Nonautonomous Hénon–Heiles systems

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Abstract: We obtain nonautonomous generalizations of the Hénon–Heiles system by considering scaling similarity solutions of certain fifth-order nonlinear PDEs. The resulting equations should be integrable in the sense of having the Painlevé property, and we exhibit Bäcklund transformations for them and produce some rational solutions as well as others related to the first Painlevé transcendent.

1 Introduction

In the following we will be looking at some similarity reductions of infinite-dimensional integrable systems, which yield ODEs that should have the Painlevé property. One might wonder why such reductions of PDEs should be worthy of interest. There are various reasons to consider them. The first point to note is that, as described in Ragnisco's lecture (appearing elsewhere in this volume), if an ODE is the stationary flow of a soliton equation then it inherits some of the integrable structure of that soliton equation e.g. conserved quantities, Bäcklund transformations (see [1, 2] for examples of stationary flows related to the systems described below). More generally, given a PDE in 1+1 dimensions with independent variables x, t and dependent variable $u(x,t)$, the problem of finding solutions is considerably simplified if we seek a solution in the similarity form, that is

$$u(x,t) = U(w(z), x, t), \tag{1}$$

where $z = z(x,t)$ is the similarity variable, and on substituting $U(w,x,t)$ into the PDE, an ODE for $w(z)$ is obtained. There are various ways of finding similarity forms, the most common being the classical Lie symmetry approach (although this method does not yield all possible similarity solutions; see [3] for a case where it fails, as well as references to the other techniques). Obviously a stationary flow is just a special sort of similarity solution. One may similarly find similarity solutions to systems of equations, and for PDEs with more independent variables, but we will not be interested in such complications here.

Once we have a similarity form, we are left with an ODE to solve. The fact is that if we start with a PDE which has lots of nice properties (such as solvability by inverse scattering) then the resulting ODE should be correspondingly manageable. This is expressed more precisely in the conjecture of Ablowitz, Ramani and Segur (ARS), which states that all similarity reductions of integrable PDEs have the Painlevé property (although due to the fact that no universally accepted definition exists, we are being deliberately imprecise about what we mean by "integrable"). For a good account of the Painlevé property, refer to Goldstein's talk (in these Proceedings), and for further details as well as some theorems which support the ARS conjecture, see [4].

For comparison with what follows it is worth looking at a well-known example. If we start with the modified Korteweg–de Vries (mKdV) equation

$$v_t = v_{xxx} - 6v^2 v_x, \tag{2}$$

and notice that it has a scaling symmetry, then this gives us its scaling similarity solutions. More explicitly, (2) is invariant under $x \to \beta x$, $t \to \beta^3 t$, $v \to \beta^{-1} v$, and so this implies that there is a similarity solution

$$v(x,t) = (-3t)^{-\frac{1}{3}} y(z(x,t)), \tag{3}$$

with the similarity variable $z = (-3t)^{-\frac{1}{3}} x$. Substituting this form into (2) we find that y satisfies

$$y''' = 6y^2 y' + zy' + y \tag{4}$$

($'$ denotes $\frac{d}{dz}$), which may be integrated once to give

$$y'' = 2y^3 + zy + \alpha, \tag{5}$$

for some constant α. (5) is known as the second Painlevé equation (P2), being the second exceptional equation in Paul Painlevé's classification of second-order equations having no movable singularities other than poles (exceptional in the sense that its general solution cannot be expressed in terms of classical transcendental functions).

In the next section we will look at three integrable hierarchies of evolution equations (the Sawada–Kotera, Korteweg–de Vries and Kaup–Kupershmidt hierarchies), and derive their scaling similarity forms. The third section concerns some completely integrable finite-dimensional Hamiltonian systems known as the Hénon–Heiles systems. We describe how they are related to stationary flows of some of the PDEs looked at in the previous section, and relate the similarity solutions of these PDEs to nonautonomous versions of the Hénon–Heiles systems. In the last section we derive Bäcklund transformations for the nonautonomous equations, and show how they may be used to generate special families of solutions.

