Minimal surfaces, holomorphic curves and Toda systems

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Abstract: The Weierstrass representation of minimal surfaces in $\mathbb{E}^3$ gives rise to the general solution of the Liouville equation, which is the simplest example of the open Toda system. The origin of these classical results has its roots in parametrization of isotropic holomorphic curves. This approach is generalized to other open Toda systems corresponding to the four classical sequences of the simple Lie algebras. It is shown that the geometric counterpart of the open Toda systems is provided by certain holomorphic curves. Parametrization of these holomorphic curves yields general solutions of the open Toda systems. As a by-product one obtains minimal surfaces in real, complex and quaternionic projective spaces or complex quadrics.

1 Introduction

The theory of minimal surfaces goes back to Lagrange, and is one of the most celebrated subjects of mathematics [1]. An important tool in studying of the minimal surfaces is provided by their Weierstrass representation in terms of — roughly speaking — two meromorphic functions. Since the Gauss equation of the minimal surfaces is closely connected to the so-called Liouville equation, the Weierstrass representation allows to write down the general solution of this equation.

The Liouville equation is the simplest open Toda system. The Toda systems are integrable partial differential equations labeled by the Dynkin diagrams of simple Lie algebras [2, 3]. In the theory of integrable systems one considers also the so called periodic Toda systems, which correspond to affine Lie algebras. Both the open and periodic Toda systems are special reductions of the infinite Toda system.

The aim of this article is to present a generalization of the connection between the Weierstrass representation of the minimal surfaces and the Liouville equation to the case of the other open Toda systems, (see also [4] and references therein). In Section 2 we summarize the relevant material on minimal surfaces in $\mathbb{E}^3$. Section 3 is devoted to the exposition of the open Toda systems and establishes a relation between the $\mathfrak{su}(n+1)$ Toda system and holomorphic curves in the complex projective space $\mathbb{CP}^n$. In Section 4 we will be concerned with maximally isotropic holomorphic curves which correspond to the $\mathfrak{so}(n)$ Toda systems. The $\mathfrak{sp}(n)$ Toda systems (discussed in Section 5) are connected to the so-called super-horizontal holomorphic curves. Finally, in the last section, we touch another related topics, present some open problems and further generalizations.

2 Minimal surfaces in $\mathbb{E}^3$, their Weierstrass representation and the Liouville equation

Let $\phi : M^2 \to \mathbb{E}^3$ be a minimal immersion. The conformal structure, induced on M^2 from the ambient Euclidean space, allows to consider M^2 as a Riemann surface (one dimensional complex manifold). In the corresponding local isothermal coordinate system $z = x + iy$, characterized by

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y}\cdot\frac{\partial\phi}{\partial y}, \qquad \frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y} = 0, \tag{1}$$

the minimality condition can be expressed as

$$\frac{\partial^2\phi}{\partial z\partial\bar{z}} = 0. \tag{2}$$

In terms of the function $f = \frac{\partial\phi}{\partial z}$, the conditions (1) and (2) take the form

$$f\cdot f = 0, \quad \frac{\partial f}{\partial\bar{z}} = 0, \tag{3}$$

i.e., *f is an isotropic holomorphic curve.* The above conditions give rise to parametrization of f in terms of two meromorphic functions μ and ψ as follows

$$f = \begin{pmatrix} \mu(1-\psi^2)/2 \\ i\mu(1+\psi^2)/2 \\ \mu\psi \end{pmatrix}, \tag{4}$$

which, combined with the formula

$$\phi = 2\mathrm{Re}\int_{z_0}^{z} f\, d\zeta \; + \phi_0, \tag{5}$$

give rise to the celebrated Weierstrass representation of the minimal surfaces.

