Solitons and projectively flat affine surfaces

Włodzimierz Jelonek

Institute of Mathematics, Technical University of Cracow,
Warszawska 24, 31-155 Kraków, Poland
e-mail: `wjelon@usk.pk.edu.pl`

Abstract: The aim of this paper is to give a local description of affine surfaces, whose induced Blaschke structure is projectively flat. We show that such affine surfaces with constant Gauss affine curvature and indefinite induced Blaschke metric are described by soliton equations.

Introduction

It is well known that the sine-Gordon (s-G) equation $\partial_{12}u = \sin u$ is strictly related to Riemannian surfaces of constant negative Gauss curvature immersed in the Euclidean space $\mathbb{R}^3$. The equation s-G has ben studied in this context by many geometers since the end of the nineteenth century. This hyperbolic nonlinear equation has soliton solutions. All the surfaces of constant negative curvature immersed in $\mathbb{R}^3$ are not convex. We show in the present paper that the similar soliton equations appear in the natural way in the study of non-convex projectively flat affine surfaces with constant affine Gauss curvature. Among them are semi-Riemannian surfaces of constant Gauss curvature immersed in the Lorentz space $\mathbb{R}^3$. All the soliton equations obtained for non-convex surfaces are of great importance in the soliton theory (see [1]). In the first part of the paper we consider affine locally symmetric surfaces (with $\mathbb{C}$-diagonalizable shape operator). The similar results can be found in ([14, 12]). The second part is devoted to description of affine spheres (see [5]). The similar results were obtained by V. V. Nesterenko [8] and U. Simon and C. Wang in [13].

1 Preliminaries

Our notation is as in [10, 4]. Let (M, f) be an affine surface in $\mathbb{R}^3$ with the induced equiaffine structure (∇, h, S). It means that $f : M \to \mathbb{R}^3$ is an immersion and there exists a transversal section ξ of the vector bundle $f^*T\mathbb{R}^3$ such that

$$D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)\xi$$

and $D_X\xi = -f_*(SX)$. We call h, S the induced affine metric and shape operator respectively. We shall assume that the immersion f is nondegenerate (that is h is a nondegenerate semi-metric). By C we denote the cubic form $C = \nabla h$. By $J = \frac{1}{8}h(C, C)$ we denote the Fubini–Pick invariant. In any basis in TM we have $h(C, C) = \sum h^{ip}h^{jq}h^{kr}C_{ijk}C_{pqr}$.

For an equiaffine structure (∇, h, S) the following equations hold:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{1}$$
$$C(X,Y,Z) = C(Y,X,Z), \tag{2}$$
$$\nabla S(X,Y) = \nabla S(Y,X), \tag{3}$$
$$\hat{\nabla} S(X,Y) - \hat{\nabla} S(Y,X) = K(SX,Y) - K(SY,X),$$
$$h(SX,Y) = h(Y,SX),$$

where $\hat{\nabla}$ is the Levi-Civitá connection of the affine metric h and K, defined by $K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y$, is the difference tensor. The Radon theorem says that the structures (∇, h, S) on surface M satisfying the above equations are in one-to-one correspondence (up to an affine transformation) with equiaffine surfaces (M, f, ξ) in $\mathbb{R}^3$. Equation (2) is equivalent to

$$h(K_X Y, Z) = h(Y, K_X Z). \tag{4}$$

The Blaschke structure (∇, h, S) is characterized additionally by the apolarity condition

$$\mathrm{tr}_h K = 0. \tag{5}$$

For a Blaschke structure on a surface M we denote $H = \frac{1}{2}\mathrm{tr}\, S$, $\tau = \det S$. H, τ are called respectively an affine mean and Gauss curvatures of the surface (M, f). For the Blaschke structure on a surface (M, f) the Affine Theorema Egregium holds:

$$K_h = H + J,$$

where K_h denotes the Gauss curvature of the metric h.

