Painlevé test: standard gun of integrability hunter

Piotr P. Goldstein

Sołtan Institute for Nuclear Studies, Hoża 69, 00-681 Warsaw, Poland
e-mail: `goldstei@fuw.edu.pl`

Abstract: The lecture is a review of commonly used tests for the Painlevé property and various applications of those tests. In particular, application to classification of nonlinear differential equations (especially selection of the integrable cases) and construction of transformations for integrable systems is treated in more detail.
After a short review of the classical Painlevé method, the approach of Kovalevskaya (recently developed into an algorithm called the Painlevé test) is presented through a couple of examples. The extra flexibility inherent in the generalization of the test to partial differential equations is used in another example to construct the Bäcklund transformation. Various extensions of the test: to equations whose solutions possess singularities of fractional power type, to solutions with essential singularities and to an integro-differential system are discussed in a separate section. The review is completed by an example of discrete counterparts to the Painlevé property and the Painlevé test.

Introduction

According to Stanisław Ulam [1], science of nonlinear phenomena may be compared to zoology of non-elephant animals. Integrable nonlinear systems are not so rare as the linear ones therefore the title of my talk suggests comparison of those systems to game rather than elephants. There is no need of convincing this audience that this game is worth poaching. The test recently became standard equipment of its hunters. It is applied

– to distinguish between integrable and non-integrable,
– to provide further classification of the non-integrable,
– to process the integrable by construction of Bäcklund and Darboux transformations, Lax pair etc.

The method common to various versions of the test is based on extension of the unknown and the parameter functions to the complex plane (or the Cartesian product of complex planes) and examination of behaviour of the solutions in the vicinity of their singular points.

The singularities of solutions to ordinary differential equations (ODE's) are classified by their type: poles, branch points, essential singular points (the last two are known as the *critical points*), and by their dependence on integration constants: singularities whose location does not depend on initial (or boundary) conditions are **fixed** singularities while those singularities which depend on these conditions are called **movable.** Linear differential equations and systems have merely fixed singularities as the their general solutions depend linearly on the integration constants.

Let us begin with the classical Cauchy theorem:

If f is an analytic function in some Cartesian product of two disks centered at (z_0, u_0) then the initial value problem $u_{,z}(z) = f(z,u)$, $u(z_0) = u_0$ has a unique solution in some disk about z_0.

Hence the solution cannot have fixed singular points other than singularities of f. However, existence of solutions for which $u(z)$ becomes infinite and/or multivalued at some z_0, does not contradict the theorem. As the equation is invariant upon a shift in z_0, the position of that singularity depends on initial conditions. Hence it is a movable singularity. Extension to systems of equations is straightforward.

Imagine that we start from initial conditions and propagate the solution, step by step, according to the differrential equation (DE). Fixed singularities of any kind may be avoided by means of appropriate cuts on the complex plane. Poles prove to be harmless unless we step exactly on them. It is the movable critical points which cause difficulties in solving differential equations. A value of the solution at a point may depend on the path through which we get to it, i.e. on which side we pass the singular point. But the position of the singularity is not known! We cannot escape non-uniqueness: the unknown function may turn into a multivalued entity at an unpredictable position. Even the domains of solutions possessing movable critical points are unpredictable.

There definitely exists a connection between movable critical points and sensitivity to initial conditions, even if the points lie outside the real axis. Namely, small change in initial conditions may throw us onto a different sheet of the Riemann surface of the solution. On the other hand, we are far from full understanding of the connection between these properties and there are few exact results [2–6].

Absence of movable critical points is called the Painlevé property after Paul Painlevé, French mathematician and statesman, a colourful person: first passenger of an aeroplane, minister of war (partly during the first world war), minister of finances, also the prime minister of France for some time. He was the first one to stress the importance of absence of movable critical points for solvability of equations. Moreover, with his students and collaborators he performed complete classification of 2-nd order ODE's of the form

$$d^2u/dz^2 = F(z, u, p), \tag{1a}$$

$$p = du/dz \tag{1b}$$

where F was rational in p, algebraic in u and analytic in z.

