Schlesinger transformations for Bonnet surfaces

Ulrich Eitner

TU Berlin, MA-8-5, Strasse des 17. Juni 136, D-10623 Berlin, Germany
e-mail: uli@sfb288.math.tu-berlin.de

Abstract: Bonnet surfaces, i.e. surfaces in Euclidean 3-space, which admit a one-parameter family of isometries preserving the mean curvature function, can be described in terms of solutions of some special Painlevé equations. The goal of this work is to use the well-known Schlesinger transformations for solutions of Painlevé VI equations to create new Bonnet surfaces from a known one.

1 Introduction

Let us consider a surface in a Euclidean 3-space which admits a family of isometric surfaces having the same mean curvature function. It was the French mathematician O. Bonnet who first discussed whether there exist such surfaces or not. He stated that excluding surfaces of revolution, helicoids and constant mean curvature surfaces there is a class of such surfaces, which can be described by ordinary differential equations. We call them Bonnet surfaces.

A. Hazzidakis formulated the problem in terms of an ordinary differential equation of third order for the mean curvature function on a Bonnet surface and integrated this equation once. He also discussed the first (non-geometrical) solution of it. Forty years later it was E. Cartan, who gave a classification of all Bonnet surfaces, which is up to now the state of the art. He showed that Bonnet surfaces depend — up to a rigid motion — on at most 3 real parameters.

In [1], it is shown how to describe Bonnet surfaces in terms of solutions of some special Painlevé V(III) and VI equations. The first integral found by Hazzidakis is thereby a parameter in the Painlevé equations. The main tool hereby is the so-called Ψ-function, which is a matrix-valued solution of the corresponding linear systems (the Painlevé equations are the zero-curvature conditions for integration of the corresponding linear systems). This Ψ-function generates a moving frame on the surfaces and so have a geometrical meaning.

As it is known for specialists in Painlevé equations, there are so-called Schlesinger transformations, i.e. left-matrix-multiplications of the Ψ-functions, by which from one solution of a Painlevé equation one can generate solutions of discrete many different Painlevé equation. These transformations are discussed in [3], [4] and especially for Painlevé VI in [5]. Using the formal ansatz from [5] I produce a Schlesinger transformation for Bonnet surfaces by shifting of the first integral parameter by an integer number. Up to now this recipe can be geometrically interpreted only in the Cartan **B**-case [6], since for the Cartan **A**-case I can not characterize geometrical solutions of the corresponding Painlevé VI equation. In the Cartan **C**-case the first integral is not involved in the corresponding linear system for the Ψ-function. This comes from the fact, that this

parameter is not an essential parameter of the surfaces. It is only a scaling factor and could be fixed — unless it is not zero (all Bonnet surfaces where this parameter vanishes are described explicitly in hyper-geometric functions in [1]).

I should mention that I did not find a complete geometrical meaning of the Schlesinger transformation up to now.

2 Basics

Let $su(2)$ be the Lie algebra of $SU(2)$ and

$$\mathbb{i} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbb{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbb{k} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

be a basis of it. To find a proper description of Bonnet surfaces in terms of Painlevé functions it is useful to identify $\mathbb{R}^3$ with $su(2)$ and use the 2×2 representation of the Painlevé equations introduced in [3]. Here this identification is realized by the isomorphism

$$(e_1\,, e_2\,, e_3) \in \mathbb{R}^3 \;\to\; (\mathbb{i}\,, \mathbb{j}\,, \mathbb{k}) \in su(2). \tag{1}$$

How by this identification a local theory of surfaces in $\mathbb{R}^3$ can be formulated is stated in the following

LEMMA 1. *Let $\mathcal{F} \subset \mathbb{R}^3$ be a smooth surface and $F = F(x,y)$ be a smooth local conformal parameterization with the induced metric*

$$ds^2 = e^{u(x,y)}(d\,x^2 + d\,y^2),$$

Then there exists a smooth $SU(2)$-valued function $\Psi(x,y)$ such that

$$\begin{array}{rcl} F_x(x,y) & = & e^{u(x,y)/2}\,\Psi^{-1}(x,y)\,\mathbb{i}\,\Psi(x,y), \\ F_y(x,y) & = & e^{u(x,y)/2}\,\Psi^{-1}(x,y)\,\mathbb{j}\,\Psi(x,y), \\ N(x,y) & = & \Psi^{-1}(x,y)\,\mathbb{k}\,\Psi(x,y), \end{array} \tag{2}$$

where F_x and F_y denotes the partial derivatives of $F(x,y)$ and $N(x,y)$ is the corresponding Gauss map, each identified with a $su(2)$-valued function by the isomorphism given above.