Two surfaces $(M, f), (N, g)$ we call equiaffinely equivalent iff there exist a diffeomorphism $\phi : M \to N$ and an affine transformation $A \in ASL(3)$ such that $g \circ \phi = A \circ f$.

The semi-isothermal coordinates on a semi Riemannian surface (M, h) are the local coordinates (x_1, x_2) on M such that $h(\partial_1, \partial_1) = \epsilon e^{-2u}$, $h(\partial_2, \partial_2) = \eta e^{-2u}$, $h(\partial_1, \partial_2) = 0$ where $\partial_i = \frac{\partial}{\partial x_i}$, $u \in C^\infty(M)$ and $\epsilon, \eta \in \{-1, 1\}$. It means that

$$h = e^{-2u}(\epsilon dx_1 \otimes dx_1 + \eta dx_2 \otimes dx_2).$$

In the semi-isothermal coordinates the Christoffel symbols of the Levi-Civitá connection of h are given by:

$$\Gamma^1_{11} = -\partial_1 u, \qquad \Gamma^2_{11} = \epsilon\eta\partial_2 u, \qquad \Gamma^1_{12} = -\partial_2 u,$$
$$\Gamma^2_{12} = -\partial_1 u, \qquad \Gamma^1_{22} = \epsilon\eta\partial_1 u, \qquad \Gamma^2_{22} = -\partial_2 u.$$

The Laplacian Δ on the space (M, h) is the differential operator defined in local coordinates by the formula:

$$\Delta\phi = \Theta^{-1}\left(\Sigma\partial_j(\Theta h^{ij}\partial_i\phi)\right),$$

where $\Theta = \sqrt{|\det h|}$. In the semi-isothermal coordinates we have:

$$\Delta\phi = e^{2u}(\epsilon\partial_1^2\phi + \eta\partial_2^2\phi) = e^{2u}\Delta_0\phi,$$

where $\Delta_0 = \epsilon\partial_1^2 + \eta\partial_2^2$. If K_h is the Gauss curvature of the semi-Riemannian surface (M,h) (i.e. $R(X,Y)Z = K_h(h(Y,Z)X - h(X,Z)Y)$ where R is the curvature tensor of (M,h)) then

$$\Delta u = K_h$$

which means that $\Delta_0 u = e^{-2u}K_h$.

2 Projectively flat Blaschke structures induced on surfaces

Let us recall that an equiaffine connection ∇ is projectively flat iff the following conditions hold (for $\gamma(X,Y) = \frac{1}{n-1}\,\mathrm{Ric}\,(X,Y) = \frac{1}{n-1}\mathrm{tr}\,\{Z \to R(Z,X)Y\}$)

$$R(X,Y)Z = \gamma(Y,Z)X - \gamma(X,Z)Y, \tag{6}$$

$$\nabla\gamma(X,Y,Z) = \nabla\gamma(Y,X,Z). \tag{7}$$

If M is a surface then (6) always holds. If $\dim M > 2$ then (7) is a consequence of (6) (see [9]). We say that an equiaffine structure (∇, h, S) is projectively flat if ∇ is a projectively flat connection. The following lemma gives a description of equiaffine projectively flat structures induced by an affine immersion:

LEMMA 1. *Let (M,f) be a nondegenerate surface in $\mathbb{R}^3$ and (∇, h, S) be an equiaffine structure induced by f and an equiaffine transversal field ξ. Then the induced connection ∇ is projectively flat if and only if* $\mathrm{tr}\,_h\nabla S = 0$.