He obtained 53 types of algebraically distinct equations (50 rational in u while 3 contained square root) 47 of which were integrable in terms of known elementary and special functions. The remaining 6 defined new transcendental functions, later called "Painlevé transcendents". The first four of them read [7]

$$d^2w/dz^2 = 6w^2 + z, \tag{2a}$$

$$d^2w/dz^2 = 2w^3 + zw + \alpha, \tag{2b}$$

$$d^2w/dz^2 = (1/w)(dw/dz)^2 - (1/z)(dw/dz) + (1/z)(\alpha w^2 + \beta), \tag{2c}$$

$$d^2w/dz^2 = (1/2w)(dw/dz)^2 + (3/2)w^3 + +4zw^2 + 2(z^2 - \alpha)w + \beta/w \tag{2d}$$

(the Greek letters denote constant parameters). Equations defining the fifth and sixth transcendents are more complicated, we shall not quote them here.

1 Painlevé's method

Painlevé performed his classification by means of a parametric approach ("the alpha method"). He introduced a parameter α into the 2-nd order equation written as a system of two first order ones

$$du/dz = H(z, u, w, \alpha), \tag{3a}$$

$$dw/dz = K(z, u, w, \alpha) \tag{3b}$$

and proved a theorem stating that a solution of such a system, single-valued when $\alpha \neq 0$, remains single-valued when $\alpha = 0$. In his method, the parameter α is introduced through

$$z = z_0 + \alpha Z, \tag{4a}$$

$$u = \sum_{n=0}^{\infty} \alpha^{n+p} U^{(n)}. \tag{4b}$$

This substitution transforms equation of the form $E(z; u) = 0$ into

$$\sum_{n=0}^{\infty} \alpha^{n+q} E^{(n)} = 0, \tag{5}$$

which — due to arbitrariness of α — turns into a system of simpler equations

$$E^{(n)} = 0 \quad \text{for all } n. \tag{6}$$

The equation corresponding to $n = 1$ is often easy to solve; whenever the solution has a movable critical point the original equation is non-Painlevé. This is a convenient way of sieving out the values of arbitrary parameters or even forms of arbitrary functions. On the other hand, solving the equations for $n > 1$ is not straightforward and complexity (of the equations to solve) rapidly increases with n. Therefore the method described in the next section, although not so complete as the Painlevé approach, is much more popular nowadays. So much that it bears the name "Painlevé test" although Painlevé himself had strong objections to it.

It should also be mentioned that a more refined, easier version of the parametric test was proposed by Bureau [8].

All these methods provide only the necessary condition for the P-property. Proving that the remaining (not sieved out) values of the parameters correspond to solutions without movable critical points is an extra task which Painlevé had to carry out for his transcendents.

2 The method of Kovalevskaya

The approach based on the Laurent expansion of a solution, named "the Painlevé test" was probably known by Weierstrass' group. The first paper containing a method resembling this test was P. Hoyer's dissertation of 1879 [9]. However the modern use of this test is due to Kovalevskaya who applied it with great success to the theory of a spinning top [10]. The selection of integrable cases provided by that test enabled her to find another solvable model of the top in addition to the previously known two models of Euler and Lagrange.

The Kovalevskaya approach was eventually made an algorithm by Ablowitz, Ramani and Segur in 1980 [2]. To see how it works, we consider a simple example

$$w_{,tt} = aw^2 + t, \tag{7}$$

where (from now on) differentiation is denoted by a comma followed by subscripts corresponding to the differentiation variables (repeated as many times as the order of differentiation); a is a parameter.

Let t_0 be an arbitrary pole. Denote $\tau := t - t_0$. The Laurent expansion of the solution has the form

$$w = \sum_{r=0}^{\infty} w_r \tau^{p+r}, \qquad p < 0. \tag{8}$$

If a parameter function of the independent variable appears explicitly in the equation, we should also expand it in powers of τ. Here (7) the expansion ends after two terms

$$t = t_0 + \tau\ . \tag{9}$$

Instead of inserting the whole expression (8) into (7), we first determine the leading order (l. o.) exponent and leading order coefficient in a simpler way.

