function with known poles.

The Liouville theorem is the corner stone of the dressing method (as pointed out by Its [43]): local behaviour of analytic functions (e.g., around singularities) can define them globally. However, the matrix case is much more complicated (and more interesting) than the scalar one (compare [37, 68]).

In this paper we apply methods of the theory of integrable systems to the geometry of surfaces immersed in Euclidean spaces. The main idea of the *soliton surfaces approach* consists in associating with a given integrable system of nonlinear partial differential equations a class of surfaces (or manifolds) immersed in the Lie algebra of the corresponding linear problem using the so called Sym–Tafel formula (20) (see [79], compare also [21, 34, 42]). In this way we can unify in a natural way various nonlinear models (spins, vortices, chiral fields, etc.).

We present several examples, mostly immersions in $\mathbb{R}^3$: surfaces swept out by various vortex motions, surfaces of constant Gaussian curvature, constant mean curvature surfaces, Bianchi surfaces, isothermic surfaces. The Sym-Tafel formula can be also applied to manifolds immersed in spaces of higher dimensions like orthogonal coordinates in $\mathbb{R}^n$ [23] and n-dimensional space forms immersed in $\mathbb{R}^{2n-1}$ [22].

The *Darboux–Bianchi* transformations (DBB transformations on the level of surfaces) coincide, as a rule, with classical transformations studied by Luigi Bianchi and other great geometers of 19th century [8, 49, 79].

2 The Darboux–Bäcklund transformation

There are several approaches to the construction of the Darboux matrix. All these methods are closely related but usually not much attention is given to this fact. The discussion of the relationship between the Matveev method and the Zakharov–Shabat approach ([6], see also [59]) is a positive exception. In this paper we compare the Zakharov–Shabat method with the approach of Neugebauer.

The Neugebauer–Meinel approach

An especially effective method to construct N-fold Darboux–Bäcklund transformation has been proposed by Neugebauer ([61, 65, 66], see also [40]). The Darboux matrix is assumed to be a polynomial in λ,

$$T = \sum_{j=0}^{N} C_j(x^1, x^2)\lambda^j \tag{7}$$

(polynomial Darboux matrices will be denoted by T). Any matrix D rational in λ can be replaced by an equivalent matrix T of the form (7) (compare Remark 2 where for f we should take the least common denominator). One can consider other equivalent matrices, e.g., polynomials in $1/\lambda$.

From the elementary linear algebra we know that $T^{-1} = (\det T)^{-1}T'$, where T' is the matrix of cofactors of T. Obviously T' is a polynomial in λ. Therefore, if U_k are rational functions of λ, then $\tilde{U}_k$ given by (6) are rational as well. What is more, the only candidates for poles of $\tilde{U}_k$ are poles of U_k and zeros of $\det T$.

LEMMA 3. *If* $\mathrm{Tr}U_k = 0$ *then the determinant of the Darboux matrix does not depend on* x^1, x^2. *In particular, the zeros of* $\det T$ *are constant.*

The proof is based on a very useful identity, $(\log\det A)_{,x} = \mathrm{Tr}(A_{,x} A^{-1})$, which holds for any nondegenerate matrix function $A = A(x)$. The conditions $\mathrm{Tr}U_k = 0$, assumed throughout the paper, are not particularly restrictive: practically all linear problems associated with integrable systems are traceless.

The necessary condition for T to be a Darboux matrix is the requirement that $\tilde{U}_k$ have no poles in zeros of $\det T$. We assume that $\det C_0 \neq 0$ and all zeros of $\det T$ (denoted by λ_i) are simple and pairwise different. The total number of zeros is Nn. We assume also that U_1, U_2 and Φ are regular in λ_i. Because $\tilde{U}_k = \tilde{\Phi}_{,k}\,\tilde{\Phi}^{-1} = \tilde{\Phi}_{,k}\,\tilde{\Phi}'(\det\Phi\det T)^{-1}$, then the condition for $\tilde{U}_k$ to have no pole in $\lambda = \lambda_i$ is given by

$$\tilde{\Phi}_{,k}(\lambda_i)\tilde{\Phi}'(\lambda_i) = 0. \tag{8}$$

We proceed to construct $\tilde{\Phi}$ in order to satisfy the equations (8). The matrix $\tilde{\Phi}(\lambda_i)$ is degenerate. Hence there exists a vector p_i such that $\tilde{\Phi}(\lambda_i)p_i = 0$. Then, by (1) it follows that $\tilde{\Phi}_{,k}(\lambda_i)p_i = 0$. Differentiating the equation $\tilde{\Phi}(\lambda_i)p_i = 0$ we obtain $\tilde{\Phi}(\lambda_i)p_{i,k} = 0$. Therefore, taking into account that zeros λ_i are simple, we obtain that $p_{i,k}$ is proportional to p_i. In fact, using an additional freedom (p_i are defined up to a scalar factor) we can choose p_i to be constant vectors.

Corollary 4. *If λ_i is a simple zero of $\det T$, then there exists a constant vector $p_i \neq 0$ such that $\tilde{\Phi}(\lambda_i)p_i = \tilde{\Phi}_{,k}(\lambda_i)p_i = 0$.*

The corollary implies the equation (8). Indeed, one has only take into account the following fact of the elementary linear algebra (we suggest to prove it as an exercise) [61].

Lemma 5. *If there exists a vector $p \neq 0$ such that $Xp = 0$ and $Yp = 0$ (X, Y are matrices) then $YX' = 0$.*

Finally, we can treat constants $\lambda_i \in \mathbb{C}$ and $p_i \in \mathbb{C}^n$ $(i = 1, \ldots, nN)$ as prescribed parameters. Then the equations $\tilde{\Phi}(\lambda_i)p_i = 0$ will determine the coefficients C_j of the Darboux matrix (7). Namely, the system

$$T(\lambda_k)\Phi(\lambda_k)p_k = 0 \quad (k = 1, \ldots, Nn), \tag{9}$$

is equivalent to Nn^2 scalar equations for $(N+1)n^2$ scalar coefficients. The remaining freedom corresponds to the gauge freedom. For instance we can treat C_0 as still undetermined. Then matrices $C_1,\ldots,C_N$ are uniquely determined provided that the complex parameters λ_k and constant complex vectors p_k are given.

The *degenerate case* (multiple zeros of $\det T$) is more complicated. If there exist several vectors satisfying the equation $\tilde{\Phi}(\lambda_j)p_j = 0$, then they also can be chosen to be constant.

To obtain explicit solutions of a given nonlinear system by the presented method we have to know explicitly at least one solution of the linear problem (1). Then we can obtain in an algebraic way a sequence of other solutions which can be interpreted as a "nonlinear superposition" of the given ("background") solution and some number of solitons. In particular, applying the transformation (7) to the trivial background we obtain "pure" N-soliton solutions.

The Zakharov–Shabat approach

In the papers of Zakharov, Shabat and Mikhailov another representation of the Darboux matrix was used [63, 85, 86, 87],

$$D = \mathcal{N}\left(I + \sum_{k=1}^{N} \frac{A_k}{\lambda - \lambda_k}\right), \qquad D^{-1} = \left(I + \sum_{k=1}^{N} \frac{B_k}{\lambda - \mu_k}\right)\mathcal{N}^{-1}, \tag{10}$$

where I is the unit matrix, $\mathcal{N}, A_k, B_k$ depend on x^1, x^2 and λ_k, μ_k are constants (assumed to be pairwise different).

The similar form of D and D^{-1} is a restriction on A_k and B_k. The condition $DD^{-1} = I$ is equivalent to

$$A_k\left(I + \sum_{j=1}^{N} \frac{B_j}{\lambda_k - \mu_j}\right) = 0, \qquad \left(I + \sum_{j=1}^{N} \frac{A_j}{\mu_k - \lambda_j}\right) B_k = 0, \tag{11}$$

$(k = 1, \ldots, N)$. In the case $N = 1$ ("1-soliton case") this system can be easily solved to give

$$D = \mathcal{N}\left(I + \frac{\lambda_1 - \mu_1}{\lambda - \lambda_1} P\right), \qquad D^{-1} = \left(I + \frac{\mu_1 - \lambda_1}{\lambda - \mu_1} P\right)\mathcal{N}^{-1}, \tag{12}$$

where P is a projector (i.e. $P^2 = P$).

Theorem 6. (Zakharov, Shabat [89]) *If P is given by the formulae:*

$$\ker P = \Phi(\lambda_1) V_{\text{ker}}, \quad \mathrm{im} P = \Phi(\mu_1) V_{\text{im}}, \tag{13}$$

where V_{ker} and V_{im} are constant vector spaces such that $V_{\text{ker}} \oplus V_{\text{im}} = \mathbf{C}^n$, then the transformation (6) *with D given by* (12) *preserves the divisors of poles of matrices U_k.*

Every projector is completely characterized by its image and kernel. The matrix of the projector P is explicitly given by

$$P = \{0, \Phi(\mu_1) V_{\text{im}}\} \{\Phi(\lambda_1) V_{\text{ker}}, \Phi(\mu_1) V_{\text{im}}\}^{-1},$$

where in the brackets we have $n \times n$ matrices. We use the same notation to designate a subspace of $\mathbb{R}^n$ and a matrix representing it (the columns of the matrix span the considered vector space).