Let us introduce real function θ by

$$\mathrm{e}^{2\theta} = \frac{\partial\phi}{\partial z}\cdot\frac{\partial\phi}{\partial\bar{z}}; \tag{6}$$

when $\mathbf{N}$ is the unit normal vector to the immersion, we define function g by

$$g = \frac{\partial^2\phi}{\partial z^2}\cdot\mathbf{N}. \tag{7}$$

Then the Gauss equation and the Mainardi–Codazzi equations of the minimal immersion read, correspondingly,

$$2\frac{\partial^2\theta}{\partial z\partial\bar{z}} = |g|\mathrm{e}^{-2\theta}, \tag{8}$$

$$\frac{\partial g}{\partial\bar{z}} = 0. \tag{9}$$

In terms of the functions μ, ψ the general solution of equations (8) and (9) is of the form

$$\begin{aligned} \mathrm{e}^{2\theta} &= \tfrac{1}{2}|\mu|^2(1+|\psi|^2)^2, &(10)\\ g &= -\mu\psi', &(11)\end{aligned}$$

here by $'$ we denote differentiation with respect to the complex variable z.

In particular, when $g=1$ we obtain the Liouville equation

$$2\frac{\partial^2\theta}{\partial z\partial\bar{z}} = \mathrm{e}^{-2\theta}, \qquad (12)$$

together with its general solution

$$\mathrm{e}^{2\theta} = \frac{(1+|\psi|^2)^2}{2|\psi'|}. \qquad (13)$$

3 Toda systems, holomorphic curves and Plücker formulae

The Liouville equation is the simplest open Toda system corresponding to the (simplest) simple Lie algebra $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$. With any simple Lie algebra one can associate the open Toda system as follows. Let $\mathfrak{g}$ be a real simple compact Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, $\Delta \subset \mathfrak{h}^*$ the root system and

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha \qquad (14)$$

the corresponding root space decomposition (see e.g. [5]). We denote by $\pi \subset \Delta$ a system of simple roots $\pi = \{\alpha_k\}_{k=1}^m$ ($m = \dim\mathfrak{h}$) and by $\{h_k\}_{k=1}^m$ the basis of $\mathfrak{h}$ dual to $\{\alpha_k\}_{k=1}^m$ with respect to the Killing form of $\mathfrak{g}$. The open Toda system corresponding to $\mathfrak{g}$ [2] is the following equation:

$$2\frac{\partial^2\theta}{\partial z\partial\bar{z}} = \sum_{k=1}^m h_k \mathrm{e}^{-2\alpha_k(\theta)}, \qquad (15)$$

where the unknown function $\theta \in \sum_{k=1}^m \mathbb{R}h_k$ takes values in the real part of the Cartan algebra. General solutions to the open Toda systems were found in [3] using the theory of representation of the Lie algebras.

As an example, let us consider the $A_n \simeq \mathfrak{su}(n+1)$ Lie algebra, whose simple roots are of the form

$$\pi = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{e}_n - \mathbf{e}_{n+1}\} \cong \{h_1, h_2, \dots, h_n\}; \qquad (16)$$

the corresponding Toda system reads

$$
\begin{aligned}
2\frac{\partial^2\theta^1}{\partial z\partial\bar{z}} &= \mathrm{e}^{2(\theta^2-\theta^1)} \\
2\frac{\partial^2\theta^k}{\partial z\partial\bar{z}} &= \mathrm{e}^{2(\theta^{k+1}-\theta^k)} - \mathrm{e}^{2(\theta^k-\theta^{k-1})}, \qquad k = 2,\ldots,n \\
2\frac{\partial^2\theta^{n+1}}{\partial z\partial\bar{z}} &= -\mathrm{e}^{2(\theta^{n+1}-\theta^n)},
\end{aligned}
\tag{17}
$$

with the constraint

$$
\sum_{k=1}^{n+1}\theta^k = 0. \tag{18}
$$

Given holomorphic curve, i.e. a holomorphic map $\phi : \mathbb{R} \to \mathbb{CP}^n$ from the Riemann surface $\mathbb{R}$ into the complex projective space [6], consider its local representation in terms of a meromorphic vector function $f : \mathbb{R} \supset \mathcal{O} \to \mathbb{C}^{n+1}$. By $\phi_k : \mathbb{R} \to \mathbb{G}(k, n+1)$, $k = 1,\ldots,n$, we denote its associated curves ($\phi_1 = \phi$) represented locally by k-planes with matrix homogeneous coordinates