Proof: The proof is given in [7]. It is stated there, that Ψ has to solve the following linear system

$$\begin{array}{rcl} \Psi_x\,\Psi^{-1} & = & \begin{pmatrix} -\frac{1}{4}iu_y & -Q\,e^{-u/2} - \frac{1}{2}He^{u/2} \\ \bar{Q}\,e^{-u/2} + \frac{1}{2}He^{u/2} & \frac{1}{4}iu_y \end{pmatrix} \\ \Psi_y\,\Psi^{-1} & = & i \begin{pmatrix} \frac{1}{4}u_x & -Q\,e^{-u/2} + \frac{1}{2}He^{u/2} \\ -\bar{Q}\,e^{-u/2} + \frac{1}{2}He^{u/2} & -\frac{1}{4}u_x \end{pmatrix}, \end{array} \tag{3}$$

where $u = u(x,y)$ is the metric function as above, $H = H(x,y)$ is the mean curvature function and $Q = Q(x,y)$ is the Hopf differential defined by

$$Q(x,y) = \tfrac{1}{4}\,\left(\langle F_{xx}(x,y) - F_{yy}(x,y), N(x,y)\rangle - 2\,i\,\langle F_{xy}(x,y), N(x,y)\rangle\right).$$

$\bar{Q}(x,y)$ is the complex conjugate of $Q(x,y)$. □

As in the classical theory there are integrability condition for moving frames, here the same conditions are the compatibility condition for (3). This is what is stated in the following

LEMMA 2. *Let $u(x,y)$, $H(x,y)$ and $Q(x,y)$ be a solution of*

$$\begin{aligned} \Delta\, u + 2\,H^2\,e^u - 8|Q|^2\,e^{-u} &= 0, \\ (Q+\bar{Q})_x + i\,(Q-\bar{Q})_y &= H_x\,e^u, \\ i\,(Q-\bar{Q})_x - (Q+\bar{Q})_y &= H_y\,e^u. \end{aligned} \tag{4}$$

Then there exists (up to left-multiplication with a constant $SU(2)$-matrix) a unique $SU(2)$-valued function Ψ, which solves (3). Moreover, by (2) a unique (up to a constant rotation) moving frame on a surface is defined. The corresponding surface is unique (up to a translation) integral of this moving frame.

Proof: One observe that (4) are the compatibility conditions for (3). The solution of this equation is unique up to left-multiplication with a constant $SU(2)$ matrix (if the initial data matrix is in $SU(2)$). Now, defining F_x, F_y, N by (2) we have $(F_x)_y = (F_y)_x$. Therefore they can be integrated to one function $F = F(x,y)$. This integration is unique up to an additional term in $su(2)$. Finally using the isomorphism (1) we get a surface in $\mathbb{R}^3$ unique up to rigid motion. □

3 General results on Bonnet A and B surfaces

Following the general results formulated in [1] we find

THEOREM 1. *Let $\mathcal{F}$ be a Bonnet surface of Cartan-type A or B. Then there exists a (local) conformal coordinate $z = x + i\,y$, such that the fundamental functions read as follows:*

$$\begin{aligned} Q_A(x,y) &= -2\frac{\sin(2(x-iy))}{\sin(4x)\sin(2(x+iy))}, & e^{u_A(x)} &= -\frac{16}{\sinh^2(4x)H_A'(x)}, \\ Q_B(x,y) &= -2\frac{\sinh(2(x-iy))}{\sinh(4x)\sinh(2(x+iy))}, & e^{u_B(x)} &= -\frac{16}{\sin^2(4x)\,H_B'(x)}, \end{aligned} \tag{5}$$

and $H_\nu(x)$, $\nu = A, B$, the mean curvature function on the corresponding surface, solve the differential equation

$$\left(\frac{H_\nu''(x)}{H_\nu'(x)}\right)' - 2\,H_\nu'(x) = \frac{8\,\alpha_\nu^2}{\sinh^2(\alpha_\nu\,x)}\,\left(1 - \frac{{H_\nu}^2(x)}{H_\nu'(x)}\right), \tag{6}$$

where $\alpha_A = 4\,i$ and $\alpha_B = 4$.

