Geometry of surfaces in S^4

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Abstract: The geometry of immersions f of a Riemann surface S into the four-sphere S^4 may be explored via the geometry of certain related maps which can be described naturally in terms of the tangent and normal bundles of f. These are lifts into quotient spaces of the form $SO(5)/K$, where $K \subset SO(5)$ is a subgroup of maximal rank. For conformal, minimal immersions the differential equations satisfied by these lifts are the $SO(5)$-Toda equations. A particularly interesting case to look at is given by $K = U(2)$, since $SO(5)/U(2)$ is $\mathbb{CP}^3$, the twistor space of S^4. As is well-known, the twistor fibration has been used by Bryant to give a one-to-one correspondence between superminimal immersions f and holomorphic, horizontal curves in $\mathbb{CP}^3$. The talk gives results for minimal f and indicates the link with the $SO(5)$-Toda equations.

1 Introduction

In this article we study immersions $f : S \to S^4$ of a surface S into the four-sphere in terms of certain related maps constructed in a natural way from f. Identifying S^4 with the homogeneous space $SO(5)/SO(4)$ of the special orthogonal group $SO(5)$, these related maps are lifts into $SO(5)/T^2$, where T^2 is a maximal torus of $SO(5)$. Properties of the immersion are then reflected by properties of its lift.

The approach is particularly fruitful and interesting in the case of minimal surfaces in S^4. This is because the space $SO(5)/T^2$ has 4-fold symmetry at each point and minimal surfaces (or more generally, weakly conformal harmonic maps) are characterised as having lifts which respect this symmetry in a simple geometrical way. The symmetry aspects are also reflected in the fact that the lifts correspond to solutions of the $SO(5)$-Toda equations. Furthermore, a lift $\hat{f} : S \to SO(5)/T^2$ is weakly conformal and harmonic and has the property that if K is a subgroup of $SO(5)$ containing T^2 then $\sigma_K \hat{f} : S \to SO(5)/K$ is also weakly conformal and harmonic, where $\sigma_K : SO(5)/T^2 \to SO(5)/K$ is the natural projection. The case where K is the unitary group $U(2)$ is particularly interesting since $SO(5)/U(2)$ may be identified with the complex projective space $\mathbb{CP}^3$ and the resulting map $\sigma_{U(2)}\hat{f} : S \to SO(5)/U(2)$ is the twistor lift of f studied by Bryant and others.

Extensions of these ideas to other types of immersions are the subject of ongoing research for the author's doctoral thesis.

In order to motivate the ideas involved we begin with a simple preliminary example in Section 2, followed in Section 3 by a description of the use of moving frames to construct a local lift to $SO(5)$ and an explanation of how this gives rise to a global lift into $SO(5)/T^2$. The case of subgroups K of maximal rank and the corresponding maps into $SO(5)/K$ are discussed in Section 4. We identify these homogeneous spaces with certain flag manifolds and this gives a more geometrical way of describing the lifts. Section 5 takes the special case where K is $U(2)$ and the twistor theory for $\mathbb{CP}^3$ is discussed. Results for minimal immersions are given in Section 6 and in Section 7 the link with the Toda equations is explained.

2 A preliminary example

As a preliminary example, we consider a curve on the surface of the two-sphere in $\mathbb{R}^3$, i.e. an immersion $\alpha : I \to S^2$ (α parametrised by arc length). There is a natural moving frame description of the geometry of α given by $e_1 = \alpha'$, $e_2 = N_\alpha$, $e_3 = \alpha$, where N_α is the unit normal to α in S^2. Since the only choice involved in this description is the choice of orientation of the curve α, this frame is essentially unique and thus the lift of α into the space $SO(3)$ via

$$\alpha(t) \longmapsto E = (e_1 \mid e_2 \mid e_3) = (\alpha'(t) \mid N_\alpha(t) \mid \alpha(t))$$

is globally defined. The lift has a single invariant, namely the geodesic curvature κ_g of the curve α. This comes from the rate of change of frame $E = (e_1 \mid e_2 \mid e_3)$, for

$$E^{-1}dE = \begin{pmatrix} 0 & -\kappa_g & 1 \\ \kappa_g & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

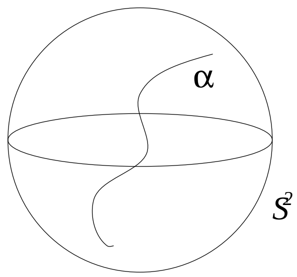

Fig. 1: A curve α on the surface of the 2-sphere.