Multiplying D given by (10) by the common denominator we obtain an equivalent polynomial matrix like (7). However (10) and (7) are not strictly equivalent. One should remember about restrictions (11). An equivalent matrix D of the form (10) exists only for some particular polynomial matrices T.

3 Algebraic representation of the linear problem

The techniques presented above guarantee that a linear problem subject to the Darboux-Bäcklund transformation is transformed into a linear problem of the same analytical dependence on λ. However, a given integrable system usually needs much more restrictions imposed on the associated linear problem. The most important reductions consist in confining U_k to some Lie algebra. Then Φ is confined to the corresponding group and, obviously, D is confined to the same group. An example is in order. Let $U_k^\dagger(\overline{\lambda}) = -U_k(\lambda)$ for $k = 1, 2$ (this property holds in the case of the linear problem (4) where the coefficients by powers of λ are $\mathbf{su}(2)$-valued). Then one can check that $((\Phi^\dagger(\overline{\lambda}))^{-1})_{,k} = U_k(\lambda)(\Phi^\dagger(\overline{\lambda}))^{-1}$ and, by virtue of Remark 1, $(\Phi^\dagger(\overline{\lambda}))^{-1} = \Phi(\lambda)C(\lambda)$ where $C(\lambda)$ does not depend on x^1, x^2. Elementary calculations show that $C^\dagger(\overline{\lambda}) = C(\lambda)$. In fact C depends only on the initial condition $\Phi(x_0^1, x_0^2; \lambda)$ and by an appropriate choice of the condition we can put $C = I$. Thus Φ satisfies $\Phi^\dagger(\overline{\lambda})\Phi(\lambda) = I$. In general, if U_k are confined to some loop algebra then, by an appropriate choice of initial conditions, Φ, $\tilde{\Phi}$ and D can be restricted to the corresponding group.

The *reduction groups* had already been introduced in the pioneering paper of Zakharov and Shabat ([89], see also [85]). Then the subject was developed by Mikhailov [62, 63, 86, 87]. He classified a large class of reductions considering groups of 4 types [62].

The restrictions on the matrix D given by (12) have been discussed in detail in [18]. Actually, because of geometrical applications, it is convenient to consider the unimodular Darboux matrix $\mathcal{D}$ [18, 79] (by Lemma 3 it is clear that in the case of traceless linear problems one can always impose the condition $\det D = 1$). To confine the Darboux matrix (12) to the group $\mathbf{SL}(n, \mathbb{C})$ it is sufficient to multiply it by the scalar coefficient $f = (\det D)^{1/n}$:

$$\mathcal{D} = \left(\frac{\lambda - \lambda_1}{\lambda - \mu_1}\right)^{d/n} \mathcal{N}\left(I + \frac{\lambda_1 - \mu_1}{\lambda - \lambda_1} P\right), \tag{14}$$

where $\det \mathcal{N} = 1$, $d = \dim(\mathrm{im} P)$ and $\lambda_1 \neq \mu_1$. The matrices (12) and (14) are equivalent (compare Remark 2). The paper [18] contains the full list of restrictions on $\lambda_1, \mu_1, f, P, \mathcal{N}$ implied by the Mikhailov reductions provided that $\det \mathcal{N} \neq 0$.

Here we are going to present briefly some results concerning Darboux matrices in the polynomial form (7).

Let $U_k(-\lambda) = JU(\lambda)J^{-1}$ where J can depend on λ. One can easily see that J has to satisfy $J(-\lambda)J(\lambda) = \theta(\lambda)I$ where θ is a scalar function. The wave function and the Darboux matrix can be confined to the group $\Phi(-\lambda) = J\Phi(\lambda)J^{-1}$, $T(-\lambda) = JT(\lambda)J^{-1}$. Therefore $\det T(-\lambda) = \det T(\lambda)$ which means that zeros of $\det T(\lambda)$ appear in pairs $\lambda_j = -\lambda_k$. The corresponding constant eigenvectors p_j and p_k satisfy $\tilde{\Phi}(\lambda_k)p_k = 0$ and $\tilde{\Phi}(\lambda_j)p_j = 0$ which implies $\tilde{\Phi}(\lambda_k)J(-\lambda_k)p_j = 0$. If the zeros λ_j are simple then $p_j = J(\lambda_k)p_k$.

The unitary reductions $U_k^\dagger(\overline{\lambda}) = -HU_k(\lambda)H^{-1}$ are more complicated. If we admit a (rational) dependence of H on λ then $H^\dagger(\overline{\lambda}) = \eta(\lambda)H(\lambda)$ where η is a scalar function. One can easily check that $H(\Phi^\dagger(\overline{\lambda}))^{-1}$ satisfies the system (1) provided that Φ is a solution. $\tilde{\Phi}$ is subject to an analogical constraint. Thus we can derive the condition

$$T(\lambda) = k(\lambda)H\left(T^{\dagger}(\overline{\lambda})\right)^{-1}H^{-1}, \tag{15}$$

(k is a scalar function of λ) which is sufficient for T to be the Darboux matrix. We point out that now the simplest choice $k(\lambda) \equiv 1$ is not possible because it is incompatible with the form (7). If H is rational then k has to be rational as well. From (15) we obtain

$$(k(\lambda))^n = \overline{\det T(\overline{\lambda})} \det T(\lambda). \tag{16}$$

Therefore $k(\lambda)$ is a polynomial with real coefficients. We confine ourselves to the case of $k(\lambda)$ without real roots, i.e.

$$k(\lambda) = (\lambda - \lambda_1)(\lambda - \overline{\lambda}_1)\ldots(\lambda - \lambda_N)(\lambda - \overline{\lambda}_N)$$

where N is the degree of the polynomial $T(\lambda)$. Then

$$\det T = (\lambda - \lambda_1)^{n-d_1}(\lambda - \overline{\lambda}_1)^{d_1}\ldots(\lambda - \lambda_N)^{n-d_N}(\lambda - \overline{\lambda}_N)^{d_N},$$

where d_k are some integers. Note that for $n > 2$ the zeros of $\det T$ are, as a rule, degenerate. If λ_k are pairwise different, then dividing T by $(\lambda - \lambda_1)\ldots(\lambda - \lambda_N)$ we obtain the matrix D (equivalent to T) bounded for $\lambda \to \infty$:

$$D = \frac{T(\lambda)}{(\lambda - \lambda_1)\ldots(\lambda - \lambda_N)} = S_0 + \sum_{k=1}^{N} \frac{S_k}{\lambda - \lambda_k}$$

where S_k are some matrices dependent on x^1, x^2. From (16) we can deduce that the matrix S_0 is not degenerated. Therefore D is exactly of the form (10). Using (15) we compute D^{-1} and, as a result, we have $\mu_k = \overline{\lambda}_k$. The case of $k(\lambda)$ with real roots is left as an exercise. We just mention that this case is possible only for even n.

General conclusion is very interesting: in the important case of unitary reductions the Darboux matrix can always be represented in the form (10).

Multilinear invariants of the Darboux matrix

Let λ_0 be a pole of matrices U_1 and U_2 of the order J and K respectively. The matrices $U := (\lambda - \lambda_0)^J U_1$ and $W := (\lambda - \lambda_0)^K U_2$ are holomorphic in the neighbourhood of λ_0:

$$U = \sum_{i=0}^{\infty} u_i (\lambda - \lambda_0)^i, \quad W = \sum_{i=0}^{\infty} w_i (\lambda - \lambda_0)^i.$$

We assume that u_j and w_j are $\mathbf{sl}(n, \mathbf{C})$-valued functions of x^1, x^2. In many interesting cases including (3) the only singularity is $\lambda_0 = \infty$, i.e. U_1 and U_2 are polynomials in λ while U, W are polynomials in $1/\lambda$.

The transformation (6) for U, W reads:

$$\tilde{U} = (\lambda - \lambda_0)^J D_{,1} D^{-1} + DUD^{-1}, \quad \tilde{W} = (\lambda - \lambda_0)^K D_{,2} D^{-1} + DWD^{-1}. \tag{17}$$

Considering the leading terms in equations (17) we can obtain a number of invariants of DBB transformation. In the sequel we assume $J \leq K$ and D is regular at $\lambda = \lambda_0$.

Linear invariants. Consider $\alpha U + \beta W$ where $\alpha = \alpha(x^1, x^2)$ and $\beta = \beta(x^1, x^2)$ are given functions. We have

$$\alpha \tilde{U} + \beta \tilde{W} = \alpha(\lambda - \lambda_0)^J D_{,1} D^{-1} + \beta(\lambda - \lambda_0)^K D_{,2} D^{-1} + D(\alpha U + \beta W) D^{-1}.$$

Let $L < J \leq K$. If $\alpha U + \beta W$ has zero of L-th order in $\lambda = \lambda_0$ then $\alpha \tilde{U} + \beta \tilde{W}$ has zero of at least the same order. Thus we have shown the following result [18].