STEP 1. The l. o. exponent. Substitute just

$$w = w_0 \tau^p + O(\tau^{p+1}) \qquad w_0 \neq 0. \tag{10}$$

The substituted terms read respectively

$$w_{,tt} = p(p-1) w_0 \tau^{p-2} + O(\tau^{p-1}), \tag{11a}$$

$$aw^2 + t = a{w_0}^2 \tau^{2p} + \tau + t_0 + O(\tau^{2p-1}). \tag{11b}$$

Since the leading order terms should cancel, we have

$$2p = p - 2, \qquad \text{whence} \quad p = -2. \tag{12}$$

STEP 2. The l. o. coefficient. From the same condition of cancellation we obtain (after substituting $p = -2$ into (11))

$$(-2)(-2-1)w_0 = a{w_0}^2 \qquad \text{whence} \quad w_0 = 6/a. \tag{13}$$

STEP 3. Calculate the resonances (see below) and

STEP 4. Check compatibility of the recurrence formulae at each resonance:

Substitute the expansions of both the solution (8) and the explicit independent variable (9) into the original equation

$$\sum_{r=0}^{\infty}(r-2)(r-3)w_r\tau^{r-4} = a\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} w_{r_1}w_{r_2}\tau^{r_1+r_2-4} + \tau + t_0. \tag{14}$$

After rearrangement of the double sum on the r.h.s. in order to compare coefficients at the same powers of τ, we obtain recurrence formulae from that comparison

$$(r+1)(r-6)w_r = \sum_{s=1}^{r-1} w_s w_{r-s} + \delta_{r,5} + t_0\delta_{r,4}. \tag{15}$$

where $\delta_{m,n}$ is the Kronecker symbol, $r = 1, 2....$

The above equation becomes indeterminate when $r = 6$. Such values of the coefficient number r are called "resonances" (recently the name "indices" was used in a couple of papers [11]). The values of the coefficients determined from the recurrence formulae are:

$$w_1 = w_2 = w_3 = 0; \qquad w_4 = t_0/10; \qquad w_5 = -1/6. \tag{16}$$

At the resonance $r = 6$ we obtain a compatibility condition: the r.h.s. should be zero. Trivial calculation shows that this condition is indeed satisfied. Hence w_6 is an arbitrary constant; bearing in mind that t_0 was also arbitrary, we have two arbitrary constants necessary to construct the general solution. The solution is just the series (8) as the higher recurrence formulae are all solvable. The equation has the P-property. Indeed it can be obtained from Painlevé-I equation (2a) by simple rescaling of variables. Note that the question of convergence of the Laurent series is not addressed in this approach (*Note added in proof:* Several interesting papers containing proofs of convergence were published in the meantime [5, 6]).

The second example will be the famous Lorenz model [12]. The calculation is based on the work of Levine and Tabor [13]. Now we have a system of 3 ODE's which is integrable for a few special values of its parameters σ, R, B only (for other values of these parameters the model has the well-known strange attractor).

$$X_{,t} = \sigma(Y - X), \tag{17a}$$

$$Y_{,t} = -XZ + RX - Y, \tag{17b}$$

$$Z_{,t} = XY - BZ. \tag{17c}$$

We assume as usually

$$X = \sum_{j=0}^{\infty} a_j \tau^{j+\alpha}, \quad Y = \sum_{j=0}^{\infty} b_j \tau^{j+\beta}, \quad Z = \sum_{j=0}^{\infty} c_j \tau^{j+\gamma}, \tag{18}$$

where $\alpha,\ \beta,\ \gamma < 0$.

1. equations for l. o. exponents

$$\alpha - 1 = \beta, \qquad \beta - 1 = \alpha + \gamma, \qquad \gamma - 1 = \alpha + \beta \tag{19}$$

whence

$$\alpha = -1 \qquad \beta = \gamma = -2. \tag{20}$$

2. The l. o. coefficients

$$a_0 = \pm 2i; \quad b_0 = \mp 2i/\sigma; \quad c_0 = -2/\sigma\ . \tag{21}$$

Analysis of the two families of solutions, differing by the $\pm$ signs, is equivalent. In the following calculation we consider the upper sign family.

