Self-dual metrics from soliton equations

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Abstract: We propose a construction that relates solutions of various two-dimensional integrable systems to self-dual Einstein equations. Both systems are embedded in self-dual Yang–Mills equations and therefore possess isomorphic Lax formulations. In effect, this leads to curved metrics on $\mathbb{R}^2 \times \Sigma$, where Σ is one of simply-connected Riemann surfaces.

1 Introduction

In the 1st NOSONGE, the most widely discussed type of geometry associated with solitons is that which goes back to the nineteenth century. However, it should be emphasised that the theory of integrable systems has been very much influenced by more recent geometrical ideas — those of twistor theory. Prototypes for applications of twistor geometry to solving nonlinear PDEs were the Nonlinear Graviton construction for self-dual Einstein equations [9] and the Ward construction for the self-dual Yang–Mills equations (SDYM) [10]. Some time later Ward [11, 12] has conjectured that most integrable systems may be obtained from the self-dual Yang–Mills equations by reduction. A review of thus far obtained results can be found in [1, 8].

In this talk I shall explain how isomorphic Lax formulations of two-dimensional integrable systems and SD Einstein equations enable us to costruct explicit forms of SD vacuum metrics out of the solutions of various nonlinear integrable equations. It is achieved by representing a reduced SDYM connection by a hamiltonian vector field on a homogeneous space for a gauge group.

In the next section we briefly review the classification of 2D-integrable systems arising from $SL(2, C)$ SDYM. Section 3 is devoted to the construction of normalised null tetrad and hence also a metric on $\mathbb{R}^2 \times \Sigma$. The results described in this section have been obtained jointly with Lionel Mason. Full details will be given in [3] (see also [4]).

2 The connection with integrable systems

Consider the Yang–Mills (YM) vector bundle over M ($\mathbb{R}^4$ or $\mathbb{C}^4$) with the connection one-form $A = A_\mu(x^\nu)dx^\mu \in T^*M \otimes LG$, where LG is the Lie algebra of some gauge group G. The corresponding curvature $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$ is given by

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{1}$$

where $D_\mu = \partial_\mu - A_\mu$ is the covariant derivative.

SDYM equations on a connection A are the Hodge self-duality conditions

$$F = *F, \qquad \text{or in coordinate form } F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}. \tag{2}$$

SDYM equations are conformally invariant. The property (2) is preserved under the gauge transformation

$$A \to g^{-1}Ag + g^{-1}dg, \qquad g \in \mathrm{Map}(M, G). \tag{3}$$

Let us introduce double-null coordinates $w, \tilde{w}, z, \tilde{z}$, in which the metric on M is $ds^2 = dw\,d\tilde{w} + dz\,d\tilde{z}$. In these coordinates SDYM may be rewritten as

$$F_{wz} = 0, \qquad F_{\tilde{w}\tilde{z}} = 0, \qquad F_{w\tilde{w}} - F_{z\tilde{z}} = 0 \tag{4}$$

which are the compatibility conditions $[L, M] = 0$ for the linear system of equations $L\Phi = 0,\ M\Phi = 0$ where

$$L = D_w - \lambda D_{\tilde{z}}, \quad M = D_z - \lambda D_{\tilde{w}} \tag{5}$$

is the Lax pair for the system. Here $\lambda \in \mathbb{CP}^1$ and Φ is an n-component column vector.

We shall be interested in real ultra-hyperbolic spaces which are recovered by imposing reality conditions

$$w = x - y, \quad z = t + v, \quad \tilde{w} = x + y, \quad \tilde{z} = t - v.$$

We fix the gauge group to be $SL(2, C)$ or some of its real subgroups. Conformal reduction of SDYM involves the choice of the group H of conformal isometries. We shall restrict ourselves to the simplest case and suppose that a connection A is invariant under the flow of two independent translational Killing vectors X and Y. These reductions are classified partially by the signature of the metric restricted to 2-plane spanned by the translations.