2 Some properties of the KdV, Sawada–Kotera and Kaup–Kupershmidt hierarchies

Before looking at the nonautonomous systems of the title, we will consider some aspects of three different hierarchies of PDEs, known as the KdV, Sawada–Kotera (SK) and Kaup–Kupershmidt (KK) hierarchies, that are needed in what follows. Each hierarchy is a sequence of evolution equations or flows with respect to times t_n $(n = 1, 2, 3, \ldots)$, which can all be put into Hamiltonian form. SK and KK have only one Hamiltonian structure, but KdV is bi-Hamiltonian, and here we will be using the second Hamiltonian structure. The n-th flow in each of the hierarchies can be written as

$$\frac{\partial u}{\partial t_n} = (\partial_x^3 + 8au\partial_x + 4au_x)\delta_u H_n[u], \tag{6}$$

where $a = 1/2$ for SK and KdV, $a = 1/4$ for KK, and H_n is the n-th Hamiltonian for the hierarchy in question. For the purposes of computing variational derivatives, we make no distinction between a Hamiltonian and its corresponding Hamiltonian density. For more details on these hierarchies and ways of calculating the sequence of Hamiltonians, see e.g. [7, 8].

There is also a Miura map from the modified versions of the hierarchies, given by

$$u = -v_x - 2av^2 =: M[v].$$

Then rewriting the Hamiltonian in terms of v and derivatives, the n-th modified flow may be expressed as

$$\frac{\partial v}{\partial t_n} = (-\partial_x)\delta_v H_n[M[v]]. \tag{7}$$

The Miura map means that given v satisfying (7) for each n, the corresponding $u = -v_x - 2av^2$ satisfies (6).

The n-th flow of the hierarchy is unchanged by the scaling

$$x \to \beta x, \qquad t_n \to \beta^m t_n, \qquad u \to \beta^{-2} u,$$

where $m = m(n)$ is a scale weight dependent on the hierarchy. Similarly the modified flow is invariant under the same scaling but with

$$v \to \beta^{-1} v.$$

Hence there are scaling similarity solutions looking like $u = t_n^{-\frac{2}{m}}\, w(xt_n^{-\frac{1}{m}})$. For convenience in what follows we scale the similarity variable $\beta = \theta(t_n)$ so that

$$u(x, t_n) = \theta^2(t_n) w(z),$$

where $z = x\theta(t_n)$ and $\frac{d\theta}{dt_n} = \theta^{m+1}$. The corresponding similarity solution for the modified flow is

$$v = \theta(t_n) y(z),$$

with the scaled Miura map giving $w = -y' - 2ay^2$.

Henceforth we will drop the suffix n. Substituting the similarity forms into the equations of motion (6) and (7) (and cancelling out powers of θ on either side) yields the ODEs for w and y. If we let $\tilde{H}$ denote the scaled Hamiltonian (expressed in terms of w with powers of θ divided out) then we have simply

$$(\partial_z^3 + 8aw\partial_z + 4aw')(\delta_w \tilde{H} - \frac{1}{4a} z) = 0, \tag{8}$$

$$\partial(\delta_y \tilde{H} + zy) = 0. \tag{9}$$

Both of these equations can be integrated once, and are conveniently written in terms of

$$f := \delta_w \tilde{H} - \frac{1}{4a} z.$$

Integration of (8) yields immediately

$$\frac{d^2 f}{dz^2} + 4awf + \frac{\lambda^2 - (\frac{df}{dz})^2}{2f} = 0. \tag{10}$$

For (9), note that

$$\delta_v H = (M')^\dagger \delta_u H = (\partial_x - 4av)\delta_u H, \tag{11}$$

where M' is the Fréchet derivative of M. The scaled similarity form of this relation (involving y and $\delta_w \tilde{H}$) allows (9) to be written in terms of the quantity f and integrated to

$$\frac{df}{dz} - 4ayf + \lambda = 0. \tag{12}$$

In (10) f is to be thought of as a function of w and its derivatives, while in (12) it is expressed instead in terms of y and derivatives of y (replacing each w by $-y' - 2ay^2$). λ is a constant of integration, and is in fact the same constant in both cases, as the Miura map becomes a one-to-one correspondence between the two equations. Note that the equations for the stationary flows are simply obtained by removing the $-\frac{1}{4a}z$ terms from f. The form of the equations makes it particularly simple to see the relationship between them. The scaled Miura map means that if y satisfies (12) then w must satisfy (10), and this is obtained directly by substituting $y = \frac{f'+\lambda}{4af}$ into $w = -y' - 2ay^2$. Conversely, given w satisfying (10), then the same substitution for y (with f expressed in terms of w) rearranges to give (12).