3. Solving of the recurrence formulae reduces to a simple linear algebraic problem.

$$\begin{pmatrix} j-1 & -\sigma & 0 \\ -2/\sigma & j-2 & 2i \\ 2i/\sigma & -2i & j-2 \end{pmatrix} \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} = \begin{pmatrix} & r.h.s. & \end{pmatrix}. \tag{22}$$

The determinant of the matrix M of coefficients in (22) equals

$$\det M = (j+1)(j-2)(j-4). \tag{23}$$

As a rule, the determinant turns into zero at $j = -1$ and this fact is connected with arbitrariness of location of the singularity, i.e. with the constant t_0. The other zeros of the determinant correspond to the following linear dependence of the rows of M

$$\begin{array}{lll} \text{For} \quad j = 2 & \text{3rd row} = & (2i/\sigma)\times \text{1st row}, \\ \phantom{\text{For}} \quad j = 4 & \text{3rd row} = & (-i)\times \text{2nd row}. \end{array} \tag{24}$$

The same linear dependence should hold for the r.h.s. matrix. Hence the first two compatibility conditions follow

$$\begin{aligned} j = 2: \qquad & 6\sigma^2 - B\sigma - 2\sigma = B(B-1), \\ j = 4: \qquad & \sigma^2(21 - 3B) + B^2(13 + 2\sigma) + 44\sigma + 44\sigma + \\ & +4B - 55B\sigma - 17 = (\text{two line expression})\, a_2. \end{aligned} \tag{25}$$

Note that the coefficient a_2 which may be taken arbitrary, provided that the compatibility condition at $j = 2$ is satisfied, is inherent in the compatibility condition at $j = 4$. This makes us split the equation of compatibility (25) in two conditions because the condition at $j = 4$ should be satisfied for all values of a_2. Hence each hand side of (25) must vanish separately. Eventually we get three cases in which the P-test is passed (the equations (17) are "Painlevé integrable")

$$\begin{array}{llll} \sigma = 1/2, & B = 1, & R = 0 & \text{solution}: \text{ Jacobi elliptic,} \\ \sigma = 1, & B = 2, & R = 1/9 & \qquad\qquad \text{Painlevé II eq.,} \\ \sigma = 1/3, & B = 1, & R \text{ arbitrary} & \qquad\qquad \text{Painlevé III eq.} \end{array} \tag{26}$$

This example illustrates application of the P-test for selecting values of arbitrary parameters for which the equations are integrable. It may also help to determine those values of the parameters for which the equations have smaller number of first integrals than necessary for complete integration (see [13] for details).

In all cases (P or non-P) the solution may be represented in the form of double series containing logarithmic terms. For the Lorenz model (17), the double expansion reads [13]

$$\begin{aligned} X &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}\tau^{j-1}(\tau^2\ln\tau)^k, \\ Y &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{jk}\tau^{j-2}(\tau^2\ln\tau)^k, \\ Z &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\tau^{j-2}(\tau^2\ln\tau)^k. \end{aligned} \tag{27}$$

Substitution of this expansion into (17) may also provide useful information on behaviour of the solution but that analysis lies beyond the scope of our lecture.

3 Main generalizations of the test

The most important generalization of the P-test is its extension to partial differential equations (PDE's) performed by Weiss Tabor and Carnevale in 1983 [14]. They were able to use the test not only to check PDE's for integrability but also to construct the Bäcklund transformation, Lax pair and other tools of integration.

It follows from a well-known theorem on many-variable complex functions that singularities of functions of d variables lie on manifolds of dimension $d - 1$. An equation is said to have the P-property when its general solution has no movable critical manifold. If we have an evolution equation in which x represents the spatial independent variables, then the singular manifold may be defined by equation

$$\Phi(x, t) = 0. \tag{28}$$

The Laurent series of the unknown function is replaced by expansion in powers of Φ. In principle, the manifold should be non-characteristic [15], however cases, in which behaviour in a vicinity of a characteristic manifold is critical, have been pointed out by Weiss [16].