Proof: The proof is given in [1]. □

The parameter α_ν is not an essential parameter of the differential equation. This will be shown in the following

LEMMA 3. *Let $H(x) = H_\nu(x)$ be an arbitrary solution of (6) with arbitrary $\alpha = \alpha_\nu \in \mathbb{C} \setminus \{0\}$. Then the function $H_\theta(x)$ defined by*

$$H(x) = \frac{1}{\theta} H_\theta \left(\frac{\theta}{x} \right)$$

solves (6) but with $\alpha_\nu = \alpha\,\theta$.

Proof: This follows from the chain rule. □

REMARK 1. Lemma 3 states, that one can express the general solutions of (6) in the Bonnet A-case by the general solution in the Bonnet B-case and vice versa. Of course by general solution I mean general complex-valued solutions. So Lemma 3 has only an analytical meaning but not a geometrical one.

4 A Schlesinger transformation for Bonnet B surfaces

Hazzidakis [8] found a first integral of (6). It was shown in [1] that there is a one-to-one-relation between the solutions of (6) and the solutions of a Painlevé VI equation. For that it was shown that the moving frame equation (3) for Bonnet A and B surfaces are the zero-curvature representation for such a Painlevé equation, described in [3]. The solution of this linear system gives the Ψ-function.

For the rest of this text I fix my view on the Bonnet B-case. By Lemma 3 it is clear that all analytical properties found here hold as well in the A-case.

LEMMA 4. *Equation (6) for $\alpha_B = 4$ has the first integral*

$$\left(\frac{H''(x)}{8\,H'(x)} + \coth(4\,x) \right)^2 - \frac{H^2(x)}{H'(x)\,\sinh^2(4\,x)} - \frac{H'(x)}{16} + \\ - \frac{H(x)}{2}\,\coth(4\,x) \quad = \quad \mu^2. \tag{7}$$

Proof: This is a simple calculation. □

REMARK 2. If $H(x)$ is a solution of (6) with the first integral (7) then $-H(-x)$ is also a solution of (6) with the same first integral.

Let us use another way to prove Lemma 4, namely via the Lax representation of (6). Introducing the similarity variables

$$s = e^{8\,x}, \qquad \lambda = e^{4\,(x+i\,y)}, \qquad \Psi(s,\lambda) = \Psi(x(s,\lambda), y(s,\lambda)) = \Psi(x,y), \tag{8}$$

and inserting them into the moving frame equation (3) together with (5) we arrive at

$$\begin{aligned}\Psi_\lambda\,\Psi^{-1} &= \frac{1}{\lambda}\begin{pmatrix} a(s) & \varphi_1(s) \\ \varphi_2(s) & -a(s)\end{pmatrix} + \frac{\varphi_2(s)-\varphi_1(s)}{\lambda-1}\begin{pmatrix}0&1\\0&0\end{pmatrix} \\ &\quad + \frac{\varphi_1(s)-\varphi_2(s)}{\lambda-s}\begin{pmatrix}0&0\\1&0\end{pmatrix}, \qquad (9)\\ \Psi_s\,\Psi^{-1} &= \frac{\varphi_2(s)-\varphi_1(s)}{\lambda-s}\begin{pmatrix}0&0\\1&0\end{pmatrix} + \begin{pmatrix} -\dfrac{a(s)}{2\,s} & \dfrac{\varphi_1(s)-s\,\varphi_2(s)}{s\,(s-1)} \\ \dfrac{\varphi_2(s)-\varphi_1(s)}{s-1} & \dfrac{a(s)}{2\,s}\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned} a(s) &= -\frac{1}{2}\left(\frac{H''(x)}{8\,H'(x)} + \coth(4\,x)\right), \\ \varphi_1(s) &= \frac{H(x)}{2\,\sinh(4\,x)\,\sqrt{-H'(x)}} - e^{4\,x}\,\frac{\sqrt{-H'(x)}}{8}, \qquad (10)\\ \varphi_2(s) &= \frac{H(x)}{2\,\sinh(4\,x)\,\sqrt{-H'(x)}} - e^{-4\,x}\,\frac{\sqrt{-H'(x)}}{8}. \end{aligned}$$

Here $'$ denotes the differentiation with respect to x. Now the compatibility conditions for the systems (9) lead to

$$a^2(s) + \varphi_1(s)\,\varphi_2(s) = \left(\frac{\mu}{2}\right)^2 \qquad (11)$$

which coincides with (7).