PROPOSITION 7. *If* $\lambda = \text{const}$ *then any system of* $L+1$ *linear constraints* $(L < J)$ *of the form*

$$\alpha u_m + \beta w_m = 0 \qquad (m = 0, 1, \ldots, L),$$

where $\alpha = \alpha(x^1, x^2)$, $\beta = \beta(x^1, x^2)$ *are given functions, is preserved by DBB transformation.*

The proposition can be extended for $L = J$ provided that the Darboux matrix satisfies the condition $D_{,1}(\lambda_0) = 0$ (in the case $\lambda_0 = \infty$ it means that the normalization matrix does not depend on x^1).

In the case of the nonlinear Schrödinger equation we have (by inspection): $u_0 + 2w_0 = 0$ and $u_1 + 2w_1 = 0$ (see (4)). These constraints are invariants of DBB transformation provided that $\mathcal{N} = \mathcal{N}(x^2)$.

Bilinear invariants. Let the center dot denotes an invariant scalar product in $\mathbf{sl}(n, \mathbf{C})$, namely $a \cdot b := \mathrm{Tr}(ab)$. Consider the transformation of $W \cdot W$:

$$\tilde{W} \cdot \tilde{W} = W \cdot W + 2(\lambda - \lambda_0)^K (D^{-1} D_{,2}) \cdot W + (\lambda - \lambda_0)^{2K} (D^{-1} D_{,2}) \cdot (D^{-1} D_{,2}).$$

On the right-hand side the leading terms (of order less than K) are contained in $W \cdot W$.

PROPOSITION 8. *If* $\lambda = \text{const}$ *then the constraint*

$$\sum_{k=0}^{m} w_k \cdot w_{m-k} = \gamma_m \qquad (0 < m < K - 1)$$

(where $\gamma_m = \gamma_m(x^1, x^2)$ *is a given function) is preserved by DBB transformation.*

The proposition holds for $m = K$ if and only if $(D^{-1}(\lambda_0) D_{,2}(\lambda_0)) \cdot w_0 = 0$ (in the polynomial case $(\lambda_0 = \infty)$ this condition reduces to $\mathcal{N}_{,2} = 0$).

The multilinear invariants can be used to obtain reductions of the linear problem (1). The reductions are fixed by the choice of α, β and γ_m.

Similar results can be obtained considering the expansions of $U \cdot U$ and $U \cdot W$ around $\lambda = \lambda_0$. One can expect to get analogical results for $\mathrm{Tr}(UU \ldots U)$, $\mathrm{Tr}(WW \ldots W)$, $\mathrm{Tr}(UW \ldots W)$ etc.

The linear problem as a system of algebraic constraints on two matrices

Let us summarize the results of the preceding sections. The knowledge of the linear problem like (1) is crucial to solve a given integrable system. One can describe the linear problem parameterizing it explicitly by soliton fields, their derivatives and integrals. However, if the the construction of the Darboux matrix is concerned, much more convenient description is by a system of relatively simple (non-differential) constraints (*algebraic representation of the linear problem* [18, 19]): matrices U_1 and U_2 are rational functions of λ (with prescribed poles) restricted to some loop algebra (in other words, their coefficients by powers of λ are restricted to a Lie algebra) and some multilinear constraints are imposed on the coefficients by powers of λ. Considering nonisospectral linear problems [13] we should also prescribe an explicit dependence of variable spectral parameter on a constant parameter [18].

The algebraic representation of the linear problem allows us to introduce a very useful *definition of the Darboux matrix*. The matrix D is called the Darboux matrix for the linear problem (1) if the transformation (6) preserves the system of algebraic constraints equivalent to this linear problem [18].

It is very convenient to divide the construction of the Darboux matrix into two separate steps ([18, 19], similar idea can be found also in [43]). First, we represent the linear problem under consideration as a system of algebraic constraints on two matrices. This step may turn out to be rather non-standard (i.e. each case has to be treated individually) but involves no cumbersome calculations. Usually the algebraic constraints can be found by straightforward inspection of the given linear problem. The crucial point is to find the reduction group. Second, we derive the constraints on the Darboux matrix imposed by the requirement that the matrix has to preserve the algebraic constraints. The second step is technically much more difficult but there exist many general propositions which make the construction practically algorithmic (the 1-soliton case is discussed in detail in [18]).

4 Soliton surfaces approach

We proceed to apply the soliton theory to the geometry of surfaces immersed in Euclidean spaces. Let us consider a surface $\Sigma \subset \mathbb{R}^3$ equipped (locally) with coordinates x^1, x^2, and defined by the position vector $\mathbf{r} = \mathbf{r}(x^1, x^2) \in \mathbb{R}^3$. The first fundamental form ("metric tensor") is given by $I := d\mathbf{r} \cdot d\mathbf{r}$ (the center dot means a scalar product in the ambient space $\mathbb{R}^3$). Therefore, in local coordinates, $I := g_{ij} dx^i dx^j$ (we assume the Einstein convention, i.e., summation over repeating indices), where $g_{ij} := \mathbf{r},_i \cdot \mathbf{r},_j$. The matrix inverse to (g_{ij}) will be denoted by (g^{ij}) and $g := \det(g_{ij})$. Let $\mathbf{n}$ be the unit normal vector, $\mathbf{n} := g^{-1/2} \mathbf{r},_1 \times \mathbf{r},_2$, where the cross denotes skew (or vector) product in $\mathbb{R}^3$. The second fundamental form is defined as $II := -d\mathbf{n} \cdot d\mathbf{r}$, or equivalently, $II = b_{ij} dx^i dx^j$ where $b_{ij} := \mathbf{r},_{ij} \cdot \mathbf{n}$. The matrix with coefficients $b^i_j := g^{ik} b_{kj}$ represents the "shape operator" [11]. The principal curvatures, k_1 and k_2, are defined as eigenvalues of (b^i_j). The corresponding eigenvectors ("principal directions") can also be characterized by the requirement to diagonalize both fundamental forms. The Gaussian curvature is defined as $K := k_1 k_2$ and the mean curvature as $H := \frac{1}{2}(k_1 + k_2)$. They can be expressed by the fundamental forms: $K = \det(b_{ij})/\det(g_{ij})$, $H = g^{ij} b_{ij}$. In fact, K can be expressed solely in terms of g_{ij} (the Gauss "Theorema Egregium").

Differentiating tangent vectors we obtain *Gauss–Weingarten (GW)* equations $\mathbf{r},_{ij} = \Gamma^k_{ij}\mathbf{r},_k + b_{ij}\mathbf{n}$, where Γ^k_{ij} are some x^1, x^2-dependent coefficients known as "Christoffel symbols". It is well known that they can be expressed entirely by g_{ij}, namely $\Gamma^k_{ij} = \frac{1}{2} g^{ks}(g_{is},_j + g_{sj},_i - g_{ij},_s)$. The GW equations can be rewritten as kinematic equations of the orthogonal frame $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{n}$, where $\mathbf{e}_k$ form an orthogonal basis in the tangent space:

$$\frac{\partial}{\partial x^k} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix} = \Omega_k \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{n} \end{pmatrix}, \tag{18}$$

where Ω_1 and Ω_2 are $\mathbf{so}(3)$ matrices depending on g_{ij} and b_{ij}. It is convenient to use the well known isomorphism $\mathbf{so}(3) \simeq \mathbf{su}(2)$ (see, for instance, [28]) and to rewrite the equations (18) in terms of 2×2 matrices. The explicit form of the matrices reads, for example, as follows (compare [21, 77]):

$$\Phi,_k = \frac{i}{2\sqrt{g_{11}}} \begin{pmatrix} \frac{g_{11}b_{2k} - g_{12}b_{1k}}{\sqrt{g}} & b_{1k} + i\Gamma^2_{1k}\sqrt{\frac{g}{g_{11}}} \\ b_{1k} - i\Gamma^2_{1k}\sqrt{\frac{g}{g_{11}}} & \frac{g_{12}b_{1k} - g_{11}b_{2k}}{\sqrt{g}} \end{pmatrix} \Phi \qquad (k = 1, 2). \tag{19}$$

Integrability conditions for GW equations can be derived from $\mathbf{r},_{ijk} = \mathbf{r},_{ikj}$. They are expressed explicitly in terms of matrices Ω_k:

$$\Omega_1,_2 - \Omega_2,_1 + [\Omega_1, \Omega_2] = 0,$$

and are equivalent to a system of 3 nonlinear equations for the coefficients g_{ij}, b_{ij}: the *Gauss–Mainardi–Codazzi (GMC)* equations.