We begin with a simple example in two dimensions: a family of equations

$$u_{,t} + u^k u_{,x} - u_{,xx} = 0, \qquad k \ - \ \text{a positive integer.} \tag{29}$$

1) The l. o. exponent. Substitute

$$u = u_0 \Phi^p + O(\Phi^{p+1}), \tag{30}$$

u, u_0, Φ depend on (x, t). We get

$$(k+1)p - 1 = p - 2. \tag{31}$$

The solution $p = -1/k$ corresponds to a k-fold branch of the solution. Hence, only for $k = 1$ equation (29) may have the Painlevé property. This case is the classical Burgers equation. It has the P-property indeed: for $k = 1$ the l. o. exponent is $p = -1$, the l. o. coefficient $u_0 = -2\Phi_{,x}$, the remaining coefficients are calculated from the recurrence formula

$$\begin{aligned}\Phi_{,x}{}^2 (j+1)(j-2) u_j = {} & (j-2)(\Phi_{,t} - \Phi_{,xx}) u_{j-1} - 2(j-1)\Phi_{,x} u_{j-1,x} + \\ & u_{j-2,t} - u_{j-2,xx} + \sum_{s=1}^{j-1} u_s \left[u_{j-s-1,x} + (j-s-1) u_{j-s} \Phi_{,x} \right]\end{aligned} \tag{32}$$

where convention $u_k = 0$ whenever $k < 0$ was assumed for uniform notation. For $j = 1$ the recurrence formula (32) takes the form

$$\Phi_{,x} u_1 + \Phi_{,t} - \Phi_{,xx} = 0 \tag{33}$$

while the only compatibility condition $(j = 2)$ may be written as a requirement that the x-derivative of (33) be zero and thus it is obviously satisfied.

Arbitrary Φ has more degrees of freedom than the single number t_0 of ODE's. This extra flexibility may be used to construct e.g. the Bäcklund transformation by imposing some conditions on u_j. Put the arbitrary coefficient $u_2 = 0$ and assume that the coefficient multiplying the zeroth power of Φ (the free term) satisfies the original equation

$$u_{1,t} + u_1 u_{1,x} - u_{1,xx} = 0 \tag{34}$$

then it follows from the recurrence formula (32) that $u_j = 0$ for all $j \geq 2$. The truncated expansion $u = -2\Phi_{,x}/\Phi + u_1$ together with the formula defining u_1 (33) constitute the Bäcklund transformation between two functions u and u_1 satisfying the same Burgers equation. Among others, if we start from $u_1 = 0$, i.e. $u = -2\Phi_{,x}/\Phi$, then (33) takes the form $\Phi_{,t} - \Phi_{,xx} = 0$ which is the linear diffusion equation. This is the well-known Cole–Hopf transformation linearizing the Burgers equation.

Another example, from our own work [17]

$$
\begin{aligned}
iq_t + (fq)_{,xx} + 2qR &= 0, \\
R_{,x} - (f \mid q \mid^2)_{,x} - f_{,x} \mid q \mid^2 &= 0,
\end{aligned} \tag{35}
$$

which for $f = const$ becomes the cubic Schrödinger equation, contains two problems which have not been considered so far:

1. Non-analytic function $\mid q \mid^2$. The complex conjugate of q should be treated as a separate dependent variable as its analytic continuation into the complex planes may be totally unconnected with that of q (except for the real axes).

2. The parameter function f must also be expanded in powers of Φ.

Summary of the test: Functions $q,\ q^*$ are of order Φ^{-1}, while R is of order Φ^{-2}. The resonances occur at $j = -1,\ 0,\ 2,\ 3,\ 4$. The compatibility condition at $j = 2$

$$
f_{,xx} = 0, \qquad \text{i. e. } \ f(x,t) = A(t)x + B(t), \tag{36}
$$

is also sufficient for the other resonances. In this way we have selected those f for which the system is integrable.

The result of the P-test in case of PDE's is not limited to "yes or no." It may have a form of a condition imposed on the singularity manifold and/or first integrals, thus selecting a class of Painlevé solutions of equations which are not completely integrable. Examples of such "partial integrability" may be found in [18].

The same singularity manifold may be defined by various "expansion variables" Φ. A proper choice may help to simplify tedious calculations. For instance, we may locally solve $\Phi(x,t) = 0$ with respect to x (provided that $\Phi_{,x} \neq 0$ in the vicinity of the considered point) to get the manifold from a simple equation

$$
\chi \equiv x - \xi(t) = 0. \tag{37}
$$

This choice usually leads to simplest possible calculations but it is too restrictive (or rather primitive) to leave room for the Bäcklund transformation. The other choice, which also simplifies calculations, is free from this deficiency.