We recall that geodesics, which are great circles, are curves which are locally length minimising. These are characterised by $\kappa_g = 0$ and the geodesic equations arise as Euler–Lagrange equations from a variational problem.

3 Moving frames for S^4

Let us now extend these ideas to immersions $f : S \to S^4$ of oriented surfaces into the four-sphere, $S^4 = \{x \in \mathbb{R}^5 \mid x \cdot x = 1\}$. We notice that the dimension of such an immersion is the same as the co-dimension, so this is a natural higher-dimensional analogue of the above example and we can again produce a moving frame for f from the geometry of the map.

At each point $p \in S$, the tangent bundle to S^4 restricted to S (considered at the point p) splits as

$$T_{f(p)}S^4 = T_pS \oplus N_pS$$

where T_pS is the tangent space and N_pS the normal space to S in S^4 and both T_pS and N_pS are 2-planes. Since the normal to S^4 is f itself we write

$$\mathbb{R}^5 = [f(p)] \oplus T_pS \oplus N_pS$$

where $[f]$ is the line in $\mathbf{R}^5$ determined by f.

Now choose an oriented orthonormal basis e_1, e_2 for T_pS and e_3, e_4 for N_pS so that e_1, e_2, e_3, e_4 gives the standard orientation on S^4, i.e.

$$\mathbb{R}^5 = [f] \oplus \operatorname{span}\{e_1, e_2\} \oplus \operatorname{span}\{e_3, e_4\}.$$

Thus given any $f : S \to S^4$, we have a moving frame

$$E = (f \mid e_1 \mid e_2 \mid e_3 \mid e_4) \in SO(5).$$

Compared with the case of a curve on S^2 above, this frame is more complicated and has more invariants, but the significant difference is that we cannot choose this frame globally since we have picked an orthonormal basis in each of the 2-planes T_pS and N_pS. However the frame E is unique up to rotations in T_pS and N_pS.

The compact Lie group $SO(5)$ has maximal torus given by

$$T^2 = \left\{ \begin{pmatrix} 1 & & & & \\ & \cos\theta & -\sin\theta & & \\ & \sin\theta & \cos\theta & & \\ & & & \cos\phi & -\sin\phi \\ & & & \sin\phi & \cos\phi \end{pmatrix} \mid \theta, \phi \in \mathbb{R} \right\}.$$

Thus we can obtain a lift which is globally defined simply by lifting f to $SO(5)/T^2$ instead of all the way up to $SO(5)$. Then the properties of f determine those of the lift, and vice versa. We interpret this lift in another way in the following section.

4 Subgroups of maximal rank

As remarked in the previous section, the maximal torus of $SO(5)$ is $T^2 = S^1 \times S^1$, so we see that $SO(5)$ has rank 2. A connected subgroup of a Lie group is of *maximal rank* if it contains a maximal torus. For $SO(5)$ the subgroups of maximal rank, up to conjugacy, are T^2, $U(2)$, $SO(2) \times SO(3)$ and $SO(4)$. These may be obtained using Lie algebra methods — see for example [15, 20]. Then

$$T^2 \subset U(2) \subset SO(4) \subset SO(5) \quad \text{and} \quad T^2 \subset SO(2) \times SO(3) \subset SO(5)$$

and, for any subgroup K of maximal rank, the inclusion $T^2 \subset K$ induces a projection $\sigma_K : SO(5)/T^2 \to SO(5)/K$. Thus given a lift $\hat{f} : S \to SO(5)/T^2$ we have a map $\tilde{f}_K = \sigma_K \hat{f} : S \to SO(5)/K$. The spaces $SO(5)/T^2$ and $SO(5)/K$ for $K = SO(2) \times SO(3)$ and $U(2)$ are complex manifolds and σ_K is complex analytic (see [17]).

$$\begin{tikzcd}
 & SO(5) \arrow[d] & \\
 & SO(5)/T^2 \arrow[d] \arrow[dr] & \\
 & SO(5)/U(2) \arrow[d] & SO(5)/SO(2)\times SO(3) \\
S \arrow[r] & SO(5)/SO(4) &
\end{tikzcd}$$

Fig. 2: The homogeneous spaces of $SO(5)$.

The homogeneous spaces $SO(5)/K$ may be identified as different types of generalised flag manifolds. Their elements are given in terms of different descriptions of $\mathbb{R}^5$ as orthogonal direct sum decompositions of oriented subspaces. In each case $SO(5)$ acts on the flags and K is the stabiliser of a typical flag.