To make things more concrete, it is worth looking at some particular cases. The first example to consider is the scaling similarity solutions of the ordinary KdV equation. Putting $H = \frac{1}{2}u^2$ (the first nontrivial Hamiltonian in the KdV hierarchy) into (6) with $a = \frac{1}{2}$ we obtain KdV: $u_t = u_{xxx} + 6uu_x$. The scaling similarity solutions are given by $u(x,t) = (-3t)^{-\frac{2}{3}}w(z)$, with the similarity variable $z = (-3t)^{-\frac{1}{3}}x$. After substituting into KdV and integrating once we find the ODE for w:

$$w'' + 2w^2 - zw + \frac{\alpha(\alpha+1) + w' - (w')^2}{2w - z} = 0. \tag{13}$$

Using the scaled Hamiltonian $\tilde{H} = \frac{1}{2}w^2$ we find

$$f = w - \frac{z}{2},$$

and substituting into (12) with this f and $a = \frac{1}{2}$ does indeed give the equation (13) on setting $\lambda = \alpha + \frac{1}{2}$. Also the Miura map $u = -v_x - v^2$ goes from mKdV to KdV. For the scaling similarity solutions of mKdV (3) we find that there is a one-to-one correspondence between solutions of P2 (equation (5) of the previous section) and (13), given by

$$w = -y' - y^2$$

(the scaled Miura map) and

$$y = \frac{w' + \alpha}{2w - z}.$$

Note that in terms of y, we have

$$f = -y' - y^2 - \frac{z}{2},$$

and on putting this into (12) with $a = \frac{1}{2}$, P2 results. This particular case is considered in [9] for example.

As our second example we take the fifth-order equations in each of the hierarchies, which following [1] may be written as

$$u_t = \left(u_{xxxx} + (8a - 2b) u u_{xx} - 2(a+b) u_x^2 - \frac{20}{3} ab u^3 \right)_x, \tag{14}$$

where we have three cases (i), (ii), (iii) corresponding to $a = \frac{1}{2}, \frac{1}{2}, \frac{1}{4}$ and $b = -\frac{1}{2}, -3, -4$ respectively. These are the only values of a, b for which an equation of the form (14) is integrable, and (i) and (iii) are respectively the Sawada–Kotera and Kaup–Kupershmidt equations (see references in [8]), while (ii) is the fifth-order KdV equation. (14) may be obtained from the Hamiltonian formalism described above, by taking

$$H = -\frac{1}{2}(u_x)^2 - \frac{1}{3} b u^3$$

with the correct values of a and b for each case. Then the scaling similarity form is

$$u(x,t) = (-5t)^{-\frac{2}{5}} w(z),$$

where now $z = (-5t)^{-\frac{1}{5}} x$. We find

$$f = w'' - bw^2 - \frac{z}{4a},$$

and putting this into (10) we find a fourth-order ODE for w. Using the scaled Miura map, we may express f in terms of y:

$$f = -y''' - 4ayy'' - (4a+b)(y')^2 - 4aby^2y' - 4a^2by^4 - \frac{z}{4a}.$$

Then the ODE for y is (from (12))

$$y^{(iv)} = -2(6a+b)y'y'' + 4a(4a-b)(y^2y'' + y(y')^2) + 16a^3by^5 + zy + \alpha, \tag{15}$$

with $\alpha = \lambda - \frac{1}{4a}$. Given a solution to (15), we can then obtain a solution to the fourth order ODE for w via $w = -y' - 2ay^2$. Notice that on substituting in the relevant values of a and b the resulting equations (15) for cases (i) and (iii) differ only by a sign in the even $(y'y'')$ terms. So if y_α is a solution to the equation in case (i), then $Y_{-\alpha} = -y_\alpha$ is a solution to the equation in case (iii) at parameter value $-\alpha$. This is because the modified hierarchies in these two cases are essentially the same. We shall be using this property in the final section to derive the Bäcklund transformation.