1. **Nondegenerate case ($\mathbf{H_1}$)**

   $G = SU(2), \ \ X = \partial_w, \ Y = \partial_{\tilde{w}}$:

$$\begin{aligned} &A_{\tilde{z}} = 0, && A_w = \cos\phi \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \sin\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ &A_{\tilde{w}} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, && A_z = \tfrac{1}{2}(\phi_v - \phi_t) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned} \tag{6}$$

   SDYM yield $\phi_{tt} - \phi_{vv} = 4\sin\phi$. This is a hiperbolic sine–Gordon equation [1].

2. **Partially degenerate case ($\mathbf{H_2}$)**

   $X = \partial_w - \partial_{\tilde{w}}, \ Y = \partial_{\tilde{z}}$:

   a)

$$
\begin{array}{ll}
A_w = \begin{pmatrix} q & 1 \\ b & -q \end{pmatrix}, & A_{\tilde{w}} = 0, \\
A_z = \frac{1}{2}\begin{pmatrix} b_x & -2q_x \\ 2w & -b_x \end{pmatrix}, & A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},
\end{array} \tag{7}
$$

where $4w = q_{xxx} - 4qq_x - 2{q_x}^2 + 4q^2q_x$ and $b = q_x - q^2$. SDYM (with the definition $u = -q_x$) are equivalent to the Korteweg–de Vries equation $4u_z = u_{xxx} + 12uu_x$.

b)

$$
\begin{array}{ll}
A_w = \begin{pmatrix} 0 & \phi \\ \mp\bar{\phi} & 0 \end{pmatrix}, & A_{\tilde{w}} = 0, \\
A_z = i\begin{pmatrix} |\phi|^2 & \pm\phi_x \\ \bar{\phi}_x & -|\phi|^2 \end{pmatrix}, & A_{\tilde{z}} = \pm\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{array} \tag{8}
$$

Here the upper (lower) sign corresponds to $G = SU(2)$ (or $SU(1,1)$). SDYM become $i\phi_z = -\phi_{xx} \mp 2|\phi|^2\phi$ which is the nonlinear Schrödinger equation with an attractive (respectively repulsive) self interaction [6].

3 Self-dual metrics on $\mathbb{R}^2 \times \Sigma$

Self-duality conditions (SD) on the curvature are naturaly expressed in terms of Lax pairs of vector fields. Let $\mathcal{M}$ be a real four-manifold with a volume form ν. Let $V_i = (W, \widetilde{W}, Z, \widetilde{Z})$ be real, independent, volume preserving vector fields on $\mathcal{M}$. Define $f^2 = \nu(W,\ \widetilde{W},\ Z,\ \widetilde{Z})$. Mason and Newman's [5] form of the nonlinear graviton theorem states that if

$$
L = W - \lambda\widetilde{Z}, \qquad M = Z - \lambda\widetilde{W} \tag{9}
$$

commute for each value of $\lambda \in \mathbb{CP}^1$, then $f^{-1}V_i$ is a normalized null tetrad for an anti self dual vacuum metric. Introduce dual frame:

$$
\begin{array}{ll}
e_W = f^{-2}\nu(\ldots, \widetilde{W}, Z, \widetilde{Z}), & e_{\widetilde{W}} = f^{-2}\nu(W, \ldots, Z, \widetilde{Z}), \\
e_Z = f^{-2}\nu(W, \widetilde{W}, \ldots, \widetilde{Z}), & e_{\widetilde{Z}} = f^{-2}\nu(W, \widetilde{W}, Z, \ldots).
\end{array} \tag{10}
$$

A covariant metric can be expressed in terms of these one forms as

$$
g = f^2(e_W \odot e_{\widetilde{W}} - e_Z \odot e_{\widetilde{Z}}). \tag{11}
$$

We can make a number of connections between the SDYM equations and the SD conditions on a four-dimensional metric by considering gauge potentials that take values in a Lie algebra of vector fields on some manifold. Let $(\Sigma,\ \Omega_\Sigma)$ be a 2-dimensional symplectic manifold and let SDiff(Σ) be the group of canonical transformations of Σ. Consider