$$
M_k = (f, f', \ldots, f^{(k-1)}) \in \mathbb{C}^{n+1}_k. \tag{19}
$$

Denoting by

$$
\Lambda_k^2 = \det M_k^+ M_k, \tag{20}
$$

and using the Laplace expansion of the corresponding determinants we obtain the so-called Plücker formulae

$$
\frac{\partial^2}{\partial z\partial\bar{z}}\log\Lambda_k^2 = \frac{\Lambda_{k-1}^2\Lambda_{k+1}^2}{\Lambda_k^4}, \qquad k = 1,\ldots,n \tag{21}
$$

where, by definition, $\Lambda_0^2 = 1$.

Rewriting the Plücker formulae (21) using functions

$$
\theta^k = \log\frac{\Lambda_k}{\Lambda_{k-1}}, \tag{22}
$$

we obtain the $\mathfrak{su}(n+1)$ Toda system (17); the constraint (18) means that the local holomorphic lift f of the curve ϕ is subjected to the normalization

$$
\det M_{n+1} = \det(f, f', \ldots, f^{(n)}) = 1. \tag{23}
$$

Using the Gram–Schmidt orthonormalization procedure one can obtain from the natural basis $f, f', \ldots, f^{(n)}$ along the nondegenerate (normalized) curve the unitary basis

$\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_{n+1}$ along the curve; $\langle \mathbf{E}_i | \mathbf{E}_j \rangle = \delta_{ij}$. This new basis (called the Frenet basis) satisfies the set of linear equations

$$\frac{\partial}{\partial z}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \vdots \\ \mathbf{E}_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta^1}{\partial z} & e^{\theta^2-\theta^1} & 0 & \cdots & 0 \\ 0 & \frac{\partial \theta^2}{\partial z} & e^{\theta^3-\theta^2} & & \vdots \\ & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \frac{\partial \theta^n}{\partial z} & e^{\theta^{n+1}-\theta^n} \\ 0 & \cdots & & 0 & \frac{\partial \theta^{n+1}}{\partial z} \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \vdots \\ \mathbf{E}_{n+1} \end{pmatrix} \tag{24}$$

$$\frac{\partial}{\partial \bar{z}}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \vdots \\ \mathbf{E}_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta^1}{\partial \bar{z}} & e^{\theta^2-\theta^1} & 0 & \cdots & 0 \\ 0 & \frac{\partial \theta^2}{\partial \bar{z}} & e^{\theta^3-\theta^2} & & \vdots \\ & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \frac{\partial \theta^n}{\partial \bar{z}} & e^{\theta^{n+1}-\theta^n} \\ 0 & \cdots & & 0 & \frac{\partial \theta^{n+1}}{\partial \bar{z}} \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \vdots \\ \mathbf{E}_{n+1} \end{pmatrix},$$

and (17) are compatibility conditions of the above linear system.

One can show [8, 9] that all the vectors $E_k \in \mathbb{C}^{n+1}$ represent minimal surfaces in the complex projective space $\mathbb{CP}^n$.

4 Holomorphic curves in orthogonal geometries

The Euclidean geometry can be embedded into the projective geometry, in the spirit of Cayley and Klein, by fixing a nondegenerate quadric and considering only those linear transformations which leave the quadric invariant. Following Cartan [12] we know that the basic (spinor) representation of the orthogonal transformations makes use of the isotropic (i.e. contained in the quadric) linear spaces of the maximal possible dimension.