THEOREM 2. *Let $H(x)$ be a solution of* (6) *in the case $\alpha_B = 4$ with the first integral μ given by* (7). *Then*

$$y(\tau) = \frac{1}{4\,H'(x)}\left(\frac{8\,H'(x)\,(\mu\,\sinh(4\,x)+\cosh(4\,x)) + \sinh(4\,x)\,H''(x)}{4\,H(x) + e^{4\,x}\,\sinh(4\,x)\,H'(x)}\right)^2, \qquad (12)$$

where $\tau = e^{-8x}$, solves the Painlevé VI equation

$$\begin{aligned} y'' &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-\tau}\right)y'^2 - \left(\frac{1}{\tau}+\frac{1}{\tau-1}+\frac{1}{y-\tau}\right)y' \\ &\quad + \frac{y(y-1)(y-\tau)}{\tau^2(\tau-1)^2}\left(\alpha + \beta\,\frac{\tau}{y^2} + \gamma\,\frac{\tau-1}{(y-1)^2} + \delta\,\frac{\tau(\tau-1)}{(y-\tau)^2}\right) \end{aligned} \qquad (13)$$

with

$$\alpha = 0, \quad \beta = 0, \quad \gamma = \frac{\mu^2}{2}, \quad \delta = -\frac{\mu(\mu+2)}{2}. \qquad (14)$$

Inversely let $y(\tau)$ be an arbitrary solution of (13) *with* (14)*, then*

$$H(x) = -\frac{2\,(\tau-1)(\tau^2\,y'^2(\tau) - \mu^2\,y^2(\tau))}{y(\tau)\,(y(\tau)-1)(y(\tau)-\tau)}, \qquad \tau = e^{-8\,x} \tag{15}$$

solves (6)*. Moreover if $\mu \in \mathbb{R}$ and $y(\tau) < 0$ then there exists a parameterization of a Bonnet B type surface, with the mean curvature $H(x)$ as in (15) and the metric* (5) *given by*

$$e^{u(x)} = -\frac{4\,y(\tau)\,(y(\tau)-\tau)^2}{(\tau-1)^2(\mu\,y(\tau)+\tau\,y'(\tau))^2}. \tag{16}$$

REMARK 3. It is simple to prove, that in the Bonnet B case μ could be real as well as pure imaginary. The above theorem holds in general. This is not explicitly formulated, since I am not able to characterize geometrical solutions of (13) in the general case.

Proof: For the proof let us introduce the auxiliary function

$$\xi(s) = \left(\frac{a(s)-\mu/2}{\varphi_1(s)}\right)^2, \tag{17}$$

which, according to (10), for given solution of (6) is expressible in terms of $H(x)$, $H'(x)$ and $H''(x)$. The compatibility conditions of (9) read as follows:

$$\begin{aligned} a'(s) &= \frac{(\mu^2+4\,a^2(s))(1-s\,\xi^2(s)) - 4\,\mu\,a(s)(1+s\,\xi^2(s))}{4\,s(s-1)\,\xi(s)}, \\ \xi'(s) &= \frac{2\,a(s)(1+s\,\xi(s))(1+\xi(s)) + \mu\,(s\xi^2(s)-1)}{s(s-1)}. \end{aligned} \tag{18}$$

This can be formulated as second-order equation for $\xi(s)$ only:

$$\begin{aligned} \xi'' &= \frac{1}{2}\left(\frac{1}{\xi}+\frac{1}{\xi+1}+\frac{s}{1+s\,\xi}\right)\xi'^2 - \left(\frac{1}{s}+\frac{1}{s-1}-\frac{\xi}{1+s\,\xi}\right)\xi' + \\ &\quad \frac{\xi(\xi+1)(1+s\,\xi)}{s^2(s-1)^2}\left(\frac{\mu\,(\mu+2)}{2}\,\frac{s(s-1)}{(1+s\,\xi)^2} - \frac{\mu^2}{2}\,\frac{s-1}{(\xi+1)^2}\right). \end{aligned} \tag{19}$$

In view that

$$y(\tau) = -\xi(s), \quad \tau = \frac{1}{s}, \tag{20}$$

we find the first part of the theorem.