3 Hénon–Heiles systems

The original Hénon–Heiles system is given by a Hamiltonian with two degrees of freedom:

$$h = \frac{1}{2}(p_1^2 + p_2^2) + aq_1q_2^2 - \frac{1}{3}bq_1^3, \tag{16}$$

The equations of motion are just Hamilton's equations

$$\frac{dq_j}{dz} = \frac{\partial h}{\partial p_j}, \qquad \frac{dp_j}{dz} = -\frac{\partial h}{\partial q_j} \tag{17}$$

(we are denoting the time by z here to make connection with results which appear below). It was known for some time from Painlevé analysis [5] that this system is integrable for three values of the ratio $r = a/b$ (because of a scaling symmetry of the equations the integrability only depends on this ratio), i.e. $r = -1, -1/6, -1/16$.

More recently, Fordy [1] showed that for these integrable cases the equations of motion were just disguised versions of the stationary flows of some fifth-order soliton equations — the Sawada–Kotera, fifth order Korteweg–de Vries and Kaup–Kupershmidt equations (hence the choice of values for a and b as in the previous section gives the right values for the ratio r in the cases (i), (ii), (iii)). The zero curvature form of these PDEs yields a matrix Lax representation of the stationary flows, and then traces of powers of the Lax matrix give the Hamiltonian and the second constant of motion (which shows that these systems are indeed Liouville integrable). It was subsequently shown that all three systems are completely separable in suitable coordinates, and may be integrated in terms of theta functions of genus one (cases (i) and (iii)) or genus two (case (ii))[6].

Instead of looking at the stationary flows of these three PDEs, we will take the equations for the scaling similarity solutions, and rewrite them in Hamiltonian form. So we have $f = w'' - bw^2 - \frac{z}{4a}$, and f must satisfy the equation (10). On setting $w = q_1$, $f = -aq_2^2$, we find

$$q_1'' = bq_1^2 - aq_2^2 + \frac{z}{4a}, \qquad q_2'' = -2aq_1q_2 - \frac{\lambda^2}{4a^2q_2^3}. \tag{18}$$

These are just Hamilton's equations for the system with Hamiltonian

$$h = \frac{1}{2}(p_1^2 + p_2^2) + aq_1q_2^2 - \frac{1}{3}bq_1^3 - \frac{\lambda^2}{8a^2}q_2^{-2} - \frac{1}{4a}zq_1,$$

which is just (16) with an extra inverse square term and a nonautonomous (time-dependent) term in the potential. So the similarity solutions introduced at the end of Section 2 may be viewed as nonautonomous Hénon–Heiles systems. Because of the explicit time-dependence, the Hamiltonian is no longer a constant of motion, and we don't get a matrix Lax representation any more. Below we discuss the integrability properties of these equations, as well as Bäcklund transformations which generate some special families of solutions.

4 Integrability, Bäcklund transformations and some special solutions

Given the zero curvature representation of the fifth-order PDEs considered above, it is straightforward to obtain the matrix Lax representation of the equations for the stationary flows. It is no longer possible to do this for the corresponding nonautonomous systems, so we have to use other methods to test their integrability. The most obvious thing to try is the Painlevé test, and indeed this does find principal balances (with four arbitrary constants in the power series solutions) in each of the cases (i), (ii), (iii) (for more details see [10]). A more useful approach from the point of view of finding exact solutions is to look at the similarity equations coming form the modified hierarchy.

The variable w appearing as q_1 is in one-to-one correspondence with a solution y of (15), and the latter is a lot more tractable for various reasons. The zero curvature representation of the PDEs in the modified hierarchy scales nicely, so that on restricting it to the scaling similarity forms it produces a zero curvature representation of (15). This then means that the initial value problem for y can be solved in terms of an inverse monodromy problem (an ODE analogue of inverse scattering). This method has been pursued in detail for the second Painlevé transcendent P2 (see [11, 12]). In [11] the scheme is also outlined for the scaling similarity solutions of the higher-order equations in the modified KdV hierarchy (referred to as the Painlevé II Family). The equation (15) in case (ii) is the next equation up from P2 in this family.