Looking for the geometric reduction of the $\mathfrak{su}(n+1)$ Toda systems to the $\mathfrak{so}(n+1)$ systems we consider in the complex projective space $\mathbb{CP}^n$ a nondegenerate quadric $\mathcal{Q}_{n-1}$. Holomorphic curves $\phi : \mathbb{R} \to \mathcal{Q}_{n-1}$ contained in the quadric are usually called isotropic curves. A nondegenerate holomorphic curve ϕ is called k-isotropic, when its k-th associated curve ϕ_k is made out of isotropic k-planes (we write $\phi_k \subset \mathcal{Q}_{n-1}$); the maximal dimension of isotropic subspaces is equal to the integer part of $n/2$. This fact forces us to consider separately even and odd n.

We shall concentrate on the algebra $D_m \simeq \mathfrak{so}(2m)$, which corresponds to the $(2m-2)$-dimensional quadric $\mathcal{Q}_{2m-2} \subset \mathbb{CP}^{2m-1}$. A *maximally isotropic* curve $\phi : \mathbb{R} \to \mathcal{Q}_{2m-2}$ is characterized by the condition $\phi_{m-2} \subset \mathcal{Q}_{2m-2}$. In terms of the local holomorphic lift $f : \mathcal{O} \to \mathbb{C}^{2m}$ this condition is equivalent to

$$f^{(k)} \cdot f^{(k)} = 0, \qquad k = 0, \dots, m-2, \tag{25}$$

where by $\cdot$ we denote the bilinear symmetric form defined by the quadric. Let us take the first $m-1$ vectors $\mathbf{E}_1, \ldots, \mathbf{E}_{m-1}$ of the Frenet frame along the curve. Since these vectors are isotropic, they are orthogonal to their conjugate $\bar{\mathbf{E}}_1, \ldots, \bar{\mathbf{E}}_{m-1}$. Consider the isotropic vector $\mathbf{F}$ of unit length (in the hermitian sense) and orthogonal to the space spanned by $\mathbf{E}_1, \ldots, \mathbf{E}_{m-1}, \bar{\mathbf{E}}_1, \ldots, \bar{\mathbf{E}}_{m-1}$; such a vector is given up to a phase factor. The next vector $\mathbf{E}_m$ of the Frenet frame can be therefore decomposed as follows

$$\mathbf{E}_m = a\mathbf{F} + b\bar{\mathbf{F}}. \tag{26}$$

We fix the phase factor in the definition of $\mathbf{F}$ by the condition $a \in \mathbb{R}_+$, and normalize the lift f by

$$f^{(m-1)} \cdot f^{(m-1)} = 2, \tag{27}$$

which implies $b = a^{-1}$.

Let us define function $\tilde{\theta}^m$ by

$$\mathrm{e}^{2\theta^m} = 2\cosh 2\tilde{\theta}^m; \tag{28}$$

then

$$\mathrm{e}^{\theta^m}\mathbf{E}_m = \mathrm{e}^{\tilde{\theta}^m}\mathbf{F} + \mathrm{e}^{-\tilde{\theta}^m}\bar{\mathbf{F}}, \tag{29}$$

The new vector $\mathbf{F}$ is subject to the following equations

$$\begin{aligned}
\frac{\partial \mathbf{F}}{\partial z} &= \frac{\partial \tilde{\theta}^m}{\partial z}\mathbf{F} - \mathrm{e}^{-\tilde{\theta}^m - \theta^{m-1}}\bar{\mathbf{E}}_{m-1}, \\
\frac{\partial \mathbf{F}}{\partial \bar{z}} &= -\frac{\partial \tilde{\theta}^m}{\partial \bar{z}}\mathbf{F} - \mathrm{e}^{\tilde{\theta}^m - \theta^{m-1}}\mathbf{E}_{m-1}.
\end{aligned} \tag{30}$$