Inversely, let us start with an arbitrary solution $y(\tau)$ of (13) with the parameters (14) given by an arbitrary $\mu \in \mathbb{C}$. Define $\xi(s)$ and $a(s)$ by (20) and (18). Now set $\varphi_1(s)$ and $\varphi_2(s)$ as

$$\varphi_1(s) = \frac{a(s)-\mu/2}{\sqrt{\xi(s)}}, \qquad \varphi_2(s) = -\sqrt{\xi(s)}\,(a(s)+\mu/2). \tag{21}$$

By (10) we can express $H(x)$, $H'(x)$ and $H''(x)$ in terms of $a(s)$, $\varphi_1(s)$ and $\varphi_2(s)$. The ambiguity of the definition of $\varphi_1(s)$ and $\varphi_2(s)$ (coming form the square root of ξ) is cancelled in this expressions. Using (21) we find (15) and an expression for $H'(x)$, which is compatible to the derivation of (15) by use of (13) and (14). It is simple to check that (15) solves (6) if $y(\tau)$ solves (13), (14). Finally by taking (formally) (5) for the metric we reach at (16). For $\mu \in \mathbb{R}$ this is strictly positive, iff $y(\tau) < 0$. □

The last part of the proof may motivate the following

DEFINITION 1. A solution $y(\tau)$ of (13),(14) for $\mu \in \mathbb{R}$ is said to be *geometrical* on a non-empty interval $\mathcal{I} \subset \mathbb{R}$, if $y(\tau) \in \mathbb{R}^-$, $\forall \tau \in \mathcal{I}$.

REMARK 4. Lemma 4 and Theorem 2 show, that for any solution $H(x)$ of (6) there exist two first integrals, namely $\pm\mu$. Consequently, to each solution of (6) two functions $y(\tau)$ solving two different Painlevé VI equations (13), (14) are related.

REMARK 5. Formulae (12) and (16) show, that for $\mu \in \mathbb{R}$ in the Bonnet B case a solution $y(\tau)$ can be characterized to be geometrical, which means that the corresponding function $H(x)$ given by (15) is realized on a Bonnet B surface. It is obvious that a solution $y(\tau)$ of (13), (14) is geometrical if and only if $y(\tau) < 0$ for all τ in an interval in $\mathbb{R}^+$.

For the next theorem it will be convenient for me to use the auxiliary function $\xi(s)$ instead of $y(\tau)$. From (20) the relation to $y(\tau)$ is clear.

THEOREM 3. *Let $\mu \in \mathbb{C}$, $\xi_\mu = \xi_\mu(s)$ be an arbitrary solution of* (19)*, $a(s)$, $\varphi_1(s)$ and $\varphi_2(s)$ be defined by* **(18)** *and* **(21)***, respectively, and $\Psi_\mu(\lambda, s)$ the solution of the corresponding linear system* (9)*. Then*

$$\Psi_{\mu+1}(\lambda, s) = L(\lambda, s)\, \Psi_\mu(\lambda, s), \tag{22}$$

with

$$\begin{aligned} L(\lambda, s) = \frac{i\sqrt{\lambda}}{\sqrt{(1+\xi_\mu)(1+s\,\xi_\mu)}} \Bigg\{ \begin{pmatrix} -s^{-1/4} & \sqrt{\xi_\mu}\, s^{-1/4} \\ -\sqrt{\xi_\mu}\, s^{1/4} & \xi_\mu\, s^{1/4} \end{pmatrix} + \\ \frac{1}{\lambda} \begin{pmatrix} -\xi_\mu\, s^{3/4} & -\sqrt{\xi_\mu}\, s^{3/4} \\ \sqrt{\xi_\mu}\, s^{1/4} & s^{1/4} \end{pmatrix} \Bigg\}, \end{aligned} \tag{23}$$

solves the linear system (9) *corresponding to the solutions*

$$\xi_{\mu+1}(s) = \frac{1}{s\,\xi_\mu} \left(\frac{\mu\,\xi_\mu\,(1+s+2\,s\,\xi_\mu) + 2\,s\,\xi_\mu(\xi_\mu+1) + s\,\xi'_\mu\,(s-1)}{2\,(\mu+1) + \mu\,\xi_\mu\,(s+1) + 2\,\xi_\mu - s\,\xi'_\mu\,(s-1)} \right)^2 \tag{24}$$