The Sym–Tafel formula

Let matrices U_1, U_2 for $\lambda \in \mathbb{R}$ assume values in a Lie algebra g (in other words, matrices U_k are meromorphic functions of λ with g-valued coefficients). Then one can confine Φ to the corresponding Lie group which implies that the matrix $\mathbf{r} = \mathbf{r}(x^1, x^2; \lambda)$ given by

$$\mathbf{r} = \Phi^{-1}\Phi,_\lambda \tag{20}$$

is g-valued. The formula (20) is known as the Sym formula or the Sym-Tafel formula (the final step in derivation of this formula was done by Tafel, see [72]). We assume that the Lie algebra g is equipped with a scalar product invariant with respect to adjoint transformations, $\langle Ad_\Phi a \,|\, Ad_\Phi b\rangle = \langle a \,|\, b\rangle$ (in a matrix representation $Ad_\Phi a := \Phi^{-1} a \Phi$). A well known example is the Killing-Cartan form, $\langle a \,|\, b\rangle := \mathrm{Tr}(ad_a \circ ad_b)$, which is non-degenerate for semi-simple algebras. In general, $\mathrm{Tr}(f(a)f(b))$ defines an invariant bilinear form for any matrix representation f of the Lie algebra g, and can be used as a scalar product provided that the bilinear form is non-degenerate [39]. Then g can be identified with a pseudo-Euclidean space $\mathbb{R}^m$ and the function $\mathbf{r}$ represents a λ-family of parametric surfaces [83] (*"soliton surfaces"*) immersed in $g \cong \mathbb{R}^m$. Kinematics of $\mathbf{r}$ defines an integrable nonlinear model which can be interesting in itself [73, 79].

Starting from the formula (20) we compute tangent vectors $\mathbf{r},_k = \Phi^{-1} U_k,_\lambda \Phi$ and the metric tensor $g_{ij} = \langle \mathbf{r},_i | \mathbf{r},_j \rangle = \langle U_i,_\lambda | U_j,_\lambda \rangle$ (the invariance of the scalar product was used). We obtained g_{ij} without solving differential equations (1). It was sufficient to know only the matrices U_1, U_2. GW equations read $\mathbf{r},_{kj} = \Phi^{-1}(U_k,_{\lambda j} + [U_k,_\lambda, U_j])\Phi$ and GMC equations ($\mathbf{r},_{ijk} = \mathbf{r},_{ikj}$) are given by $\Phi^{-1}[U_i,_\lambda, U_j,_k - U_k,_j + [U_j, U_k]]\Phi = 0$.

The Sym–Tafel formula gives an interesting connection between the classical geometry of manifolds (with possible singularities) immersed in $\mathbb{R}^m$ and the theory of solitons [79]. The soliton surfaces approach is very useful in construction of the so-called "integrable geometries" [8, 15, 25, 79]. Indeed, any class of soliton surfaces given by (20) is integrable. Geometrical objects associated with soliton surfaces (tangent vectors, normal vectors, foliations by curves etc.) usually can be identified with solutions to some nonlinear models (spins, chiral models, strings, vortices etc.) [73, 79]. The transformation (5) can be immediately extended on the models associated with soliton surfaces. For instance, the transformation (5) applied to the position vector (20) assumes the form

$$\tilde{\mathbf{r}} = \mathbf{r} + \Phi^{-1}\mathcal{D}^{-1}\mathcal{D},_\lambda \Phi \tag{21}$$

and we propose to call it the *Darboux-Bianchi transformation.* Usually $\mathbf{r} \in g \subset \mathbf{sl}(n, \mathbb{C})$ and we may consider the unimodular Darboux matrix (14). In the $\mathbf{SU}(n)$ case we have the additional restriction $\mu_1 = \overline{\lambda}_1$. The transformation (21) for soliton surfaces immersed in $\mathbf{su}(n)$ is given by [73]:

$$\tilde{\mathbf{r}} = \mathbf{r} + \frac{2\mathrm{Im}\lambda_1}{|\lambda - \lambda_1|^2}\Phi^{-1} i \left(\frac{d}{n} - P\right)\Phi, \tag{22}$$

where λ_1 is a complex parameter and P is the hermitian projector (i.e., $P^2 = P$ and $P^\dagger = P$) onto the space $\Phi(\overline{\lambda}_1)V_{\mathrm{im}}$, where V_{im} is a constant vector space of a given dimension d. The length of the segment $\tilde{\mathbf{r}} - \mathbf{r}$ does not depend on x^1, x^2.

Geometry from spectral problems

The Euclidean 3-dimensional space $\mathbb{E}^3$ can be identified with the Lie algebra $\mathbf{su}(2)$ endowed with the scalar product given by the Killing–Cartan form

$$a \cdot b := \langle a \,|\, b \rangle = -2\mathrm{Tr}(ab). \tag{23}$$

Note that $\mathrm{Tr}(a^2) \leq 0$ for $a \in \mathbf{su}(2)$. What is more, the skew product in $\mathbb{R}^3$ can be identified with the commutator of $\mathbf{su}(2)$ matrices. The standard basis, orthonormal with respect to (23), is given by

$$\mathbf{e}_1 = -\frac{1}{2}i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = -\frac{1}{2}i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{e}_3 = -\frac{1}{2}i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{24}$$

In other words, we identify

$$\mathbf{su}(2) \ni \frac{1}{2}\begin{pmatrix} -iz & -y-ix \\ y-ix & iz \end{pmatrix} \quad \longleftrightarrow \quad (x,\ y,\ z) \in \mathbb{E}^3 .$$

Below we present several examples of integrable geometries in $\mathbb{E}^3$. We start from given linear problems and use the formula (20). The normal vector to $\mathbf{r}$ is given by $\mathbf{n} = (-2\mathrm{Tr}([U_{1,\lambda}, U_{2,\lambda}]^2))^{-1/2}\Phi^{-1}[U_{1,\lambda}\ \ U_{2,\lambda}]\Phi$. Then we compute the fundamental forms and the mean or Gaussian curvature of the resulting surface.

In order to put the result into a more elegant form we have chosen a special dependence between λ and the spectral parameter $\zeta = \zeta(\lambda)$. In two cases (constant mean curvature surfaces and spherical surfaces) the linear problem depends on the spectral parameter through trigonometric functions $\sin\kappa$ and $\cos\kappa$, where κ is proportional to λ. It is equivalent to using the spectral parameter ζ confined to the unit circle, i.e., $\sin\kappa = \frac{1}{2i}(\zeta - \frac{1}{\zeta})$ and $\cos\kappa = \frac{1}{2}(\zeta + \frac{1}{\zeta})$. We use notation $x^1 \equiv u$, $x^2 \equiv v$.

- **Pseudospherical surfaces**

$$\begin{aligned} \Phi_{,1} &= (-\zeta\mathbf{e}_3 - \varphi_{,1}\mathbf{e}_2)\Phi, \\ \Phi_{,2} &= \zeta^{-1}(\cos\varphi\ \mathbf{e}_3 - \sin\varphi\ \mathbf{e}_1)\Phi, \end{aligned} \tag{25}$$

where $\zeta = \exp(-R\lambda)$. The compatibility conditions (2) are equivalent to the sine–Gordon equation $\varphi_{,12} = \sin\varphi$ and the fundamental forms are given by

$$\begin{aligned} I &= R^2(\zeta^2 du^2 + 2\cos\varphi\, du dv + \zeta^{-2}dv^2), \\ II &= 2R\sin\varphi\, du dv. \end{aligned} \tag{26}$$

The Gaussian curvature is given by $K = -R^2$ (compare [73, 79]).

- **Constant mean curvature surfaces**

$$\begin{aligned} \Phi_{,1} &= \left(-(e^{\varphi/2} + e^{-\varphi/2}\cos 2\kappa)\mathbf{e}_1 - \tfrac{1}{2}\varphi_{,2}\,\mathbf{e}_2 - e^{-\varphi/2}\sin 2\kappa\mathbf{e}_3\right)\Phi, \\ \Phi_{,2} &= \left(-e^{-\varphi/2}\sin 2\kappa\ \mathbf{e}_1 + \tfrac{1}{2}\varphi_{,1}\,\mathbf{e}_2 - (e^{\varphi/2} - e^{-\varphi/2}\cos 2\kappa)\mathbf{e}_3\right)\Phi, \end{aligned} \tag{27}$$

where $\kappa := \frac{1}{2}H^{-1}\lambda$. GMC eqs. are reduced to the single equation

$$\varphi_{,11} + \varphi_{,22} + 4\sinh\varphi = 0, \tag{28}$$

known as the elliptic sinh-Gordon equation. The fundamental forms of (20) read

$$\begin{aligned} I &= H^{-2}e^{\varphi}(du^2 + dv^2), \\ II &= H^{-1}\left((e^{\varphi} + \cos 2\kappa)du^2 + 2\sin 2\kappa\ du dv + (e^{\varphi} - \cos 2\kappa)dv^2\right), \end{aligned} \tag{29}$$

which means that the surface has the constant mean curvature H [31].