Therefore, for maximally isotropic curves $\phi : \mathbb{R} \to \mathcal{Q}_{2m-2}$ we can reduce the linear system (24) to a new system involving the isotropic vectors $\mathbf{E}_1, ..., \mathbf{E}_{m-1}, \mathbf{F}$ and their conjugate, and the functions $\theta^1, ..., \theta^{m-1}, \tilde{\theta}^m$ only. Taking the compatibility condition of the new reduced linear system we obtain the $\mathfrak{so}(2m)$ Toda system

$$\begin{aligned}
2\frac{\partial^2 \theta^1}{\partial z \partial \bar{z}} &= \mathrm{e}^{2(\theta^2 - \theta^1)}, \\
2\frac{\partial^2 \theta^k}{\partial z \partial \bar{z}} &= \mathrm{e}^{2(\theta^{k+1} - \theta^k)} - \mathrm{e}^{2(\theta^k - \theta^{k-1})}, \qquad k = 2, \ldots, m-2 \\
2\frac{\partial^2 \theta^{m-1}}{\partial z \partial \bar{z}} &= \mathrm{e}^{2(\tilde{\theta}^m - \theta^{m-1})} + \mathrm{e}^{-2(\tilde{\theta}^m + \theta^{m-1})} - \mathrm{e}^{2(\theta^{m-1} - \theta^{m-2})}, \\
2\frac{\partial^2 \tilde{\theta}^m}{\partial z \partial \bar{z}} &= -\mathrm{e}^{2(\tilde{\theta}^m - \theta^{m-1})} + \mathrm{e}^{-2(\tilde{\theta}^m + \theta^{m-1})}.
\end{aligned} \tag{31}$$

To obtain the general solution of the $\mathfrak{so}(2m)$ Toda system we have to to take into account the isotropy conditions (25). It is convenient to use the unitary basis of $\mathbb{C}^{2m}$

consisting of isotropic vectors, i.e. in the corresponding coordinates the equation of the quadric $\mathcal{Q}_{2m-2}$ is of the form

$$\mathcal{Q}_{2m-2} = \{[Z] \in \mathbb{CP}^{2m-1}; Z \cdot Z = \sum_{k=1}^{m} Z^{2k-1} Z^{2k} = 0\}. \tag{32}$$

When $f_1 : \mathcal{O} \to \mathbb{C}^{2m}$ is a local holomorphic lift of the maximally isotropic curve then the isotropy condition

$$f_1 \cdot f_1 = 0 \tag{33}$$

allows to reduce the number of functions necessary to describe the curve

$$f_1 = u_1 \, (1, f_2, -f_2 \cdot f_2 \,) \,, \tag{34}$$

where $f_2 : \mathcal{O} \to \mathbb{C}^{2m-2}$ is a local meromorphic vector function, and $\cdot$ is here the 'old' scalar product but now restricted to $\mathbb{C}^{2m-2}$. Moreover one can show that

$$f_1' \cdot f_1' = u_1^2 f_2' \cdot f_2', \tag{35}$$

what, when the next isotropy condition holds (see (25)), allows to parameterize f_2' in the similar form as above

$$f_2' = u_2 \, (1, f_3, -f_3 \cdot f_3 \,) \,. \tag{36}$$

This procedure can be applied on each step, and allows to reconstruct the maximally isotropic curve and the corresponding solution of the Toda system, once a system of m meromorphic functions $u_1, \ldots, u_m$ is given.

It should be mentioned here that all the vectors $\mathbf{E}_1, \ldots, \mathbf{E}_{m-1}, \mathbf{F}$ of the Frenet frame considered in this section represent minimal surfaces in the quadric $\mathcal{Q}_{2m-2}$ [11] .

The $\mathfrak{so}(2m-1)$ Toda system can be viewed as a reduction of the $\mathfrak{so}(2m)$ Toda system; geometrically this reduction is induced by embedding of the quadric $\mathcal{Q}_{2m-3}$ into $\mathcal{Q}_{2m-2}$. This implies that the vector $\mathbf{E}_m$ of the Frenet frame represents a minimal surface in the real projective space $\mathbb{RP}^{2m-2}$, and $\theta^m = 0$.