of (19) *with $\mu+1$ instead of μ.*

Proof: This is a direct calculation. □

REMARK 6. I should remark that one can get the transformation law for the ξ-functions directly. Therefore one has to see that

$$\xi_{-(\mu+1)}(s) = \frac{1}{s\,\xi_\mu(s)} \tag{25}$$

solves (19) for $-(\mu+1)$ instead of μ. On the other hand

$$\xi_{-\mu}(s) = \xi_\mu(s)\left(\frac{2\,a_\mu(s)+\mu}{2\,a_\mu(s)-\mu}\right)^2$$

solves (19) but with $-\mu$ instead of μ. Both together give the simple relation between ξ_μ and $\xi_{\mu+1}$ which is (24).

EXAMPLE 1. For $\mu = 0$ one finds $\xi_0(s) = \alpha \neq 0$ as a 1-parameter family of solutions of (18). By use of (24), one gets for any integer number $\mu = n \in \mathbb{Z}$

$$\xi_n(s) = \alpha\, s^n, \quad y_n(\tau) = -\frac{\alpha}{\tau^n}. \tag{26}$$

Finally one can directly prove that this is a solution of (18) for any $\mu \in \mathbb{C}$ (instead of n).

REMARK 7. From the viewpoint of the equation (19), we found, that any solution of the linear differential equation

$$\tau\, y'(\tau) = -\mu\, y(\tau) \tag{27}$$

is at the same time a solution of the corresponding Painlevé VI (13), (14). This behaviour is similar to that in the Painlevé V case discussed for the Bonnet **C** case in [1]. All solution of (27) corresponds to the solution $H(x) \equiv 0$ of (6).

COROLLARY 1. *Let $H(x)$ be a real solution of* (6)*, $H'(x) < 0$ and $\mu \in \mathbb{R}$ be defined (up to the sign) by* (7)*. Then $L \in SU(2)$.*

Proof: First one observes that $L \in SL(2)$. Moreover since $a(s)\,, \varphi_1^2(s), \mu \in \mathbb{R}$ ξ_μ must be positive (e.g. see (17)). With $s > 1$ we get $(1+\xi_\mu)(1+s\,\xi_\mu) > 0$. Now it is simple to see, that L is of the form

$$L = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

□

REMARK 8. Corollary **1** together with (24) means that the transformation $\Psi \to L\,\Psi$ transforms geometrical solutions of (19) into geometrical.

LEMMA 5. *Let $H(x) = H_\mu(x)$ be a solution of* (6) *($\alpha_B = 4$) with the first integral μ given by* (7). *Then*

$$H_{\mu+1}(x) = \left(1 + \frac{2\, y_\mu(\tau)}{\mu\, y_\mu(\tau) - \tau\, y'_\mu(\tau)}\right) H_\mu(x), \quad \tau = e^{-8\,x}, \tag{28}$$

with $y_\mu(\tau)$ as in (12) *solves* (6), *too. The first integral (7) for this solution is $\pm(\mu+1)$. Moreover if $H_\mu(x)$ satisfies the conditions of Corollary 1, then H_μ and $H_{\mu+1}$ are both geometrical. The metrics* (5) *of the corresponding surfaces are related by:*

$$e^{u(x)}_{\mu+1} = \frac{\tau\,(y_\mu(\tau)-1)^2}{(y_\mu(\tau)-\tau)^2}\, e^{u(x)}_{\mu}. \tag{29}$$

Proof: Since $H_\mu(x)$ does not depend on the sign of μ (see (7)), we can take (25) to define $y_{-(\mu+1)}(\tau)$ via (20). Now using (15) but with $-(\mu+1)$ instead of μ leads to an expression for $H_{\mu+1}(x)$. Cancelling the part $H_\mu(x)$ (see (15)) gives (28).