SDYM with the gauge group G being the subgroup of SDiff(Σ). We can represent the components of D by hamiltonian vector fields on $\mathbb{R}^4 \times \Sigma$:

$$D_w = \partial_w - X_{H_w},\ D_{\tilde{w}} = \partial_{\tilde{w}} - X_{H_{\tilde{w}}},\ D_z = \partial_z - X_{H_z},\ D_{\tilde{z}} = \partial_{\tilde{z}} - X_{H_{\tilde{z}}}. \tag{12}$$

H_μ denotes a hamiltonian which generates a vector field corresponding to A_μ. Now suppose that the connection is invariant under two translations and identify $W = D_w$, $\widetilde{W} = D_{\tilde{w}}$, $Z = D_z$, $\widetilde{Z} = D_{\tilde{z}}$. From isomorphic formulations of SDYM (5) and SDG (self-dual gravity) (9) follows that the reduced Lax pair will give rise to a half flat metric on $\mathbb{R}^2 \times \Sigma$, so long as we find a way of representing its gauge group by canonical transformations of Σ. It is known that

$$LSDiff(\Sigma) = \lim_{n\to\infty} LSL(n, C). \tag{13}$$

However, it seems that $SL(n, C)$ is a subgroup of such defined $SL(\infty)$ only for $n = 2$ [7]. In this case we can take the linear action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ or a Möbius action of $SU(2)$ and $SU(1,1)$ on $\mathbb{CP}^1$ or D (Poincaré half plane) respectively. We restrict ourselves to real parts of vector fields, which implies that obtained SD metrics have ultrahyperbolic signature. To be more explicit we write down hamiltonians corresponding to matrix

$$A_\mu = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in LSL(2,\mathbb{C}),$$

$$\Sigma = \mathbb{R}^2, \qquad \Omega_\Sigma = dm \wedge dn, \qquad H_\mu = \left(\frac{bn^2}{2} + amn - \frac{cm^2}{2}\right), \tag{14}$$

$$\Sigma = \mathbb{CP}^1, \quad \Omega_\Sigma = \frac{id\xi \wedge d\bar{\xi}}{(1+\xi\bar{\xi})^2}, \quad H_\mu = -i\frac{\xi\bar{b} - \bar{\xi}b + 2a}{1+\xi\bar{\xi}}, \tag{15}$$

$$\Sigma = D, \qquad \Omega_\Sigma = \frac{id\xi \wedge d\bar{\xi}}{(1-\xi\bar{\xi})^2}, \quad H_\mu = -i\frac{\xi\bar{b} - \bar{\xi}b - 2a}{1-\xi\bar{\xi}}. \tag{16}$$

Now we shall elaborate on the form of a dual null tetrad (10) obtained after the two-dimensional reductions of SDYM.

1. $\mathbf{H_1}$, $(X = \partial_w, Y = \partial_{\tilde{w}})$, $\nu = dz \wedge d\tilde{z} \wedge \Omega_\Sigma$**:**

$$\begin{aligned}
f^2 &= \nu(W, \widetilde{W}, Z, \widetilde{Z}) = \Omega_\Sigma(W, \widetilde{W}) = \{H_w, H_{\tilde{w}}\}, \\
e_W &= \frac{\Omega_\Sigma(\widetilde{W}, Z)}{\Omega_\Sigma(W, \widetilde{W})}dz + \frac{\Omega_\Sigma(\widetilde{W}, \widetilde{Z})}{\Omega_\Sigma(W, \widetilde{W})}d\tilde{z} + \frac{\Omega_\Sigma(\ldots, \widetilde{W})}{\Omega_\Sigma(W, \widetilde{W})} \\
&= \frac{\{H_{\tilde{w}}, H_z\}}{\{H_w, H_{\tilde{w}}\}}dz + \frac{\{H_{\tilde{w}}, H_{\tilde{z}}\}}{\{H_w, H_{\tilde{w}}\}}d\tilde{z} - \frac{d_\Sigma H_{\tilde{w}}}{\{H_w, H_{\tilde{w}}\}}, \\
e_{\widetilde{W}} &= -\frac{\{H_w, H_z\}}{\{H_w, H_{\tilde{w}}\}}dz - \frac{\{H_w, H_{\tilde{z}}\}}{\{H_w, H_{\tilde{w}}\}}d\tilde{z} + \frac{d_\Sigma H_w}{\{H_w, H_{\tilde{w}}\}}, \\
e_Z &= dz, \\
e_{\widetilde{Z}} &= d\tilde{z}.
\end{aligned} \tag{17}$$