5 Holomorphic curves in symplectic geometries

Geometry of the $C_m \simeq \mathfrak{sp}(2m)$ Toda systems can be described in terms of the celebrated Penrose fibration $\mathbb{CP}^{2m-1} \xrightarrow{\mathbb{CP}^1} \mathbb{HP}^{m-1}$ (here we consider the right quaternionic projective space) obtained from the standard identification of the quaternionic space $\mathbb{H}$ with $\mathbb{C}^2$

$$\mathbb{H} \ni q = a + jb \mapsto (a, b) \in \mathbb{C}^2. \tag{37}$$

In terms of the homogeneous coordinates $(Z_1, Z_2) \in \mathbb{C}^{2m} = \mathbb{C}^m \times \mathbb{C}^m$ the corresponding projection $\pi : \mathbb{CP}^{2m-1} \to \mathbb{HP}^{m-1}$ is given as follows

$$\pi[(Z_1, Z_2)]_{\mathbb{C}} = [Z_1 + jZ_2]_{\mathbb{H}}, \qquad Z_1, Z_2 \in \mathbb{C}^m. \tag{38}$$

The fiber over $\pi[(Z_1, Z_2)]_{\mathbb{C}}$ is the projective line in $\mathbb{CP}^{2m-1}$ passing through the points $[(Z_1, Z_2)]_{\mathbb{C}}$ and $[(-\bar{Z}_2, \bar{Z}_1)]_{\mathbb{C}}$; note that $(Z_1 + jZ_2)j = -\bar{Z}_2 + j\bar{Z}_1$. The complement to the fiber orthogonal with respect to the Fubini–Study metric in $\mathbb{CP}^{2m-1}$ defines the horizontal distribution $\mathcal{H}$. The vector $(V_1, V_2) \in T_{[(Z_1,Z_2)]}\mathbb{CP}^{2m-1}$ is horizontal iff

$$\langle (Z_1, Z_2)j | (V_1, V_2)\rangle = \langle (-\bar{Z}_2, \bar{Z}_1) | (v_1, V_2)\rangle = \langle \bar{Z}_1 | V_2\rangle - \langle \bar{Z}_2 | V_1\rangle = 0. \tag{39}$$

Holomorphic curve $\phi : \mathbb{R} \to \mathbb{CP}^{2m-1}$ is called horizontal if it is tangent to the distribution $\mathcal{H}$ [13, 11]. In terms of the local holomorphic lift $f = (f_1, f_2) : \mathcal{O} \to \mathbb{C}^{2m}$ this property is equivalent to

$$\langle fj | f'\rangle = \langle \bar{f}_1 | f_2'\rangle - \langle \bar{f}_2 | f_1'\rangle = 0. \tag{40}$$

One can generalize the notion of horizontality to the associated curves. A nondegenerate holomorphic curve $\phi : \mathbb{R} \to \mathbb{CP}^{2m-1}$ is called super-horizontal if its $(m-1)$th associated curve $\phi_{m-1} : \mathbb{R} \to \mathbb{G}(m-1, 2m)$ is horizontal. In terms of the local holomorphic lift it is equivalent to (compare with (25))

$$\langle f^{(k)} j | f^{(k+1)}\rangle = 0, \qquad k = 0, \dots, m-2. \tag{41}$$

The super-horizontal curve defines a special type of the complex frame (called the symplectic frame) as follows. Let us take the first m vectors $\mathbf{E}_1, \dots, \mathbf{E}_m$ of the Frenet frame. The super-horizontality of the curve implies that, together with the vectors $\mathbf{E}_1 j, \dots, \mathbf{E}_m j$, they form a unitary frame along the curve. Moreover, the equations satisfied by $\mathbf{E}_k j$ can be found from those of $\mathbf{E}_l$, for example

$$\frac{\partial}{\partial z}(E_k j) = -\frac{\partial \theta^k}{\partial z}\mathbf{E}_k j - \mathrm{e}^{\theta^k - \theta^{k-1}}\mathbf{E}_{k-1} j. \tag{42}$$

From the above considerations it follows that the vector $\mathbf{E}_{m+1}$ of the natural unitary Frenet frame is proportional to $\mathbf{E}_m j$. The phase factor can be fixed by the following normalization of the local holomorphic lift

$$\langle f^{(m-1)} j | f^{(m)}\rangle = 1, \tag{43}$$

which implies $\mathbf{E}_{m+1} = \mathbf{E}_m j$.