For the second part of the lemma one should take Corollary 1 and (16), where $y_{-(\mu+1)}(\tau)$ is inserted and the term $e^{u_\mu(x)}$ is cancelled. This means (29). □

To illustrate that one can use Lemma 5 to create new solutions of (6) (as well as of (13), (14)) we give the following (non-geometrical)

EXAMPLE 2. Taking the one-parameter family of solutions of (6) ($\alpha_B = 4$)

$$H_{\pm1}(x) = \frac{1}{\alpha - x}$$

which have the first integral $\mu = \pm 1$. The corresponding solution family of the Painlevé VI (13), (14) with $\mu = 1$ is

$$y_1(\tau) = \tau\, \left(\frac{1-\tau+8\,\alpha+\log(\tau)}{1-\tau+\tau(8\,\alpha+\log(\tau))}\right)^2.$$

Now (28) leads the one-parameter family of solutions for $\mu = 2$

$$H_{\pm2}(x) = 8\,\frac{1 - e^{16\,x} - 16\,e^{8\,x}(\alpha - x)}{(e^{8\,x}-1)^2 - 64\,e^{8\,x}\,(\alpha-x)^2}.$$

By view of (25) we find directly

$$y_{-2}(\tau) = \left(\frac{1-\tau+\tau(8\,\alpha+\log(\tau))}{1-\tau+8\,\alpha+\log(\tau)}\right)^2.$$

Recursively one can create a discrete set of one-parameter families of solutions of (6) rational in $e^{8\,x}$ and x and simultaneously solutions of (13), (14) for any $\mu \in \mathbb{Z}$.

Finally let me describe the Schlesinger transformation on the level of the moving frame. This will be handled in the following

LEMMA 6. *Let $y(\tau) = y_\mu(\tau)$ be a geometrical solution of* (13), (14) *with arbitrary $\mu \in \mathbb{R}$. $\Psi(x,y) = \Psi^\mu(x,y)$ denotes the solution of the corresponding linear system* (9). *Define a moving frame $F_x^\mu(x,y)$, $F_y^\mu(x,y)$ and $N^\mu(x,y)$ by* (2). *Then for the Schlesinger-transformed solution $y_{\mu+1}(\tau)$ the moving frame defined by the similar procedure $F_x^{\mu+1}(x,y)$, $F_y^{\mu+1}(x,y)$ and $N^{\mu+1}(x,y)$ is related to initial frame as follows:*

$$\begin{pmatrix} F_y^{\mu+1}(x,y) \\ F_y^{\mu+1}(x,y) \\ N^{\mu+1}(x,y) \end{pmatrix} = S(x,y) \begin{pmatrix} F_y^{\mu}(x,y) \\ F_y^{\mu}(x,y) \\ N^{\mu}(x,y) \end{pmatrix}, \tag{30}$$

where

$$S(x,y) = \begin{pmatrix} -\sqrt{(1-\omega_1^2)(1-\omega_2^2)} & 0 & -\omega_1\sqrt{1-\omega_2^2} \\ 0 & 0 & 0 \\ -\omega_2\sqrt{1-\omega_1^2} & 0 & -\omega_1\,\omega_2 \end{pmatrix}$$

$$+ \begin{pmatrix} -\omega_1\,\omega_2\cos(4\,y) & \omega_2\sin(4\,y) & \omega_2\sqrt{1-\omega_1^2}\cos(4\,y) \\ \omega_1\sin(4\,y) & \cos(4\,y) & -\sqrt{1-\omega_1^2}\sin(4\,y) \\ \omega_1\sqrt{1-\omega_2^2}\cos(4\,y) & -\sqrt{1-\omega_2^2}\sin(4\,y) & -\sqrt{(1-\omega_1^2)(1-\omega_2^2)}\cos(4\,y) \end{pmatrix},$$

with

$$\omega_1 = \omega_1(\tau) = \frac{1+y(\tau)}{1-y(\tau)}, \qquad \omega_2 = \omega_2(\tau) = \frac{\tau+y(\tau)}{\tau-y(\tau)}$$

and $y(\tau)$ a geometrical solution of (13), (14).

Proof: The proof follows directly from (23) with (2) and the isomorphism (1) between $su(2)$ and $\mathbb{R}^3$. Of course the moving frames are given in a natural way by the partial derivatives (and there cross product) along the parameterization $F^\mu(x,y)$ and $F^{\mu+1}(x,y)$ respectively. □

REMARK 9. In general the matrix $S(x,y)$ for arbitrary $y(\tau)$ has determinant 1 and the columns (rows) are orthogonal to each other. But they are real if and only if the function $y(\tau)$ is geometrical.