Let $\Phi \equiv \phi - \phi_0$, where the constant ϕ_0 is introduced to leave room for a one-parameter transformation. Then the expansion variable reads

$$
\chi \equiv \left(\frac{\phi_{,x}}{\phi - \phi_0} - \frac{\phi_{,xx}}{2\phi_{,x}} \right)^{-1} \to 0 \ \text{ when } \ \phi \to \phi_0. \tag{38}
$$

This is so called invariant Painlevé analysis of Conte [19]. The name "invariant" is due to the fact that the analysis may be expressed in terms of invariants of the homographic transformation

$$
\phi' = (\alpha\phi + \beta)/(\gamma\phi + \delta) \tag{39}
$$

where $\alpha, \beta, \gamma, \delta$ are complex constants. These invariants are

$$C = -\phi_{,t} / \phi_{,x} \tag{40}$$

and the "Schwarzian derivative" of ϕ

$$S = (\phi_{,xx} / \phi_{,x})_{,x} - (1/2)(\phi_{,xx} / \phi_{,x})^2. \tag{41}$$

In fact, we do not use the formula (38) for χ but we perform expansion in a variable χ on which the following constraints have been imposed

$$\chi_{,x} = 1 + (S/2)\chi^2, \tag{42a}$$

$$\chi_{,t} = -C + C_{,x}\chi - (1/2)(CS + C_{,xx})\chi^2, \tag{42b}$$

and for compatibility of these two

$$S_{,t} + C_{,xxx} + 2C_{,x} S + CS_{,x} = 0. \tag{43}$$

This approach may also be used for ODE's so the benefit of flexibility of the expansion variable is not limited to PDE's. The Bäcklund transformation may in most cases be represented by a shift of ϕ_0.

4 Extensions and new ideas

The "weak Painlevé property". Some systems do not have Laurent expansions. Still a solution may explicitly be constructed by means of a similar single series in fractional powers

$$u(t) = \tau^{p/s} \sum_{j=0}^{\infty} u_j \tau^{1/s}. \tag{44}$$

First integrals for the anharmonic oscillator

$$\begin{aligned} x_{,tt} &= 2xy^3 + (3/4)x^3 y, \\ y_{,tt} &= (3/16)x^4 + 3x^2y^2 + 5y^4, \end{aligned} \tag{45}$$

have been constructed this way in [20] by means of expansion in $\tau^{2/3}$. Some results indicate however that such systems may show symptoms of chaotic behaviour [21].

Essential singular points. Not all essential singular points are origins of branches, e.g. $\exp(-1/(t-t_0))$ returns to its initial value after a single complete turn about the essential singularity $t = t_0$. The Laurent series in the vicinity of such point extends to infinity in both directions of summation and it may happen that it is convergent merely outside some circle. Movability of the singular point means that the solution has a movable natural boundary [22].

Description of systems with movable essential singular points (or manifolds in case of PDE's) corresponds to the problem of negative resonances. Namely, some of those systems possess particular solutions without such point. These solutions may be constructed in the form of the Laurent series (8) by means of the P-test. However, construction of the general solution would not be possible, usually due to the fact that some of the resonances (more than one) occur at negative values of the summation index r. The P-analysis does not supply enough arbitrary constants.

An extension of the P-test which should be passed by some systems with non-branching movable essential singular points, was recently proposed by Fordy, Pickering and Conte [23, 11] (first hints were given by Kruskal in [22]). The solution is constructed in a perturbative way as a double series

$$u = \sum_{n=0}^{\infty} \varepsilon^n \sum_{j=n\rho}^{\infty} u_j{}^{(n)} \Phi^{j+p} = \sum_{j=-\infty}^{\infty} u_j \Phi^{j+p} \tag{46}$$

where $\rho < -1$ is the lowest resonance.

At $n = 0$ the usual P-analysis is performed and the solution $u^{(0)}$ is constructed as the Laurent series (8). The coefficients for larger n are obtained from equation perturbed about the solution $u^{(0)}$. In particular, movable singularity of $u^{(0)}$ becomes a fixed singularity of $u^{(1)}$, $u^{(2)}$, ... and the classical Fuchsian theory of equations with fixed singularities [24] may be applied to the perturbed equation in order to determine whether the solution is single valued in the vicinity of the essential singular point.