For example, in the case of $SO(5)/T^2$ the elements are direct sums of oriented subspaces

$$\mathbb{R}^5 = L \oplus V_1 \oplus V_2, \qquad \dim L = 1,\ \dim V_i = 2\ (i = 1, 2),$$

with the orientation induced on $\mathbb{R}^5$ by those on L, V_1 and V_2 agreeing with the standard one. For $SO(5)/U(2)$ the decompositions are of the form

$$\mathbb{R}^5 = L \oplus V, \qquad \dim L = 1,\ \dim V = 4,$$

with V having an orthogonal complex structure. This is a particularly interesting case since $SO(5)/U(2)$ may also be identified both with the total space of the bundle of orthogonal almost complex structures on S^4 and with the space of maximal isotropic subspaces of $\mathbb{C}^5$ (see [1]).

For $SO(5)/SO(2) \times SO(3)$ the decompositions are of the form

$$\mathbb{R}^5 = W \oplus W^{\perp}, \qquad \dim W = 2.$$

Finally, for $SO(5)/SO(4)$ the decompositions are of the form

$$\mathbb{R}^5 = L \oplus V, \qquad \dim L = 1,\ \dim V = 4,$$

and $SO(5)/SO(4) = S^4$.

Note that the projection maps in Fig. 2 may now be understood in terms of 'forgetting' certain properties. For example, if $(L, V_1, V_2) \in SO(5)/T^2$ then both V_1 and V_2 have a natural orthogonal complex structure and hence so does $V = V_1 \oplus V_2$. The projection $\sigma_{U(2)} : SO(5)/T^2 \to SO(5)/U(2)$ sending (L, V_1, V_2) to (L, V) 'forgets' the decomposition of V.

From the discussion in Section 3 it is now clear how the lifts may be identified. For $\hat{f} : S \to SO(5)/T^2$ we have

$$\hat{f}(p) = ([f(p)], T_p(S), N_p(S))$$

and the others can be immediately written down from this. In particular note that the map $g : S \to SO(5)/SO(2) \times SO(3)$ which is given by $g(p) = (T_p(S), T_p(S)^{\perp})$ is just the Gauss map of $f : S \to S^4$.

5 Twistor space

In the early 1970's, Roger Penrose began to develop a scheme to interpret aspects of real four-dimensional geometry in holomorphic terms which is now commonly referred to as the twistor programme. A useful overview of the work done using this programme in the study of harmonic maps is given in [2] (or see [9]) with geometrical aspects in the case of harmonic maps into 4-manifolds being discussed in the work of Eells-Salamon [12]. We give here a brief outline of the twistor formulation for the case of S^4.

Consider the quaternions

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

Notice that since $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$ the quaternion $q = a + bi + cj + dk \in \mathbb{H}$ has conjugate $\bar{q} = a - bi - cj - dk$. Also q can be written uniquely as

$$q = z + wj, \text{ where } z = a + bi, \ w = c + di \in \mathbb{C}.$$

and thus $\mathbb{H}$ is essentially $\mathbb{C}^2$. Moreover we have $zj = j\bar{z}$ and so

$$(z_1 + w_1 j)(z_2 + w_2 j) = z_1 z_2 - w_1 \bar{w}_2 + (w_1 \bar{z}_2 + z_1 w_2) j.$$

Further if $\mathbb{H}^2$ is the space of pairs (q_1, q_2), $q_i \in \mathbf{H}$, then $\mathbb{H}^2$ can be identified with $\mathbb{C}^4$ via

$$(z_0, z_1, z_2, z_3) \longleftrightarrow (z_0 + z_1 j, z_2 + z_3 j).$$

The Hopf map is given by

$$P(\mathbb{C}^4) = \mathbb{CP}^3 \xrightarrow{\rho} \mathbb{HP}^1 = P(\mathbb{H}^2)$$

and assigns a complex line in $\mathbb{C}^4$ to the corresponding quaternionic line in $\mathbb{H}^2$, i.e.