A simpler approach to finding solutions is to derive the Bäcklund transformations for the equation (15) for each of the three cases. We consider (ii) first, as it is the simplest case and the Bäcklund transformation takes the same form as for P2 (and indeed for the whole P2 family [13]). Everything is most conveniently expressed in the notation of Section 2. So we take the equation for y in the form (12). Now let y_α denote a solution to this at parameter value α, and define

$$w_\alpha = -y'_\alpha - y_\alpha^2 \tag{19}$$

to be the corresponding solution to (10), with $\lambda = \alpha + \frac{1}{2}$, $a = \frac{1}{2}$. Also take

$$f_\alpha = w''_\alpha + 3w_\alpha^2 - \frac{z}{2}.$$

Now to find the Bäcklund transformation we just need to use two facts. The first is that given a solution y_α for parameter value α, we have that $y_{-\alpha} = -y_\alpha$ is also a solution for parameter value $-\alpha$. The second thing to notice is that under the scaled Miura map (19), two solutions y_α and $y_{-(\alpha+1)}$ correspond to the same value of λ^2 and hence the same solution of (10), we obtain

$$y_{\alpha+1} = -y_\alpha + \frac{2\alpha+1}{2f_\alpha}. \tag{20}$$

This gives the standard transformation for P2 [9], on putting $f_\alpha = w_\alpha - \frac{z}{2}$. To obtain the sequence of rational solutions to (15) in case (ii), we apply (20) repeatedly starting from $y_0 = 0$. Some of these solutions (together with the corresponding w_α) are given in a table below.

α	0	1	2	3
y_α	0	$-\frac{1}{z}$	$-\frac{2}{z}$	$\frac{2}{z} - \frac{5z^4}{z^5-144}$
w_α	0	$-\frac{2}{z^2}$	$-\frac{6}{z^2}$	$-\frac{2}{z^2} - \frac{10z^3(z^5+576)}{(z^5-144)^2}$

These solutions may also be obtained from the sequence of polynomials [14] which give the rational solutions of the KdV hierarchy; this is discussed in more detail in [10].

Since, as mentioned in Section 2, the equations (15) differ only by a minus sign for (i) and (iii), these two cases may be dealt with together. We employ the same notation as before, with lower case and upper case letter y corresponding to solutions to (15) in (i) and (iii) respectively. Also we introduce $w_\alpha = -y'_\alpha - y_\alpha^2$, $f_\alpha = w''_\alpha + \frac{1}{2}w_\alpha^2 - \frac{1}{2}z$, and similarly $W_\alpha = -Y'_\alpha - \frac{1}{2}Y_\alpha^2$, $F_\alpha = W''_\alpha + 4W_\alpha^2 - z$. Using $Y_{-\alpha} = -y_\alpha$ as well as the fact that (corresponding to the same value of λ^2 in each case) $w_\alpha = w_{-\alpha-1}$ and $W_\alpha = W_{-\alpha-2}$, we find a slightly different sort of Bäcklund transformation:

$$y_{\alpha+3} = y_\alpha - \frac{2\alpha+1}{2f_\alpha} + \frac{2(\alpha+2)}{F_{\alpha+1}}.$$

Again we can find a sequence of rational solutions by applying this starting at $y_0 = 0$, but now every third integer is missed out in between. We present a few of these in a table, with the corresponding w_α and $W_{-\alpha} = y'_\alpha - \frac{1}{2}y_\alpha^2$, thus:

α	-4	-3	-1	0	2	3
y_α	$\frac{4}{z}$	$-\frac{2}{z} + \frac{5z^4}{z^5+36}$	$\frac{1}{z}$	0	$-\frac{2}{z}$	$-\frac{3}{z}$
w_α	$-\frac{12}{z^2}$	$-\frac{6}{z^2}$	0	0	$-\frac{6}{z^2}$	$-\frac{12}{z^2}$
$W_{-\alpha}$	$-\frac{12}{z^2}$	$-\frac{15z^3(z^5-144)}{2(z^5+36)^2}$	$-\frac{3}{2z^2}$	0	0	$-\frac{3}{2z^2}$

One might wonder if there was any advantage in writing the equation (10) for w as a nonautonomous Hamiltonian system. In fact in case (i) when $\alpha = -\frac{1}{2}$ ($\lambda = 0$) the same substitution that works in the ordinary (autonomous) system causes the equations of motion to separate. Putting $Q_\pm = q_1 \pm q_2$ into (18), we find $Q''_\pm = -\frac{1}{2}Q_\pm^2 + \frac{z}{2}$, which (up to a scaling) is just two separate copies of the first Painlevé equation. The corresponding solution to (15) is

$$y_{-\frac{1}{2}} = (\log(Q_+ - Q_-))',$$

where we assume that Q_+ and Q_- are not equal. So plugging this into the Bäcklund transformation we get a whole sequence of solutions in terms of the first Painlevé transcendent. For further details and results see [10].

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