Putting all the facts together we obtain that the vectors

$$\mathbf{E}_1, \dots, \mathbf{E}_m, \mathbf{E}_m j, \dots, (-1)^k \mathbf{E}_{m-k} j, \dots, (-1)^{m-1}\mathbf{E}_1 j, \tag{44}$$

form the Frenet frame along the super-horizontal curve, with the corresponding functions

$$\theta^1, \dots, \theta^m, -\theta^m, \dots, -\theta^1. \tag{45}$$

Equations (17) reduce then to the $\mathfrak{sp}(2m)$ Toda system

$$\begin{aligned}
2\frac{\partial^2 \theta^1}{\partial z \partial \bar{z}} &= \mathrm{e}^{2(\theta^2 - \theta^1)}, \\
2\frac{\partial^2 \theta^k}{\partial z \partial \bar{z}} &= \mathrm{e}^{2(\theta^{k+1} - \theta^k)} - \mathrm{e}^{2(\theta^k - \theta^{k-1})}, \qquad k = 2, \dots, m-1 \\
2\frac{\partial^2 \theta^m}{\partial z \partial \bar{z}} &= \mathrm{e}^{-4\theta^m} - \mathrm{e}^{2(\theta^m - \theta^{m-1})}.
\end{aligned} \tag{46}$$

One can show [11] that the vectors of the symplectic frame represent minimal surfaces in $\mathbb{HP}^{m-1}$.

The general solution to the system (46) can be found from the parametrization of the super-horizontal curves subjected to the normalization condition (43). When $f_1 = (f_{1,1}, f_{1,2}) : \mathcal{O} \to \mathbb{C}^{2m}$ is a local holomorphic lift of such a curve then the horizontality condition allows to represent it as

$$f_1 = u_1 \left(1, f_{2,1}, -\int \langle f_2 j | f_2' \rangle d\zeta, f_{2,2}\right), \tag{47}$$

where $f_2 = (f_{2,1}, f_{2,2}) \in \mathbb{C}^{2m-2}$. Using the next condition

$$\langle f_1' j | f_1'' \rangle = u_1^2 \langle f_2' j | f_2'' \rangle = 0 \tag{48}$$

one can represent f_2' similarly. This procedure can be applied on each step, and allows to reconstruct the super-horizontal curve and the corresponding solution of the Toda system, once a system of m meromorphic functions $u_1, \ldots, u_m$ is given.

6 Conclusions and final remarks

In this paper we have presented the geometric meaning of the open Toda systems connected to the four classical sequences of the simple Lie algebras. The general solutions of the Toda systems, found in [3], result, in this context, from parametrization of the corresponding holomorphic curves.

For the periodic Toda systems, the corresponding theory have been found in [14] (see also [15]). The Toda system corresponding to the exceptional Lie algebra G_2 has been treated from geometric point of view in [16].

The infinite dimensional Grassmann manifolds and the corresponding Plücker formulae are the basic objects in the Sato–Wilson approach to the integrable systems [17, 18].

It should be mentioned also that the Toda systems appeared in geometry in the last century in connection with the Laplace sequence of the conjugate nets [19].

Recently, a lot of attention is payed to the discrete analogues of the integrable equations. In particular, the discrete analogue of the infinite Toda system (the Hirota equation) is considered as the basic discrete soliton equation. Moreover, the Hirota equation unifies most of the known solvable quantum field theories and statistical models. The geometric meaning of the Hirota equation was found recently in [20] by "discretizing" the original approach of Darboux. The Hirota equation describes (analogues of) Laplace transforms of the discrete conjugate nets. It would be interesting to transform all the geometric theory of reductions of the Toda system to the discrete level.

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