$$v\mathbb{C} \to v\mathbb{H}, \qquad v \in \mathbb{H}^2 \backslash \{(0,0)\}.$$

We now notice that $\mathbb{HP}^1$ may be identified with S^4 ($\mathbb{HP}^1$ is compact, simply connected and is a space of constant curvature $+1$). In fact we can write down explicitly the canonical identification of these two spaces:

$$[q_1, q_2] \in \mathbb{HP}^1 \longleftrightarrow \frac{(2\bar{q}_1 q_2\ ,\ |q_1|^2 - |q_2|^2)}{|q_1|^2 + |q_2|^2} \in S^4 \subset \mathbb{H} \oplus \mathbb{R} = \mathbb{R}^5.$$

Composing this identification with ρ we have a map $\pi : \mathbb{CP}^3 \to S^4$ (in fact a Riemannian submersion) which takes

$$[z_0, z_1, z_2, z_3] \mapsto \frac{(2\overline{(z_0 + z_1 j)}(z_2 + z_3 j)\ ,\ |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2)}{|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2}$$

and this is called the twistor fibration.

Then for any immersion $f : S \to S^4$ (which is not totally geodesic on an open subset of S) there exists a unique twistor lift to the twistor space $\mathbb{CP}^3$, say $\tilde{f} : S \to \mathbb{CP}^3$ and $f = \pi\tilde{f}$. The idea is to characterise immersions in terms of their twistor lifts and this is particularly fruitful for minimal immersions.

6 Results for minimal immersions

Recall that a minimal surface is a special constant mean curvature surface, namely one for which the mean curvature $H \equiv 0$. Minimal surfaces are locally area minimising, and arise as solutions to the Euler–Lagrange equations for certain variational problems. A conformal immersion $f : S \to S^4$ of a Riemann surface S is minimal if and only if it is harmonic — that is to say that if $z = x + iy$ is a local complex coordinate on S, then f satisfies $\Delta f = -2f$, where Δ is the Laplacian with respect to the metric on S.

We recall that the second fundamental form of an immersion is a bilinear form defined on TS and taking values in NS, denoted by $I\!I_S^{S^4}$. Since the mean curvature $H =$ trace $I\!I_S^{S^4}$, the minimality condition for f is written in terms of the orthonormal basis $\{e_1, e_2\}$ (given in Section 3) as

$$I\!I_S^{S^4}(e_1, e_1) + I\!I_S^{S^4}(e_2, e_2) = 0.$$

Let $e_\theta \in TS$ with $e_\theta = \cos\theta\, e_1 + \sin\theta\, e_2$. Then

$$I\!I_S^{S^4}(e_\theta) = \cos 2\theta\, I\!I_S^{S^4}(e_1, e_1) + \sin 2\theta\, I\!I_S^{S^4}(e_1, e_2)$$

traces out an ellipse, called the *ellipse of curvature* of the immersion. It is not hard to show that $I\!I_S^{S^4}(e_1, e_1)$, $I\!I_S^{S^4}(e_1, e_2)$ are mutually orthogonal vectors in NS; indeed they correspond to the major and minor axes of the ellipse. A special case occurs when these vectors have the same length, i.e. the second fundamental form has the property

$$|I\!I_S^{S^4}(e_1, e_1)|^2 = |I\!I_S^{S^4}(e_1, e_2)|^2$$

and the ellipse is a circle. We call an immersion f with circular ellipse of curvature a *superminimal* immersion. The minimal 2-spheres in S^4 studied by Calabi [10], Barbosa [1] and in particular by Chern [11] are examples of superminimal immersions.

Lifting f to the complex manifolds $SO(5)/T^2$, $SO(5)/U(2)$ enables us to use complex analysis techniques to solve problems involving harmonic maps.

Given a compact Lie group G with maximal torus T, we recall that the homogeneous space G/T has some nice symmetry properties. In fact G/T is an m-symmetric space, where m is an integer arising naturally in the Lie algebra theory of G. Now $G = SO(5)$ has maximal torus T^2 as given in Section 3, and $SO(5)/T^2$ has a 4-symmetric space structure. This means that at each point $x \in SO(5)/T^2$ there is a geodesic symmetry of $SO(5)/T^2$ of order 4 which has x as an isolated fixed point. The symmetry comes from a canonically constructed automorphism called the Coxeter automorphism of the Lie algebra of G (cf. S^2 where $m = 2$ and the symmetry reverses the direction of geodesics through a point).