- **Spherical surfaces**

$$\begin{aligned} \Phi_{,1} &= \left(-2\sinh\tfrac{\varphi}{2}\cos\kappa\ \mathbf{e}_1 - \tfrac{1}{2}\varphi_{,2}\,\mathbf{e}_2 + 2\cosh\tfrac{\varphi}{2}\sin\kappa\ \mathbf{e}_3\right)\Phi, \\ \Phi_{,2} &= \left(-2\sinh\tfrac{\varphi}{2}\sin\kappa\ \mathbf{e}_1 + \tfrac{1}{2}\varphi_{,1}\,\mathbf{e}_2 - 2\cosh\tfrac{\varphi}{2}\cos\kappa\ \mathbf{e}_3\right)\Phi, \end{aligned} \tag{30}$$

where $\kappa := R\lambda$, and GMC eqs. are equivalent to (28).

$$\begin{aligned} I &= 2R^2\left((\cosh\varphi + \cos 2\kappa)\ du^2 + 2\sin 2\kappa\ dudv + (\cosh\varphi - \cos 2\kappa)\ dv^2\right), \\ II &= 2R\sinh\varphi\ (du^2 + dv^2). \end{aligned} \tag{31}$$

Therefore, $\mathbf{r}$ defines a spherical surface ($K = R^{-2} > 0$) [31].

- **Bianchi surfaces**

$$\begin{aligned} \Phi_{,1} &= \left(-\zeta a\mathbf{e}_3 - \left(\varphi_{,1} + \tfrac{\rho_{,2}a}{2\rho b}\sin\varphi\right)\mathbf{e}_2\right)\Phi, \\ \Phi_{,2} &= \left(\tfrac{1}{\zeta}(b\cos\varphi\ \mathbf{e}_3 - b\sin\varphi\ \mathbf{e}_1) + \tfrac{\rho_{,1}b}{2\rho a}\sin\varphi\,\mathbf{e}_2\right)\Phi, \end{aligned} \tag{32}$$

where $\rho = f(u) + g(v)$ (f, g are given functions) and

$$\zeta = \left(\frac{1 - 2\lambda g(v)}{1 + 2\lambda f(u)}\right)^{1/2} \qquad (\lambda = \text{const}),$$

is "variable spectral parameter". We obtain

$$\begin{aligned} I &= (\zeta_{,\lambda})^2\left(a^2\,du^2 + 2ab\zeta^{-2}\cos\varphi\,dudv + b^2\zeta^{-4}dv^2\right), \\ II &= -2\zeta_{,\lambda}\,\zeta^{-1}ab\sin\varphi\,dudv. \end{aligned} \tag{33}$$

The Gaussian curvature is $K = -\rho^{-2}$ where $\rho := \frac{f(u)}{1+2\lambda f(u)} + \frac{g(v)}{1-2\lambda g(v)}$. For $\lambda = 0$ we have $\rho = f + g$ and the formulae (33) assume the standard form:

$$\begin{aligned} I &= \rho^2\left(a^2\,du^2 + 2ab\cos\varphi\,dudv + b^2\,dv^2\right), \\ II &= 2\rho ab\sin\varphi\,dudv. \end{aligned} \tag{34}$$

We recognize the fundamental forms of Bianchi surfaces in asymptotic coordinates (for more details see [18, 46, 53, 80]).

Localized Induction Equations and multisoliton surfaces in $\mathbb{R}^3$

Soliton surfaces in $\mathbb{R}^3$ and several nonlinear models of physical importance [26, 55, 76, 78, 79, 27] are associated with $\mathbf{su}(2)$ algebra. The corresponding linear problems (known as $\mathbf{su}(2)$-AKNS linear problems [2], see also [79]) are parameterized by analytic functions $\omega = \omega(\lambda)$. A typical example is given by the linear problem (4) of the nonlinear

Schrödinger equation ($\omega(\lambda) = -2\lambda^2$). The function ω uniquely characterizes the asymptotic behaviour of the multisoliton solution. The linear problem corresponding to the trivial solution ($q \equiv 0$) has the form

$$\Phi_{0,x} = i\lambda\sigma_3\Phi_0, \qquad \Phi_{0,t} = i\omega(\lambda)\sigma_3\Phi_0,$$

and can be solved easily

$$\Phi_0 = \exp(i\lambda x\sigma_3 + i\omega(\lambda)t\sigma_3). \tag{35}$$

The case $\omega = -2\alpha\lambda^2 - 4\beta\lambda^3$ is associated with the Hirota equation

$$iq_{,2} + \alpha(q_{,11} + 2|\,q\,|^2 q) - i\beta(q_{,111} + 6|\,q\,|^2 q_{,1}) = 0, \tag{36}$$

where α, β are real constants and $q = q(x^1, x^2) \in \mathbb{C}$. The kinematics of the position vector to soliton surfaces (evaluated at $\lambda = 0$) of the Hirota equation describes the motion of the single thin vortex filament in the so called *localized induction approximation with axial flow* [36, 44]:

$$\mathbf{r}_{,2} = \alpha(\mathbf{r}_{,1} \times \mathbf{r}_{,11}) + \beta(\mathbf{r}_{,111} + \frac{3}{2}(\mathbf{r}_{,11})^2 \mathbf{r}_{,1}), \qquad (\mathbf{r}_{,1})^2 = 1, \tag{37}$$

where $\mathbf{r} = \mathbf{r}(x^1, x^2) \in \mathbb{R}^3$. The tangent vector $S := \mathbf{r}_{,1}$ solves the following spin model

$$S_{,2} = \alpha(S \times S_{,11}) + \beta\left(S_{,11} + \frac{3}{2}(S_{,1}^2\, S)_{,1}\right).$$

The special case $\beta = 0$ corresponds to Localized Induction Equation, nonlinear Schrödinger equation and continuum Heisenberg ferromagnet model respectively (see, for instance, [35, 50, 67, 73]). In the case $\alpha = 0$, $q \in \mathbb{R}$ the equation (36) is known as modified Korteweg–de Vries equation. Soliton surfaces are degenerate: the position vector $\mathbf{r}$ describes the evolution of a plane curve which has an interesting elastomechanical interpretation and admits so called "loop solitons" as special solutions [76].

In the $\mathbf{su}(2)$ case usually the formula (20) is used with with the factor $\frac{1}{2}$. Therefore, in this section we assume $\mathbf{r} = \frac{1}{2}\Phi^{-1}\Phi_{,\lambda}$. Iterating the Darboux–Bäcklund transformation N times we obtain the following expression for N-soliton surfaces (more precisely, N-soliton addition to any background surface) associated with $\mathbf{su}(2)$-linear problems [14, 79]:

$$\mathbf{r}_{B+N} = \mathbf{r}_B + \sum_{k=1}^{N} d_k \left(\frac{2\mathrm{Re}\Xi_k}{|\,\Xi_k\,|^2 + 1}\mathbf{e}_1 - \frac{2\mathrm{Im}\Xi_k}{|\,\Xi_k\,|^2 + 1}\mathbf{e}_2 + \frac{|\,\Xi_k\,|^2 - 1}{|\,\Xi_k\,|^2 + 1}\mathbf{e}_3 \right), \tag{38}$$

where the subscript B means "background", $\mathbf{e}_k$ are defined by (24), d_k are given by

$$d_k := \frac{\mathrm{Im}\lambda_k}{|\,\lambda - \lambda_k\,|^2}, \tag{39}$$

λ_k are constant complex parameters, and Ξ_k parameterize P_k, orthogonal projectors onto 1-dim. subspaces of $\mathbb{C}^2$, defined by $\mathrm{im}P_{k+1} = \Phi^{-1}_{B+k}(\lambda)\Phi_{B+k}(\overline{\lambda}_k)p_k$ where $p_k \in \mathbb{C}^2$ are constant vectors. Namely,

$$P_k = \frac{1}{1+|\Xi_k|^2}\begin{pmatrix} |\Xi|^2 - 1 & \Xi \\ \overline{\Xi} & 1 - |\Xi|^2 \end{pmatrix}.$$

Trying to compute $\mathbf{r}_{B+N}$ one meets serious technical problems even for $N = 2$ (compare [55]). However, it is possible to simplify the problem and to express the formula (38) in terms of functions ξ_k $(k = 1, 2, \ldots, N)$ defined by

$$\xi_k := \frac{u_{k1}}{u_{k2}}, \quad \text{where} \quad \begin{pmatrix} u_{k1} \\ u_{k2} \end{pmatrix} := \Phi_B^{-1}(\lambda)\Phi_B(\overline{\lambda}_k)\begin{pmatrix} p_{k1} \\ p_{k2} \end{pmatrix}, \tag{40}$$

where p_{k1}, p_{k2} are components of p_k (see [14]). Let us represent the complex function ξ_k by

$$\xi_k =: \exp(Q_k - i\alpha_k). \tag{41}$$

where Q_k and α_k are real functions. Then we can rewrite the formula (38) for $N = 1$ in a more explicit way:

$$\mathbf{r}_{B+1} = \mathbf{r}_B + \frac{d_1}{\cosh Q_1}\begin{pmatrix} \cos\alpha_1 \\ \sin\alpha_1 \\ \sinh Q_1 \end{pmatrix}. \tag{42}$$

The representation (41) is especially convenient in the $\mathbf{su}(2)$-AKNS case. If $\Phi_B \equiv \Phi_0$ (see (35)) then Q_k and α_k are linear in x and t ($x \equiv x^1$, $t \equiv x^2$):

$$\begin{aligned} Q_k &= 2x\mathrm{Im}\lambda_k + 2t\mathrm{Im}\omega_k + Q_{k0}, \\ \alpha_k &= 2x(\lambda - \mathrm{Re}\lambda_k) + 2t(\omega - \mathrm{Re}\omega_k) + \alpha_{k0}, \end{aligned} \tag{43}$$

where $\omega := \omega(\lambda)$, $\omega_k := \omega(\lambda_k)$ and Q_{ko}, α_{k0} are constant. The soliton surface $\mathbf{r}_0$ degenerates to the straight line, $\mathbf{r}_0 = -(x + \omega'(\lambda)t)\mathbf{e}_3$. Therefore the formula (42), valid for any background, is especially useful in the case of the trivial background (35) ($\mathbf{r}_B \equiv \mathbf{r}_0$, Q_1 and α_1 are given by (43)). In this case $\mathbf{r}_{B+N}$, denoted by $\mathbf{r}_N$, describes the interaction of N solitons. A single soliton in the LIA case, $\omega(\lambda) = -2\lambda^2$, has been first found by Hasimoto [41].