Gauge freedom is used to set $H_{\tilde{w}} = 0$:

$$
\begin{aligned}
ds^2 \quad = \quad & \frac{1}{\{H_w, H_{\tilde{w}}\}}\Big(- (\{H_{\tilde{w}}, H_z\}\{H_w, H_z\})dz^2 - \{H_w, H_{\tilde{w}}\}^2 dz d\tilde{z} \\
& -((\partial_\xi H_{\tilde{w}} \partial_{\bar{\xi}} H_w) + (\partial_{\bar{\xi}} H_{\tilde{w}} \partial_\xi H_w))d\xi d\bar{\xi} - (\partial_\xi H_w \partial_\xi H_{\tilde{w}})d\xi^2 \\
& +(\partial_\xi H_{\tilde{w}}\{H_w, H_z\} + \partial_\xi H_w \{H_{\tilde{w}}, H_z\})dz d\xi - (\partial_{\bar{\xi}} H_w \partial_{\bar{\xi}} H_{\tilde{w}})d\bar{\xi}^2 \\
& +(\partial_{\bar{\xi}} H_{\tilde{w}}\{H_w, H_z\} + \partial_{\bar{\xi}} H_w \{H_{\tilde{w}}, H_z\})dz d\bar{\xi}\Big).
\end{aligned}
$$

2. $\mathbf{H_2}$, $(X = \partial_w - \partial_{\tilde{w}}, Y = \partial_{\tilde{z}})$, $\nu = dx \wedge dz \wedge \Omega_\Sigma$:

$$
f^2 = \{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}, \tag{18}
$$

$$
\begin{aligned}
e_W &= \frac{\{H_{\tilde{z}}, H_{\tilde{w}}\}}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}dx + \frac{\{H_{\tilde{z}}, H_z\}}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}dz - \frac{d_\Sigma H_{\tilde{z}}}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}, \\
e_{\widetilde{W}} &= -\frac{\{H_{\tilde{z}}, H_w\}}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}dx - \frac{\{H_{\tilde{z}}, H_z\}}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}dz + \frac{d_\Sigma H_{\tilde{z}}}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}, \\
e_Z &= dz, \\
e_{\widetilde{Z}} &= \frac{\{H_w, H_{\tilde{w}}\}}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}dx + \frac{\{H_w - H_{\tilde{w}}, H_z\}}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}dz - \frac{d_\Sigma (H_w - H_{\tilde{w}})}{\{H_w - H_{\tilde{w}}, H_{\tilde{z}}\}}.
\end{aligned}
\tag{19}
$$

Gauge freedom $H_{\tilde{z}} = 0$:

$$
\begin{aligned}
ds^2 \quad = \quad & -\left(\frac{\{H_{\tilde{z}}, H_z\}^2}{\{H_w, H_{\tilde{z}}\}} + \{H_w, H_z\}\right)dz^2 - \frac{(\partial_\xi H_{\tilde{z}})^2}{\{H_w, H_{\tilde{z}}\}}d\xi^2 \\
& -\frac{(\partial_{\bar{\xi}} H_{\tilde{z}})^2}{\{H_w, H_{\tilde{z}}\}}d\bar{\xi}^2 + \left(\partial_\xi H_w + 2\frac{\{H_{\tilde{z}}, H_z\}}{\{H_w, H_{\tilde{z}}\}}\partial_\xi H_{\tilde{z}}\right)dz d\xi \\
& + \left(\partial_{\bar{\xi}} H_w + 2\frac{\{H_{\tilde{z}}, H_z\}}{\{H_w, H_{\tilde{z}}\}}\partial_{\bar{\xi}} H_{\tilde{z}}\right)dz d\bar{\xi} - \frac{2\partial_\xi H_{\tilde{z}} \partial_{\bar{\xi}} H_{\tilde{z}}}{\{H_w, H_{\tilde{z}}\}}d\xi d\bar{\xi} \\
& +\{H_{\tilde{z}}, H_z\}dx dz - \partial_\xi H_{\tilde{z}} dx d\xi - \partial_{\bar{\xi}} H_{\tilde{z}} dx d\bar{\xi}.
\end{aligned}
$$