A typical example of such equations is an 85 years old equation of Chazy [25]

$$u_{,t} = u_{,xxx} - 2uu_{,xx} + 3u_{,x}{}^2. \tag{47}$$

The P-analysis shows two families of solutions: first with $p = -1$, $u_0 = -6$ and resonances at $-3, \ -2, \ -1$, and the second with $p = -2$, u_0 arbitrary and resonances at $-1, \ 0$. We analyze the first family as an example.

Substituting $u + \varepsilon w$ instead of u in (47), we obtain the perturbed equation (Ist order in ε).

$$w_{,t} = w_{,xxx} - 2uw_{,xx} + 6u_{,x} w_{,x} - 2u_{,xx} w. \tag{48}$$

The "indicial equation," obtained by substitution $u = u_0\Phi^{-1}$, $w = w_0\Phi^r$ in (48), reads

$$(r+4)(r+3)(r+2) = 0. \tag{49}$$

This indicates that the expansion of w begins with Φ^{-4}. Substituting both expansions: of u and w into (48), we obtain the required compatibility conditions at each resonance. They indeed are satisfied. However we have no guarantee that the test would not fail at higher order. In fact, an example was found in [11] where the failure occured at seventh order in ε!

Recently extension to non-Fuchsian cases was proposed in [26].

An extension of the P-test to a simple Fredholm type integro-differential system was proposed by the present author in 1987 [27]. The so called reduced Maxwell-Bloch equations describe transmission of an intense electromagnetic beam through a two-level dielectric. The system reads

$$E_{,x} + E_{,t} \quad = \quad -\langle u_{,t} \rangle,$$

$$\begin{aligned} u_{,t} &= -\omega u, \\ v_{,t} &= \omega u + Ew, \\ w_{,t} &= -Ev \end{aligned} \tag{50}$$

where $\langle f(\omega)\rangle := \int d\omega f(\omega)g(\omega)$ and the weight g is a given function.

If we construct $\langle\rangle$ as the Riemann integral, from finite number n of rectangles around some points ω_i, and consider the functions $u,\ v,\ w$ at each fixed ω_i as separate dependent variables, then we obtain a system of $3n+2$ differential equations. Performing the P-analysis, we get 3 compatibility conditions for each ω_i and 2 compatibility conditions independent of ω. Those $3n+2$ compatibility conditions are satisfied whatever be the division of the ω interval. We conclude that the system should have the P-property also in the limit corresponding to the Riemann integral.

An amusing generalization to discrete systems was recently proposed by Grammaticos, Ramani and Papageorgiu [28]. Consider a mapping which describes a discretized anharmonic (quartic) oscillator

$$x_{n+1} + x_{n-1} = -\frac{a{x_n}^2 + bx_n + c}{x_n(x_n+e)}. \tag{51}$$

It may happen that for some n that $x_n = 0$. Then at the $(n+1)$-st step we encounter a singularity: $x_{n+1} = \infty$. We require that it remains isolated. As x_{n+2} is finite automatically

$$x_{n+2} = -x_n - \frac{a{x_{n+1}}^2 + bx_{n+1} + c}{x_{n+1}(x_{n+1}+e)} = -a \tag{52}$$

we postulate that x_{n+3} be finite

$$x_{n+3} = -x_{n+1} - \frac{a^3 - ab + c}{-a(-a+e)} < \infty. \tag{53}$$

Since x_{n+1} is infinite, the fraction should have zero in the denominator. Hence the necessary condition for $x_{n+3} < \infty$ is $a = e$. This is indeed the only integrable case. A slight generalization works for double lattices. (*Note added in proof:* The popularity of discrete P-test has increased significantly in the meantime. The most important of those papers treats the P-property as an invariant of the discretization [29]).

Conclusion

The review of possibilities of the P-test and its various extensions, although not complete, has definitely shown that it is not only a gun of integrability hunter. Maybe a comparison to a fisherman's net picking digestible fish from the vast sea of non-integrable would be more adequate. Anyway, it also provides quite a good kitchen equipment for making edible meals, sometimes even from non-integrable or sharp-edged discrete raw materials.