Physical characteristics of the single soliton solution $\mathbf{r}_1$ can be computed in the standard way. First of all we determine the maximum of the wave envelope ($Q_1 = 0$) which performs a helical movement. The *group velocity* v_1^g of the soliton is computed as the velocity of the maximum along $\mathbf{e}_3$ axis. The rotation rate of the maximum is denoted by Ω_1. Then we determine positions of the individual wave peaks ($\alpha_1 = 0$). Their velocity (or the *phase velocity*) is almost constant sufficiently far from the envelope maximum. The phase velocity of the single soliton wave will be denoted by v_1^{ph}. Straightforward computations yield:

$$\begin{gathered} v_k^g = \frac{\mathrm{Im}\,\omega_k}{\mathrm{Im}\,\lambda_k} - \omega'(\lambda), \qquad v_k^{ph} = \frac{\omega(\lambda) - \mathrm{Re}\,\omega_k}{\lambda - \mathrm{Re}\,\lambda_k} - \omega'(\lambda), \\ \Omega_k = 2(\omega(\lambda) - \mathrm{Re}\,\omega_k) - 2(\lambda - \mathrm{Re}\,\lambda_k)\frac{\mathrm{Im}\,\omega_k}{\mathrm{Im}\,\lambda_k}. \end{gathered} \tag{44}$$

To describe interactions of solitons (the case $N > 1$) it is convenient to introduce parameters Δ_{jk}, δ_{jk} $(j \neq k)$:

$$e^{\Delta_{jk}+i\delta_{jk}} := \frac{(\lambda_j - \lambda_k)(\lambda - \overline{\lambda}_k)}{(\lambda_j - \overline{\lambda}_k)(\lambda - \lambda_k)}.$$

In the case $N = 2$ we denote $\Delta := \Delta_{12} \equiv \Delta_{21}$, $\delta_1 := \delta_{12}$, $\delta_2 = \delta_{21}$ (compare [16]). The parameters are not independent. Indeed, $d_1(e^{\Delta - i\delta_1} - 1) \equiv d_2(e^{\Delta+i\delta_2} - 1)$. Now we can write down a compact formula for $\mathbf{r}_{B+2}$:

$$\mathbf{r}_{B+2} = \mathbf{r}_B + \\ + \frac{1}{2D}\begin{pmatrix} d_1\left(e^{Q_2}\cos\alpha_1^+ + e^{-Q_2}\cos\alpha_1^-\right) + d_2\left(e^{Q_1}\cos\alpha_2^+ + e^{-Q_1}\cos\alpha_2^-\right) \\ d_1\left(e^{Q_2}\sin\alpha_1^+ + e^{-Q_2}\sin\alpha_1^-\right) + d_2\left(e^{Q_1}\sin\alpha_2^+ + e^{-Q_1}\sin\alpha_2^-\right) \\ d_+\sinh(Q_1+Q_2) + d_-\sinh(Q_1-Q_2) + d_0\sin(\alpha_1-\alpha_2) \end{pmatrix},$$

where $D = \cosh Q_1 \cosh Q_2 \cosh\Delta + \sinh Q_1 \sinh Q_2 \sinh\Delta + \sinh\Delta\cos(\alpha_1 - \alpha_2)$, $\alpha_i^{\pm} = \alpha_i \pm \delta_i$, $d_{\pm} = (d_1 \pm d_2)e^{\pm\Delta}$ and $d_0 = d_1\sin\delta_1 = -d_2\sin\delta_2$. The asymptotic behaviour of $\mathbf{r}_2$ can be calculated easily. If $v_1^g \neq v_2^g$ then we consider the limit $Q_2 \to \pm\infty$ (assuming $\mid Q_1 \mid \ll \mid Q_2 \mid$). Thus

$$\mathbf{r}_2 \overset{Q_2\to\pm\infty}{\longrightarrow} = \begin{pmatrix} 0 \\ 0 \\ \pm d_2 - x - \omega'(\lambda)t \end{pmatrix} + \frac{d_1}{\cosh(Q_1 \pm \Delta)}\begin{pmatrix} \cos(\alpha_1 \pm \delta_1) \\ \sin(\alpha_1 \pm \delta_1) \\ \sinh(Q_1 \pm \Delta) \end{pmatrix}$$

The shape and the velocity of the soliton do not change during the interaction. The only result of the interaction is the *phase shift*. In fact we have two phase shifts: the shift Δ_{ph} along $\mathbf{e}_3$ axis and the shift of the angular variable α_1. The first one is much more important and can be measured in experiments. It can be easily calculated:

$$\Delta_{ph} = 2d_2 + \frac{\Delta}{\mathrm{Im}\lambda_1}.$$

The multi-soliton solutions are parameterized by complex eigenvalues $\lambda_1, \ldots, \lambda_N$ (in the sequel we denote $\lambda_k = a_k + ib_k$). It is important to express the solutions by a set of $2N$ parameters which admit a physical interpretation, like d_k, v_k^g, v_k^{ph}, Ω_k, Δ_{ph} etc. (compare [60]). The number of the physical parameters is much greater than $2N$ and one may expect a lot of constraints which can be checked experimentally. The definition (39) suggests the following change of variables:

$$c_k + id_k := \frac{1}{\overline{\lambda}_k - \lambda} = \frac{a_k - \lambda + ib_k}{(a_k - \lambda)^2 + b_k^2}, \qquad \lambda_k - \lambda = \frac{1}{c_k - id_k} = \frac{c_k + id_k}{c_k^2 + d_k^2}. \tag{45}$$

Thus the parameters c_k, d_k are expressed by a_k, b_k and *vice versa*. In particular we have:

$$e^{\Delta_{jk}+i\delta_{jk}} := \frac{(d_j - d_k) + i(c_j - c_k)}{(d_j + d_k) + i(c_j - c_k)}.$$

The physical meaning of d_k is clear while the interpretation of c_k is, in general, a non-trivial problem. However, in the case of the Localized Induction Equation (the case $\omega(\lambda) = -2\alpha\lambda^2$) the interpretation is quite simple. Namely, $c_k = v_k^g/\Omega_k$ which means that c_k is the distance travelled by the wave envelope during one full circle of the wave maximum [16].

Let us consider the case of Localized Induction Equation with axial flow (37), $\omega(\lambda) = -2\alpha\lambda^2 - 4\beta\lambda^3$. We have $\omega(0) = \omega'(0) = 0$ and the formulae (44) assume the form

$$\begin{aligned} v_k^g &= 4\beta(b_k^2 - 3a_k^2) - \alpha a_k, \quad v_k^{ph} = 2\alpha a_k^{-1}(b_k^2 - a_k^2) + 4\beta(3b_k^2 - a_k^2), \\ \Omega_k &= -4(a_k^2 + b_k^2)(\alpha + 4a_k\beta). \end{aligned} \tag{46}$$

Using (46) and (45) we can easily check that

$$c_k = \frac{2\alpha}{v_k^{ph} - v_k^g} - \frac{16\beta}{\Omega_k}, \qquad a_k = \frac{2(v_k^g - v_k^{ph})}{\Omega_k}.$$

Finally, let us try to parameterize the N-soliton solution by the parameters d_k and v_k^g which are most convenient from the experimental point of view [60]. It is sufficient to express c_k in terms of d_k and v_k^g. By (46) and (45) we have

$$(c_k^2 + d_k^2)^2 v_k^g = 4\beta(d_k^2 - 3c_k^2) - \alpha c_k(c_k^2 + d_k^2).$$

Therefore, c_k is a root of the algebraic equation of the 4-th order with coefficients parameterized by d_k, v_k^g, α and β.