Proof: As M is a surface we have

$$R(X,Y)Z = \gamma(Y,Z)X - \gamma(X,Z)Y.$$

Hence we obtain

$$\nabla_W R(X,Y)Z = \nabla_W\gamma(Y,Z)X - \nabla_W\gamma(X,Z)Y. \tag{8}$$

From (8) it follows that ∇ is projectively flat iff

$$\nabla_W R(X,Y)Z = \nabla_Z R(X,Y)W \tag{9}$$

for all X, Y, W, Z. On the other hand from the Gauss equation we obtain:

$$\begin{aligned}\nabla_W R(X,Y)Z &= C(W,Y,Z)SX - C(W,X,Z)SY \\ &\quad + h(Y,Z)\nabla S(W,X) - h(X,Z)\nabla S(W,Y).\end{aligned}$$

Hence from (2) and (9) it follows

$$h(Y,Z)\nabla S(W,X) - h(X,Z)\nabla S(W,Y) = h(Y,W)\nabla S(Z,X) - h(X,W)\nabla S(Z,Y). \tag{10}$$

Let $\{E_1, E_2\}$ be an orthonormal frame, $h(E_i, E_i) = \epsilon_i \in \{-1, 1\}$, and take $X = W = E_1$, $Z = Y = E_2$. We get

$$\epsilon_1 \nabla S(E_1, E_1) + \epsilon_2 \nabla S(E_2, E_2) = 0,$$

which means $\operatorname{tr}_h \nabla S = 0$. It is not difficult to see that if the last equation is satisfied then (10) holds and consequently ∇ is projectively flat. □

Our present aim is to describe projectively flat Blaschke connections ∇ on surfaces as well as Blaschke immersions f inducing such connections such that affine Gauss curvature τ of (M, f) is constant. By $\mathbb{C}$-diagonalizable endomorphism S we mean an endomorphism whose complexification is diagonalizable.

THEOREM 1. *Let $f : M \to \mathbb{R}^3$ be an affine immersion with an equiaffine structure (∇, h, S) inducing a projectively flat Blaschke connection ∇ and such that $\tau = \det S$ is constant. If S is $\mathbb{C}$-diagonalizable then (M, f) is affinely equivalent to a locally symmetric affine surface or to an affine sphere.*

Proof: Let us recall that $\gamma(X,Y) = 2Hh(X,Y) - h(X, SY)$. Hence

$$\nabla_Z \gamma(X,Y) = 2ZHh(X,Y) + 2HC(Z,X,Y) - C(Z,X,SY) - h(X, \nabla_Z S(Y)). \tag{11}$$

From (11) and Codazzi equations (2) and (3) it follows that $\nabla\gamma$ is totally symmetric iff

$$-C(Z,X,SY) + 2ZHh(X,Y) = -C(Y,X,SZ) + 2YHh(X,Z)$$

or equivalently if

$$2K(Z,SY) - 2K(SZ,Y) = 2((YH)Z - (ZH)Y). \tag{12}$$

Hence for a projectively flat structure we get

$$\hat{\nabla} S(Z,Y) - \hat{\nabla} S(Y,Z) = (ZH)Y - (YH)Z. \tag{13}$$

Next we consider two cases.

(a) The shape operator S is $\mathbb{R}$-diagonalizable on M. Let λ, μ be eigenvalues of S and $U = \{x : \lambda(x) \neq \mu(x)\}$. Let us note that $(\operatorname{int}(M \setminus U), f)$ is an affine sphere. We shall show that (U, f) is a locally symmetric affine surface. Let $\{E_1, E_2\}$ be a local orthonormal frame, $h(E_1, E_1) = \epsilon$, $h(E_2, E_2) = \eta \in \{-1, 1\}$ such that

$$SE_1 = \lambda E_1, \qquad SE_2 = \mu E_2. \tag{14}$$

We have $\hat{\nabla}_X E_1 = \epsilon\omega(X)E_2$, $\hat{\nabla}_X E_2 = -\eta\omega(X)E_1$ where $\omega_1^2 = \epsilon\omega$ is a connection form for ∇. From (14) we obtain

$$\hat{\nabla}S(E_2, E_1) = (E_2\lambda)E_1 + \epsilon(\lambda-\mu)\omega(E_2)E_2.$$