Appendix

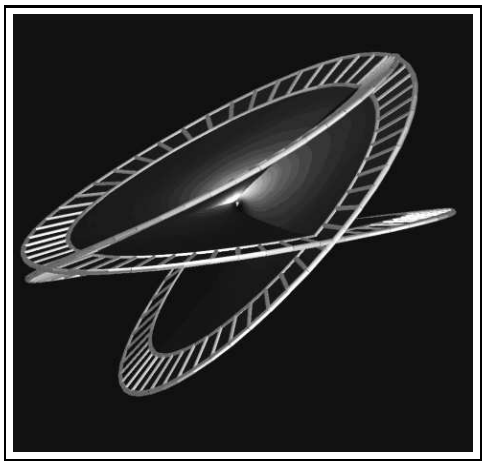

Fig. 1: An initial Bonnet B-Surface.

Figure 1 is numerically produced by a solution of (6) ($\alpha_B = 4$) the fundamental functions (5) for the Bonnet B-case and the (numerical) integration of the corresponding moving frame equation (3). The following pictures are generated (also numerically) by the Schlesinger transformation for the fundamental functions (28) and (29) and for the moving frame (30). From this pictures one can guess that periodicity properties on the surfaces (the conditions for periodicity are not analytically described up to now) are preserved by the Schlesinger transformation.

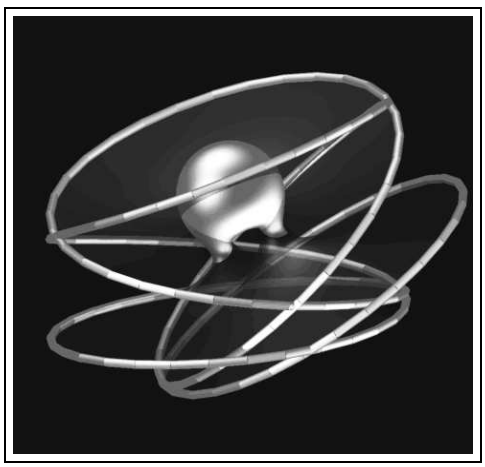

Fig. 2: 1-fold Schlesinger transform.

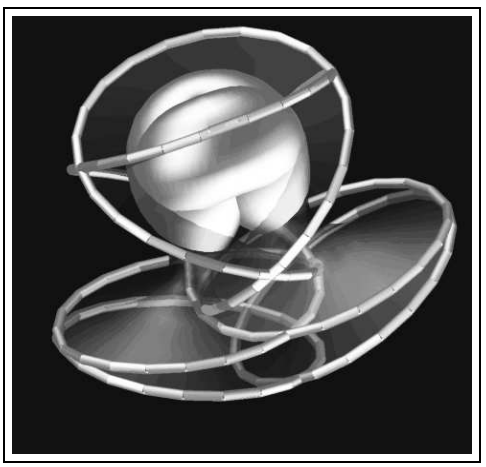

Fig. 3: 2-fold Schlesinger transform.

REFERENCES

[1] A. I. Bobenko and U. Eitner: *Bonnet Surfaces and Painlevé Equations.* to appear in *Crelles Journal.*

[2] A. I. Bobenko: *Constant Mean Curvature Surfaces and Integrable Equations. Russ. Math. Surv.* **46 (4)** (1991), 1–45.

[3] M. T. Jimbo and T. Miwa: *Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients, II. Constant Mean Curvature Surfaces and Integrable Equations Physica* **2 D** (1981), 407–448.

[4] A. S. Fokas, U. Mugan and M. J. Ablowitz: *A Method of Linearization for Painlevé Equations IV, V Physica* **D 30** (1988), 247–283.

[5] A. V. Kitaev: *An Isomonodromy Cluster of Two Regular Singularities.* SFB 288 Preprint **149**, TU Berlin, (1994).

[6] E. Cartan: *Sur les Couples de Surfaces Applicables avec Conversation des Courbieres Principles, Bull. Sc. Math* **66** (1942), 1–30.

[7] A. I. Bobenko: *Surfaces in Terms of 2 by 2 Matrices. Old and New Integrable Cases.* In: A. P. Fordy and J. Wood (eds.) *Harmonic Maps and Integrable Systems,* Braunschweig Wiesbaden, Vieweg 1994.

[8] J. N. Hazzidakis: *Biegung mit Erhaltung der Hauptkrümmungsradien. Journal für Reine und Angewandte Mathematik* **117** (1887), 42–56.