Compact formulas for N-soliton surfaces can be derived from (38) for an arbitrary N [14]:

$$\mathbf{r}_{B+N} = \mathbf{r}_B + \frac{i}{2}\left(\sum_{k=1}^{N} \frac{\mathrm{Im}\lambda_k}{|\lambda - \lambda_k|^2} - \sum_{j=1}^{N}\sum_{k=1}^{N} B_{kj}P_{kj}\right),$$

where $B := A^{-1}$ and A is $N \times N$ matrix with coefficients $A_{jk} := i(\xi_k\overline{\xi}_j + 1)(\lambda - \lambda_j)(\lambda - \overline{\lambda}_k)(\lambda_j - \overline{\lambda}_k)^{-1}$, P_{kj} are 2×2 matrices defined by

$$P_{kj} := \begin{pmatrix} \xi_k\overline{\xi}_j & \xi_k \\ \overline{\xi}_j & 1 \end{pmatrix},$$

and, finally, ξ_k are defined by (40). Note that the surface $\mathbf{r}_{B+N}$ is expressed solely in terms of the background wave function Φ_B and $2N$ complex parameters: λ_k and $\gamma_k := p_{k1}/p_{k2}$. Obviously, one can use c_k and d_k instead of λ_k.

We complete this section with few remarks on the general N-soliton case. To compute N-soliton addition to the surface $\mathbf{r} := \Phi^{-1}\Phi_{,\lambda}|_{\lambda=\lambda_0}$ let assume the Darboux matrix in a general form

$$D = \sum_{k=0}^{N} C_k(\lambda - \lambda_0)^k.$$

The matrices C_k are computed from the following linear system:

$$\sum_{k=0}^{N} C_k(\lambda_\nu - \lambda_0)^k \Phi(\lambda_\nu) p_\nu = 0 \quad (\nu = 1, \ldots, Nn),$$

where $\lambda_\nu \in \mathbb{C}$ and $p_\nu \in \mathbb{C}^n$ are constant. Of course, one should take care of reductions which can result in some constraints on λ_ν, p_ν. The formula (21) assumes the form: $\tilde{\mathbf{r}} = \mathbf{r} + \Phi^{-1} C_0^{-1} C_1 \Phi$.

Spectral problems from geometry

It would be very important to be able to discern integrable classes of surfaces. Here we present shortly how to approach this problem in a natural way using Lie symmetries. Gauss–Weingarten equations, especially in the form (18), have very similar form to the Zakharov–Shabat linear problem (1). The only difference is an absence of a spectral parameter. It is well known that in many cases the spectral parameter is a group parameter, i.e. there exists a symmetry of GMC equations (usually a simple one, like a scaling or Lorentz or Galilean boost) which changes GW equations [57, 69]. The transformed GW equations contain explicitly the group parameter. We have developed a systematic approach to study the problem [15, 17, 24, 54]. It consists in computing two algebras of Lie symmetries: the algebra A of symmetries of GMC equations and the algebra A' of symmetries of GW equations. Always $A' \subset A$. Reasonable candidates for the spectral parameter are provided by vector fields v such that $v \in A$ and $v \notin A'$. For more details and references see [15, 21, 24], compare also [47].

Let us consider the following problem. We start from a given class of surfaces (i.e., GW equations are given). Suppose that $A' \neq A$. Thus using an appropriate symmetry of GMC equations we can insert a parameter into GW equations (18) to obtain a linear problem of the form (1). Then we apply the Sym–Tafel formula (20).

Is the obtained class of surfaces identical with the class we started from? We have no general answer yet. In the following examples the answer is positive.

- **Pseudospherical surfaces.** The fundamental forms (26) with $\zeta = 1$ and the symmetry $\tilde{u} = \zeta^{-1} u$, $\tilde{v} = \zeta v$ of the sine-Gordon equation yield the linear problem (25).
- **Constant mean curvature surfaces.** The fundamental forms (29) with $\kappa = 0$, the symmetry $\tilde{u} = u \cos\kappa - v \sin\kappa$ and $\tilde{v} = v \cos\kappa - u \sin\kappa$ of (28), and the gauge transformation $\tilde{\Phi} = \exp(\kappa\, \mathbf{e}_2)\Phi$ yield the linear problem (27).
- **Spherical surfaces.** The fundamental forms (31) with $\kappa = 0$ and the symmetry $\tilde{u} = u \cos\kappa - v \sin\kappa$ and $\tilde{v} = v \cos\kappa - u \sin\kappa$ of the elliptic sinh-Gordon equation (28) yield exactly the linear problem (30).
- **Bianchi surfaces.** The fundamental forms (34) and the symmetry $\tilde{a} = a/\zeta$, $\tilde{b} = b\zeta$, $\tilde{f} = f/(1 - 2\lambda f)$ and $\tilde{g} = g/(1 + 2\lambda g)$, where $\zeta = (1 - 2\lambda f)^{1/2}(1 + 2\lambda g)^{-1/2} = (1 - 2\lambda\tilde{g})^{1/2}(1 + 2\lambda\tilde{f})^{-1/2}$, yield the linear problem (32).

In the second case it was necessary to perform a gauge transformation dependent on the spectral parameter (otherwise, starting from $H = \text{const}$ surfaces, one obtains spherical surfaces) [31].

5 Integrable geometries and Clifford algebras

The examples presented above are associated with the $\mathbf{SU}(2)$ group. Recently, the soliton surfaces approach has been applied in more complicated cases (although the word "surfaces" is slightly misleading: one can consider submanifolds of higher dimensions). It is convenient to use Clifford algebras $\mathcal{C}(p,q)$ generated by elements $\mathbf{e}_1, \ldots, \mathbf{e}_m$ ($m = p+q$) satisfying

$$\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu = 2\eta_{\mu\nu}, \tag{47}$$

where $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$, $\eta_{\mu\mu} = 1$ for $\mu = 1, \ldots, p$ and $\eta_{\mu\mu} = -1$ for $\mu = p+1, \ldots, m$. In the sequel the matrices U_j of the linear problem are linear combinations of $\mathbf{e}_\mu \mathbf{e}_\nu$. Then the function Ψ assumes values in the group $\mathrm{Spin}(p,q)$ which is the double covering of $\mathbf{SO}(p,q)$. Note that $\mathbf{SU}(2)$ can be identified with $\mathrm{Spin}(3)$.

Isothermic surfaces (isothermic immersions in $\mathbf{E}^3$) can be defined as surfaces admitting *infinitesimal* isometries preserving the mean curvature. Their curvature lines parameterized in a proper way form a conformal coordinate system. In other words, there exist local coordinates x^1, x^2 in which fundamental forms read as follows [4]:

$$\begin{aligned} I &= e^{2\vartheta}((dx^1)^2 + (dx^2)^2), \\ II &= e^{2\vartheta}(k_2(dx^1)^2 + k_1(dx^2)^2), \end{aligned} \tag{48}$$

where k_1, k_2 and ϑ depend on x^1, x^2. Recently we found the following linear problem [20, 25] which enable one to study isothermic surfaces using powerful tools of the theory of solitons:

$$\begin{aligned} \Phi_{,1} &= \tfrac{1}{2}\mathbf{e}_1 \left(-\vartheta_{,2}\,\mathbf{e}_2 - k_2 e^{\vartheta}\mathbf{e}_3 + \lambda \sinh\vartheta\ \mathbf{e}_4 + \lambda\cosh\vartheta\ \mathbf{e}_5\right)\Phi, \\ \Phi_{,2} &= \tfrac{1}{2}\mathbf{e}_2 \left(-\vartheta_{,1}\,\mathbf{e}_1 - k_1 e^{\vartheta}\mathbf{e}_3 + \lambda \cosh\vartheta\mathbf{e}_4 + \lambda\sinh\vartheta\ \mathbf{e}_5\right)\Phi, \end{aligned} \tag{49}$$

where $\mathbf{e}_k$ satisfy (47) with $(\eta_{\mu\nu}) = \mathrm{diag}(1,1,1,1,-1)$ and $i\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5 = 1$. The Sym–Tafel formula needs some modification to be applied to isothermic surfaces. Namely, the formula (20) for $\lambda = 0$ defines a surface in 6-dim. space. Projecting the surface onto appropriate orthogonal 3-dim. subspaces we obtain a pair of dual isothermic surfaces. Indeed, one can prove (see [20]) that

$$\mathbf{r}_\pm := \frac{1}{2}(1 \mp \mathbf{e}_4\mathbf{e}_5)\,\Phi^{-1}\Phi_{,\lambda}\,|_{\lambda=0} \tag{50}$$

are isothermic surfaces immersed in Euclidean spaces spanned by $\mathbf{e}_k(\mathbf{e}_4 \pm \mathbf{e}_5)$ ($k = 1,2,3$), respectively. The fundamental forms of the surface $\mathbf{r}_+$ are given exactly by (48) while the "dual surface" $\mathbb{R}_-$ is the so-called Christoffel transform of $\mathbf{r}_+$ [4, 9]. The Darboux matrix for the linear problem (49) reads

$$D = \frac{\mathbf{e}_2}{\sqrt{\lambda^2 + \kappa_1^2}}\left(\kappa_1(p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3) + \lambda(\cosh\chi\ \mathbf{e}_4 + \sinh\chi\ \mathbf{e}_5)\right),$$

where κ_1 is a real parameter and χ, p_1, p_2, p_3 are real functions which can be explicitly expressed by Φ evaluated at $\lambda = -i\kappa_1$ [18, 20]. The corresponding Darboux–Bianchi transformation for the surfaces (50),

$$\tilde{\mathbf{r}}_\pm = \mathbf{r}_\pm + \frac{2}{\kappa_1} e^{\pm\chi} \left(\pm p_1 e^{\mp\vartheta} \mathbf{r}_{\pm,1} + p_2 e^{\mp\vartheta} \mathbf{r}_{\pm,2} - p_3 \mathbf{n}_\pm \right),$$

is identical with the classical Darboux–Bianchi transformation for isothermic surfaces (compare [4]). The connection of the old branch of the classical differential geometry with the modern soliton theory *via* the linear problem (49) has already started a series of new interesting developments in this field [9, 12, 25].