Analogously we obtain:

$$\hat{\nabla}S(E_1, E_2) = \eta(\lambda-\mu)\omega(E_1)E_1 + (E_1\mu)E_2.$$

Hence we get

$$\begin{aligned}\hat{\nabla}S(E_1, E_2) - \hat{\nabla}S(E_2, E_1) &= (\eta(\lambda-\mu)\omega(E_1) - (E_2\lambda))E_1 \\ &\quad +((E_1\mu) - \epsilon(\lambda-\mu)\omega(E_2))E_2.\end{aligned} \tag{15}$$

From (13) we have also

$$\hat{\nabla}S(E_1, E_2) - \hat{\nabla}S(E_2, E_1) = (E_1H)E_2 - (E_2H)E_1. \tag{16}$$

Hence comparing (15) and (16) we obtain

$$-\eta(\mu-\lambda)\omega(E_1) = \tfrac{1}{2}E_2(\lambda-\mu), \quad \epsilon(\lambda-\mu)\omega(E_2) = -\tfrac{1}{2}E_1(\lambda-\mu)$$

and consequently

$$\eta\omega(E_1) = \tfrac{1}{2}E_2(\ln|\lambda-\mu|), \quad \epsilon\omega(E_2) = -\tfrac{1}{2}E_1(\ln|\lambda-\mu|).$$

Hence

$$\hat{\nabla}_{E_1}E_2 = -\tfrac{1}{2}E_2(\ln|\lambda-\mu|)E_1, \quad \hat{\nabla}_{E_2}E_1 = -\tfrac{1}{2}E_1(\ln|\lambda-\mu|)E_2. \tag{17}$$

Let us introduce coordinates (x_1, x_2) for which $E_1 = \phi\partial_1, E_2 = \psi\partial_2$ for some smooth functions ϕ, ψ. From (17) we obtain in those coordinates:

$$\begin{aligned}\phi\partial_1\psi\partial_2 + \phi\psi\hat{\nabla}_{\partial_1}\partial_2 &= -\tfrac{1}{2}\phi\psi\partial_2(\ln|\lambda-\mu|)\partial_1, \\ \psi\partial_2\phi\partial_1 + \phi\psi\hat{\nabla}_{\partial_2}\partial_1 &= -\tfrac{1}{2}\phi\psi\partial_1(\ln|\lambda-\mu|)\partial_2.\end{aligned}$$

As $\hat{\nabla}$ is without torsion it yields:

$$-\partial_1\ln|\psi|\partial_2 - \tfrac{1}{2}\partial_2(\ln|\lambda-\mu|)\partial_1 = -\partial_2\ln|\phi|\partial_1 - \tfrac{1}{2}\partial_1(\ln|\lambda-\mu|)\partial_2.$$

Hence

$$\partial_2\left(\ln\frac{|\phi|}{\sqrt{|\lambda-\mu|}}\right) = 0, \qquad \partial_1\left(\ln\frac{|\psi|}{\sqrt{|\lambda-\mu|}}\right) = 0. \tag{18}$$

From (18) it follows $\psi = \beta(x_2)\sqrt{|\lambda-\mu|}$ and $\phi = \alpha(x_1)\sqrt{|\lambda-\mu|}$. Let us introduce new coordinates y_1, y_2 such that:

$$y_1 = \int\frac{1}{\alpha}(x_1), \qquad y_2 = \int\frac{1}{\beta}(x_2).$$

It is clear that in new coordinates $E_1 = \sqrt{|\lambda-\mu|}\partial_1$, $E_2 = \sqrt{|\lambda-\mu|}\partial_2$ and $h_{11} = \frac{\epsilon}{|\lambda-\mu|}$, $h_{22} = \frac{\eta}{|\lambda-\mu|}$. From (12) we get

$$(\lambda-\mu)K(E_1,E_2) = (E_1H)E_2 - (E_2H)E_1. \tag{19}$$

By (19) it is clear that in introduced coordinates the following equations are satisfied:

$$\begin{aligned} K(\partial_1,\partial_2) &= \frac{1}{\lambda-\mu}(-\partial_2H\partial_1 + \partial_1H\partial_2),\\ K(\partial_1,\partial_1) &= \frac{1}{\lambda-\mu}(-\partial_1H\partial_1 - \epsilon\eta\partial_2H\partial_2),\\ K(\partial_2,\partial_2) &= \frac{1}{\lambda-\mu}(\epsilon\eta\partial_1H\partial_1 + \partial_2H\partial_2),\end{aligned} \tag{20}$$

$$h_{11} = \frac{\epsilon}{|\lambda-\mu|}, \qquad h_{22} = \frac{\eta}{|\lambda-\mu|}, \tag{21}$$

$$S\partial_1 = \lambda\partial_1, \qquad S\partial_2 = \mu\partial_2. \tag{22}$$

By (20) and (21) one can easily obtain (using equality $\nabla = \hat{\nabla} + K$) the expressions for connection coefficients of ∇ in a chart (x_1,x_2):

$$\begin{array}{llllll} \Gamma^1_{11} = -\dfrac{\partial_1\lambda}{\lambda-\mu}, & \Gamma^2_{11} = -\dfrac{\epsilon\eta\partial_2\mu}{\lambda-\mu}, & \Gamma^1_{12} = -\dfrac{\partial_2\lambda}{\lambda-\mu},\\ \Gamma^2_{12} = \dfrac{\partial_1\mu}{\lambda-\mu} & \Gamma^1_{22} = \dfrac{\epsilon\eta\partial_1\lambda}{\lambda-\mu}, & \Gamma^2_{22} = \dfrac{\partial_2\mu}{\lambda-\mu}. \end{array} \tag{23}$$

On the other hand if we define a structure (∇, h, S) by (23), (21) and (22) then (20) holds, where $\nabla = \hat{\nabla} + K$ and $\hat{\nabla}$ is the Levi-Civitá connection for h, and the Codazzi and Ricci equations are satisfied. For example the Codazzi equation $\nabla S(X,Y) = \nabla S(Y,X)$ is equivalent to

$$\Gamma^1_{12} = -\frac{\partial_2\lambda}{\lambda-\mu}, \qquad \Gamma^2_{12} = \frac{\partial_1\mu}{\lambda-\mu}$$

(see [4]). It follows that (∇, h, S) is an induced Blaschke structure of a certain projectively flat affine surface with diagonalizable shape operator in $\mathbb{R}^3$ iff the Gauss equation is satisfied. It is easy to check that the Gauss equation is equivalent to the following system of nonlinear partial differential equations of a second order

$$\begin{aligned} \Delta_0\mu + \frac{2}{\lambda-\mu}|\nabla\mu|^2 &= \alpha\mu,\\ \Delta_0\lambda - \frac{2}{\lambda-\mu}|\nabla\lambda|^2 &= -\alpha\lambda,\\ \partial_1\lambda\partial_2\mu &= \partial_1\mu\partial_2\lambda,\end{aligned} \tag{24}$$

where $\Delta_0 = \epsilon\partial_1^2 + \eta\partial_2^2$, $|\nabla f|^2 = \epsilon\partial_1 f^2 + \eta\partial_2 f^2$, $\alpha = \mathrm{sgn}(\mu - \lambda) \in \{-1, 1\}$. In the other case we can assume that $\alpha = 1$ changing ϵ by $-\epsilon$ and η by $-\eta$. So there is one-to-one correspondence between solutions of (24) and projectively flat surfaces with diagonalizable shape operator. If $\lambda\mu = \tau$ is constant, then equation (24) can be reformulated after some simple computations as (we set $\alpha = 1$)

$$\Delta_0\lambda + \frac{2\lambda}{\tau - \lambda^2}(\epsilon(\partial_1\lambda)^2 + \eta(\partial_2\lambda)^2) = -\lambda. \tag{25}$$

Next we consider three cases: $\tau > 0$, $\tau = 0$, $\tau < 0$. Note that equation (25) is hyperbolic if and only if $\epsilon\eta = -1$.