REFERENCES

[1] D. Campbell: *Nonlinear Science — from paradigms to practicalities* in *From cardinals to chaos — reflections on the life and legacy of Stanislaw Ulam,* ed. Necia Grant Cooper, Cambridge University Press, 1989, p. 218.

[2] M. J. Ablowitz, A. Ramani and H. Segur: *J. Math. Phys.* **21** (1980), 715; see also M. J. Ablowitz and H. Segur: *Phys. Rev. Lett.* **38** (1977), 1103.

[3] P. van Moerbeke: *Phil. Trans. R. Soc. Lond.* **A315** (1985), 379–390; M. Adler and P. van Moerbeke: *Comm. Math. Phys.* **83** (1982), 83–106.

[4] N. Ercolani and E. D. Siggia: *Phys. Lett.* **A119** (1986), 112–116; *Physica* **34D** (1989), 303.

[5] N. Joshi and J. A. Petersen: *Nonlinearity* **7** (1994), 595–602; *Bull. Aust. Math. Soc.* **54** (1996) 353–362.

[6] N. Joshi and G. K. Srinivasan: *Nonlinearity* **10** (1997), 71–79.

[7] B. Gambier *Acta Math.* **33** (1910), 1–55, see in particular page 4.

[8] See article by F. J. Bureau in *Proc. of the NATO Advanced Research Workshop, September 3-7 1990 Sainte-Adele, Quebec, Canada,* eds. D. Levi and P. Winternitz, Plenum, New York 1992, pp. 103–123.

[9] P. Hoyer: *Über die Integration eines Differentialgleichungssystems von der Form...*, Dissertation Königl. Friedrich-Wilhelms Univ. Berlin (1879), 1–36.

[10] S. Kovalevskaya: *Acta Mathematica* **12** (1889), 177–232, **14** (1890), 81–93.

[11] R. Conte, A. P. Fordy and A. Pickering: *Physica* **D 69** (1993), 33.

[12] E. N. Lorenz: *J. Atmos. Sci.* **20** (1963), 130.

[13] G. Levine and M. Tabor: *Physica* **33D** (1988), 189–210.

[14] J. Weiss, M. Tabor and G. Carnevale: *J. Math. Phys.* **24** (1983), 522.

[15] R. S. Ward: *Phil. Trans. R. Soc. Lond.* **A315** (1985) 451–457.

[16] see the article by Weiss in [6], p. 225.

[17] J. Cieśliński, P. Goldstein and A. Sym: *J. Phys. A, Math. Gen.* **27** (1994), 1645.

[18] J. Weiss: *J. Math. Phys.* **25** (1984), 2226.

[19] R. Conte: *Phys. Lett.* **A140** (1989), 383; see also article by R. Conte in *Proc. of the NATO Advanced Research Workshop, September 3-7 1990 Sainte-Adele, Quebec, Canada,* eds. D. Levi and P. Winternitz, Plenum, New York 1992, p. 125.

[20] A. Ramani, B. Dorizzi and B. Grammaticos: *Phys. Rev. Lett.* **49** (1982), 1539; *J. Math. Phys.* **24** (1983), 2282.

[21] K. Bajer and P. P. Goldstein, unpublished result.

[22] see the article by Kruskal in *Proc. of the NATO Advanced Research Workshop, September 3-7 1990 Sainte-Adele, Quebec, Canada,* eds. D. Levi and P. Winternitz, Plenum, New York 1992, p. 187.

[23] A. Fordy and A. Pickering: *Phys. Lett.* **A160** (1991), 347.

[24] E. L. Ince: *Ordinary Differential Equations*, Dover, New York 1956, pp 356–371.

[25] J. Chazy: *C. R. Acad. Sc. Paris* **150** (1910), 456–458; *Acta Math.* **34** (1911), 317.

[26] M. Musette and R. Conte: *Phys. Lett.* **A206** (1995), 340–346.

[27] P. P. Goldstein: *Phys. Lett.* **A121** (1987), 11.

[28] B. Grammaticos, A. Ramani and V. Papageorgiu: *Phys. Rev. Lett.* **67** (1991), 1825.

[29] R. Conte and M. Musette: *Phys. Lett.* **A223** (1996), 439–448.