Space forms of dimension n in $\mathbb{R}^{2n-1}$. The Lobachevsky plane can not be immersed globally in $\mathbb{R}^3$ and only local immersions (pseudospherical surfaces in $\mathbb{R}^3$) are possible. Analogical situation has place for immersions of n-space forms (spaces of constant curvature) for $n > 2$. There are theorems on non-existence of global immersions (see [33] and references cited therein) but local immersions are known (at least implicitly) [3, 64]. If the curvature is constant and negative then there exists a coordinate system such that all fundamental forms are diagonal. Moreover, the immersions turn out to be parameterized by an orthogonal $n \times n$ matrix function (a_{ij}) (see, for example, [1, 3, 81]). The fundamental forms read:

$$I = \sum_{j=1}^{n} a_{1j}^2 (dx^j)^2, \qquad II^m = \sum_{j=1}^{n} a_{1j} a_{mj} (dx^j)^2. \tag{51}$$

where $m = 2, \ldots, n$ and $\sum_{i=1}^{n} a_{ji} a_{jk} = \delta_{jk}$. The coefficients a_{ij} have to satisfy the Gauss–Mainardi–Codazzi–Ricci equations. The equations are integrable: the Bäcklund transformation [81], the inverse scattering method [1] and the loop group approach [33] have been applied. The following linear system with the spectral parameter λ is a modification of the $\mathbf{so}(n,n)$ spectral problem presented by Ablowitz, Beals and Tenenblat ([1], see also [33, 38]):

$$\Psi_{,j} = \frac{1}{2} \left(\frac{\lambda^2 - 1}{2\lambda} a_{1j} \mathbf{e}_1 + \frac{\lambda^2 + 1}{2\lambda} \sum_{k=2}^{n} a_{kj} \mathbf{e}_k + \sum_{k=1}^{n} \gamma_{kj} \mathbf{e}_{n+k} \right) \mathbf{e}_{n+j} \Psi, \tag{52}$$

where (a_{ij}) and (γ_{ij}) are $n \times n$ matrices, $\gamma_{kk} = 0$ and $\mathbf{e}_\mu$ satisfy (47) with $\eta_{\mu\nu} = \delta_{\mu\nu}$. The compatibility conditions for the linear system (52) yield that (a_{ij}) is orthogonal and

$$a_{lj,k} = \gamma_{kj} a_{lk}, \quad \gamma_{ij,k} = \gamma_{ik}\gamma_{kj}, \quad \gamma_{kj,k} + \gamma_{jk,j} + \textstyle\sum_{i=1}^{n} \gamma_{ij}\gamma_{ik} = a_{1j} a_{1k}, \tag{53}$$

where indices i, j, k are distinct and all indices run from 1 to n. Let us define a map F by the Sym formula

$$F := \Psi^{-1} \Psi_{,\lambda} |_{\lambda=1}. \tag{54}$$

It defines (at least locally) an n-dimensional manifold (with possible singularities) immersed, obviously, in the space of dimension $n(2n-1)$ isomorphic with the Lie algebra

$\mathbf{so}(2n)$. One can show that in fact F is confined to some Euclidean space of dimension $2n-1$. Indeed, the unit tangent vectors are given by $E_j = \frac{1}{2}\Psi^{-1}\mathbf{e}_1\mathbf{e}_{n+j}\Psi$ while the normal space is spanned by $N_j := \Psi^{-1}\mathbf{e}_1\mathbf{e}_j\Psi$ $(j = 2,\ldots,n)$. Differentiating N_j we obtain elements of the tangent space, $N_k,_j = a_{kj}E_k$. The corresponding fundamental forms are given exactly by (51). Thus we conclude that the formula (54) is an explicit expression for the n-dimensional manifolds of constant sectional curvature locally immersed in a Euclidean space of dimension $(2n-1)$ [22].

Lamé equations. We proceed to the general description of orthogonal coordinate systems in Euclidean spaces. The metric

$$I = H_1^2(dx^1)^2 + \ldots + H_n^2(dx^n)^2, \tag{55}$$

is induced by an orthogonal system in $\mathbf{E}^n$ provided that H_k satisfy the Lamé equations [32]

$$\begin{aligned} 0 &= \left(\frac{H_j,_k}{H_k}\right),_k + \left(\frac{H_k,_j}{H_j}\right),_j + \sum_{i=1}^n \frac{H_j,_i H_k,_i}{H_i^2}, \\ H_j,_{ik} &= \frac{H_j,_k H_k,_i}{H_k} + \frac{H_i,_k H_j,_i}{H_i}. \end{aligned} \tag{56}$$

Consider the following linear problem:

$$\Psi,_j = \mathbf{e}_j(\lambda\mathbf{a}_j + \mathbf{b}_j)\Psi, \tag{57}$$

where $\mathbf{e}_1, \mathbf{e}_2\ldots,\mathbf{e}_{2n}$ generate the Clifford algebra $\mathcal{C}(2n)$, $\mathbf{a}_j := \frac{1}{2}\sum_{i=1}^n \alpha_{ji}\mathbf{e}_{n+i}$, $\mathbf{b}_j := \frac{1}{2}\sum_{i=1}^n \beta_{ji}\mathbf{e}_i$ and $\beta_{jj} = 0$. The compatibility conditions for (57) imply that the matrix (α_{ij}) is orthogonal, the coefficients β_{ij} are given by $\beta_{jk} = -\alpha_{ji},_k/\alpha_{ki}$ and, finally, $H_k := \alpha_{kj}$ satisfy (for any fixed j) the Lamé equations (56).

Let us define $F := \Psi^{-1}\Psi,_\lambda|_{\lambda=0}$. The tangent vectors read $F,_j = \Psi^{-1}U_j,_\lambda\Psi|_{\lambda=0} = \Psi_0^{-1}\mathbf{e}_j\mathbf{a}_j\Psi_0$, where $\Psi_0 := \Psi(x^1,\ldots,x^n;0)$ is contained in the subalgebra generated by $\mathbf{e}_1,\ldots,\mathbf{e}_n$. It is convenient to consider projections Π^k defined by

$$X = \sum_{i,j=1}^n X_{ji}\mathbf{e}_j\mathbf{e}_{n+i} \longrightarrow \Pi^k X \equiv X^{(k)} := \sum_{j=1}^n X_{jk}\mathbf{e}_j, \tag{58}$$

and to apply them to F. As a result we obtain $F^{(1)}, F^{(2)},\ldots,F^{(n)}$. Differentiating $F,_j^{(k)} = \alpha_{jk}\Psi_0^{-1}\mathbf{e}_j\Psi_0$ we derive that $\langle F,_i^{(k)}|\, F,_j^{(k)}\rangle = \alpha_{ik}\alpha_{jk}\delta_{ij}$. Therefore, each map $F^{(k)}$ has the diagonal metric tensor (55). Computing second derivatives we can check that $F^{(k)}$ defines an immersion $\mathbf{E}^n \to \mathbf{E}^n$, i.e., coordinates in $\mathbf{E}^n$ [23]. It would be very interesting to interpret the results of [45] in our formalism. By the way, the linear problem (49) can also be derived from (57) as a consequence of the compatibility conditions. We just have to assume $\mathbf{a}_j$ as a linear combination of $\mathbf{e}_4$, $\mathbf{e}_5$ and $\mathbf{b}_j$ as a combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

6 Conclusions

The application of soliton techniques to the differential geometry revealed deep relations between these two areas, see for example [7, 8, 70, 71, 79, 82]. In fact, many ideas of the soliton theory can be traced back to 19th century. I would like to point out that the classical differential geometry studied a lot of special immersions and interesting transformations between them [5, 32, 84]. It is intriguing that the Sym–Tafel formula usually reconstructs immersions corresponding to a given GMC system. Then, using standard methods of constructing soliton solutions, one is able to derive the classical transformations of surfaces.

The other important application of the Sym–Tafel formula is the construction of discrete surfaces. There is still not so clear how to define discrete analogues of integrable classes of surfaces. Surprisingly effective way is provided by applying the formula (20) to the corresponding discretization of the linear problem (compare [9, 10, 22]).

The recent results of Doliwa and Santini suggest even more general approach. They consider immersions in a sphere S^n and the radius of the sphere is related to spectral parameter. In the limiting case (infinite radius) $S^n \to \mathbb{R}^n$ and one can derive the Sym-Tafel formula. The results concerning integrable evolutions of curves are very promising [29, 30].

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