From Section 4,

$$SO(5)/T^2 = \{L \oplus V_1 \oplus V_2 \mid \dim L = 1,\ \dim V_i = 2\}$$

and $\tau_x : SO(5)/T^2 \to SO(5)/T^2$ is conjugation by the matrix

$$\begin{pmatrix} 1 & & & & \\ & 0 & -1 & & \\ & 1 & 0 & & \\ & & & -1 & 0 \\ & & & 0 & -1 \end{pmatrix}.$$

This corresponds to a rotation in each of the 2-planes V_1 and V_2 by $\frac{\pi}{2}$ and π respectively. Thus geometrically, if $\hat{f} : S \to SO(5)/T^2$ is the lift of a harmonic map $f : S \to S^4$ then for all $p \in S$, $d\hat{f}_p(T_pS)$ is mapped into itself by the symmetry at $\hat{f}(p)$, which is a rotation through $\frac{\pi}{2}$ [2].

For minimal, superminimal $f : S \to S^4$, the lifts to $SO(5)/T^2$ and $\mathbb{CP}^3$ are holomorphic curves. In each case these curves have some additional symmetry as discussed above. In particular, for the lift to $\mathbb{CP}^3$ the holomorphic curve has a Frenet frame $(\phi_0, \phi_1, \phi_2, \phi_3)$ (see [14]) and the symmetry on $\mathbb{CP}^3$ interchanges ϕ_j and ϕ_{3-j}.

The minimal, non-superminimal case is similar except that the lifts are not holomorphic. However they are harmonic, and the lift to $\mathbb{CP}^3$ belongs to a sequence $\{\ldots, \phi_0, \phi_1, \phi_2, \phi_3, \ldots\}$ of harmonic maps, referred to in the literature as the *harmonic sequence* of the lift [4, 6]. This sequence is orthogonally periodic in the sense that any four consecutive elements are mutually orthogonal and $\phi_{4+p} = \phi_p$ for all p. This is the analogue of the Frenet frame which appears in the superminimal case.

More detailed results for lifts of superminimal f to $\mathbb{CP}^3$ have been obtained by Bryant [8]. Indeed, if $f : S \to S^4$ is superminimal then its twistor lift is a holomorphic and horizontal curve in $\mathbb{CP}^3$ and conversely if $\psi : S \to \mathbb{CP}^3$ is holomorphic and horizontal then $\pi\psi$ is superminimal. The following theorem shows that this (essentially) gives a way of describing all superminimal immersions.

THEOREM . (Bryant). *If h, g are meromorphic functions $h, g : S \to \mathbb{C}$ with g non-constant, then*

$$\Phi(h,g) = [1, h - \frac{g}{2}\frac{dh}{dg}, g, \frac{1}{2}\frac{dh}{dg}] \tag{1}$$

is holomorphic and horizontal. Moreover, apart from Gauss maps of geodesic 2-spheres in S^4 for which $\Phi(h,g) \subset \mathbb{CP}^1$, every non-constant holomorphic horizontal curve in $\mathbb{CP}^3$ is of this form.

This "Weierstrass type" formula for Φ establishes the equivalence of algebraic plane curves $\mathcal{C} \subseteq \mathbb{CP}^2$ and holomorphic, horizontal curves in $\mathbb{CP}^3$. In fact to produce examples of superminimal surfaces in S^4 it is enough to start with a curve $\mathcal{C}$ defined as $[X, Y, 1]$ satisfying $F(X, Y) = 0$. Then formula (1) shows that the set of points $[z_0, z_1, z_2, z_3]$ such that

$$z_0 = 2\frac{\partial F}{\partial Y}, \qquad z_1 = 2Y\frac{\partial F}{\partial Y} + X\frac{\partial F}{\partial X}, \qquad z_2 = 2X\frac{\partial F}{\partial Y}, \qquad z_3 = -\frac{\partial F}{\partial X},$$

is a holomorphic, horizontal curve in $\mathbb{CP}^3$ and hence projects under π to a superminimal surface in S^4.

7 Link with the Toda equations

A solution of the $SO(5)$-Toda equations corresponds to a certain harmonic map $\tilde{f}_K$ into each of the quotient spaces $SO(5)/K$, where K has maximal rank. The $\tilde{f}_K$ are all related and understanding the way in which they are related is an interesting problem. In particular, the superminimal case corresponds to the open $SO(5)$-Toda equations and the minimal, non-superminimal case to that of affine $SO(5)$-Toda.