Obtained solutions may be viewed as fibered metric on the total space of the Σ-bundle associated to a YM bundle. First notice that diffeomorphism of $\mathbb{R}^2 \times \Sigma$ given by

$$
x^a \longrightarrow x^a + \epsilon X_F(x^a) \tag{20}
$$

yields $H_\mu \longrightarrow H_\mu + \epsilon(\{H_\mu, F\} + \partial_\mu F)$ which is an infinitesimal form of full gauge transformation (3). If (20) preserves a Kähler structure of Σ then H_μ transforms under (real form of) $SL(2, C)$ and therefore our construction remains 'invariant'. Singularities in metrics corresponding to (17) and (19) which come from the base space may be removed by imposing boundary conditions on the solutions of soliton equations. However we are left with singularities of the vector fields X_H for ξ being real or purely imaginary.

3.1 Solitonic metrics

Now we are in the position to establish the connection between integrable systems reviewed in Section 2 and self-dual vacuum metrics. It resolves itself into expressing hamiltonians in terms of solutions of various nonlinear equations. From a given solution of a $2D$ nonlinear equation we can generate a null tetrad (10).

1. **Nonlinear Schrödinger equation**

$$\begin{aligned}
W &= \partial_x + (\bar{\phi}\xi^2 + \phi)\partial_\xi + (\phi\bar{\xi}^2 + \bar{\phi})\partial_{\bar{\xi}} \,, \\
\widetilde{W} &= \partial_x \,, \\
\widetilde{Z} &= -i\xi\partial_\xi + i\bar{\xi}\partial_{\bar{\xi}} \,, \\
Z &= \partial_z + i(-\bar{\phi}_x\xi^2 + 2|\,\phi\,|^2\xi + \phi_x)\partial_\xi \\
&\quad -i(-\phi_x\bar{\xi}^2 + 2|\,\phi\,|^2\bar{\xi} + \bar{\phi}_x)\partial_{\bar{\xi}} \,, \\
f^2 &= \frac{2Re(\bar{\xi}\phi)}{1+|\,\xi\,|^2} \,.
\end{aligned}$$

2. **Korteveg–de Vries equation**

$$\begin{aligned}
W &= \partial_x + (qm+n)\partial_m + (bm-qn)\partial_n, \\
\widetilde{W} &= \partial_x, \\
\widetilde{Z} &= m\partial_n, \\
Z &= \partial_z + \left(\frac{b_x}{2}m - q_x n\right)\partial_m + \left(wm - \frac{b_x}{2}n\right)\partial_n, \\
f^2 &= -m(q+mn).
\end{aligned}$$

where $b = q_x - q^2$ and $4w = q_{xxx} - 4qq_{xx} - 2{q_x}^2 + 4q^2q_x$.

3. **Sine–Gordon equation**

$$\begin{aligned}
W &= (-i\xi^2e^{-i\phi} + ie^{i\phi})\partial_\xi + (i\bar{\xi}^2e^{i\phi} - ie^{-i\phi})\partial_{\bar{\xi}} \,, \\
\widetilde{W} &= (-i\xi^2 + i)\partial_\xi + (i\bar{\xi}^2 - i)\partial_{\bar{\xi}} \,, \\
\widetilde{Z} &= \partial_{\bar{z}} \,, \\
Z &= \partial_z - i(\partial_{\bar{z}}\phi)\xi\partial_\xi + i(\partial_{\bar{z}}\phi)\bar{\xi}\partial_{\bar{\xi}} \,, \\
f^2 &= \frac{4\sin\phi(|\,\xi\,|^2 - 1)}{|\,\xi\,|^2 + 1} \,.
\end{aligned}$$

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