Given an orthonormal frame $e_0, \ldots, e_n$ in $\mathbb{R}^{n+1}$ we write

$$de_j = \omega^i_j e_i$$

where the ω^i_j are 1-forms. Then $dde_j = 0$ gives rise to the Maurer–Cartan equations

$$d\omega^i_j + \omega^i_r \wedge \omega^r_j = 0. \tag{2}$$

If $E = (e_0 \mid \ldots \mid e_n)$ is a frame, then the rate of change of this frame is given as $dE = EA$, where $A = (\omega^i_j)$ is the matrix of 1-forms whose entries are the ω^i_j described above.

The integrability condition for the frame E is the requirement that

$$\frac{\partial^2 E}{\partial z \partial \bar{z}} = \frac{\partial^2 E}{\partial \bar{z} \partial z}. \tag{3}$$

Writing $A = A_z dz + A_{\bar{z}} d\bar{z}$ where $A_z = E^{-1}\frac{\partial E}{\partial z}$, $A_{\bar{z}} = E^{-1}\frac{\partial E}{\partial \bar{z}}$, the condition (3) becomes

$$\frac{\partial A_{\bar{z}}}{\partial z} - \frac{\partial A_z}{\partial \bar{z}} + [A_z, A_{\bar{z}}] = 0. \tag{4}$$

It is not hard to show that equations (4) are in fact the Maurer–Cartan equations (2) written in terms of A. (For further results on the adapted frame E and the connection form A see [13].)

We claim that for a conformal and minimal immersion f these integrability conditions are non-other than the Toda-equations for the group $SO(5)$. To see this, we must show that the conformality and harmonicity conditions for f imply (see for example [5]) that A_z has the form

$$A_z = \frac{\partial \Omega}{\partial z} + e^{\Omega} B e^{-\Omega}$$

where

$$\Omega = \begin{pmatrix} 0 & & & & \\ & 0 & i\omega & & \\ & -i\omega & 0 & & \\ & & & 0 & i\eta \\ & & & -i\eta & 0 \end{pmatrix} \in it$$

(t is the Lie algebra of T^2) and B is a certain matrix of constants.

Since $\bar{\Omega} = -\Omega$, equations (4) become

$$\frac{\partial^2 \Omega}{\partial z \partial \bar{z}} + [e^{-\Omega} \bar{B} e^{\Omega}, e^{\Omega} B e^{-\Omega}] = 0. \tag{5}$$

In the affine case it can be shown that

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} & -1 & i & & \\ 1 & & & -\sqrt{2} & \\ -i & & & -\sqrt{2}i & \\ & \sqrt{2} & \sqrt{2}i & & \\ & & & & 0 \end{pmatrix}$$

and equations (5) are precisely the affine $SO(5)$-Toda equations:

$$\begin{aligned} \frac{\partial^2 \omega}{\partial z \partial \bar{z}} + e^{2\omega} - \frac{1}{2} e^{2(\eta-\omega)} - \frac{1}{2} e^{-2(\eta+\omega)} &= 0, \\ \frac{\partial^2 \eta}{\partial z \partial \bar{z}} + \frac{1}{2} e^{2(\eta-\omega)} - \frac{1}{2} e^{-2(\eta+\omega)} &= 0. \end{aligned}$$

The open case is similar, except that then

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} & -1 & i & & \\ 1 & & & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \\ -i & & & -\frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i & & \\ & -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} & & \end{pmatrix}$$

giving the open $SO(5)$-Toda equations:

$$\begin{aligned}\frac{\partial^2\omega}{\partial z\partial\bar z}+e^{2\omega}-\frac{1}{2}e^{2(\eta-\omega)}&=0,\\ \frac{\partial^2\eta}{\partial z\partial\bar z}+\frac{1}{2}e^{2(\eta-\omega)}&=0.\end{aligned}$$

We notice that the equations have 2 unknowns ω,η which we expect since the rank of $SO(5)$ is 2. These unknowns correspond to invariants in the geometrical setting; the induced metric is given by

$$ds^2=e^{2\omega}(dx^2+dy^2)$$

and the ellipse of curvature (see Section 6) has major and minor axes of length $\dfrac{\cosh\eta}{e^{2\omega}}$ and $\dfrac{|\sinh\eta|}{e^{2\omega}}$. Further details on this aspect may be found in [7].

Recall that in the superminimal case the ellipse of curvature is a circle and the invariants here correspond therefore to the metric and the radius of the circle. In the case where $\eta\equiv 0$, the ellipse degenerates to a line segment, and this corresponds to the case of an immersion $f:S\to S^3\subset S^4$. Minimal surfaces in S^3 have been discussed by Hitchin [16] and Lawson [18] (see also [3, 19]).

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