i) $\tau > 0$. As $\lambda \neq \mu$ on U and $\lambda\mu = \tau$ we can assume that for example $|\lambda| < \sqrt{\tau}$. Let us define a function Ψ by $\Psi = -2\mathrm{arc\ tgh}(\frac{\lambda}{\sqrt{\tau}})$. Then equation (25) is

$$\Delta_0\Psi = -\sinh\Psi.$$

ii) $\tau = 0$. In that case if we define $\Psi = \frac{1}{\lambda}$ where λ is the nonzero eigenvalue of S then equation (25) is

$$\Delta_0\Psi = -\Psi.$$

iii) $\tau < 0$. Let us define a function Ψ by $\Psi = -2\mathrm{arc\ ctg}(\frac{\lambda}{\sqrt{-\tau}})$. Then equation (25) is

$$\Delta_0\Psi = \sin\Psi.$$

In every case above the surface (M, f) with constant τ is an affine locally symmetric surface which follows from [4] and the uniqueness theorem in affine differential geometry (see [3]). These surfaces have constant Gauss curvature with respect to an appropriate nondegenerate scalar product in $\mathbb{R}^3$ (definite or indefinite).

(b) Now let us assume that (∇, h, S) is projectively flat and that a shape operator S has a complex eigenvalue. In [4] to describe locally symmetric surfaces with such a shape operator we used only assumption $\mathrm{tr}_h \nabla S = 0$ and τ is constant which holds also for projectively flat structures with a constant curvature. Hence one can repeat the proof literally for our case. In particular equation $(E4)$ is also a consequence of the Gauss equation for (∇, h, S) or of the Affine Theorema Egregium. Consequently in that case a projectively flat connection is locally symmetric. Let us note that for a general projectively flat surface in the considered case the considerations in [4] are still valid and we have in coordinates introduced in [4] using notation introduced there (we take $b' = b, c' = -b$ see [4], p. 217 as we can assume $\alpha = 1$)

$$S\partial_1 = a\partial_1 + b\partial_2, \qquad S\partial_2 = -b\partial_1 + a\partial_2,$$
$$h_{12} = \frac{1}{b}, \qquad h_{ii} = 0,$$
$$\Gamma^1_{11} = -\frac{\partial_1 b}{b}, \qquad \Gamma^2_{11} = \frac{\partial_1 a}{b},$$
$$\Gamma^1_{12} = 0, \qquad \Gamma^2_{12} = 0,$$
$$\Gamma^1_{22} = -\frac{\partial_2 a}{b}, \qquad \Gamma^2_{22} = -\frac{\partial_2 b}{b},$$

and a structure (∇, h, S) defined by these equations is integrable iff a and b satisfy the following equations (the Gauss equation)

$$b\partial_{12}a - 2\partial_1 a\partial_2 b = -b^2, \quad b\partial_{12}b + \partial_1 a\partial_2 a - \partial_1 b\partial_2 b = ab. \tag{26}$$

If the affine Gauss curvature τ of the above surface is constant then $a^2 + b^2 = \tau =$ const and let us define a function ϕ by the equations

$$a = \sqrt{\tau}\sin\phi, \qquad b = \sqrt{\tau}\cos\phi.$$

Then equation (26) is equivalent to

$$\sin\phi\,\partial^2_{12}\phi - \partial_2\phi\,\partial_2\phi\,\cos\phi = \sin\phi. \tag{27}$$

Let us define $\Psi = 2\text{arc tg}e^{\phi}$. Then equation (27) is

$$\partial_1\partial_2\Psi = \cosh\Psi.$$

Every surface with constant τ considered in that case is a semi-Riemannian surface with constant Gauss curvature immersed in the Lorentz space $\mathbb{R}^3$.

It is known that affine locally symmetric surfaces with nondegenerate γ have constant Gauss curvature with respect to some scalar product in $\mathbb{R}^3$ and that this induced semi-Riemannian structure coincides with the Blaschke structure. It follows easily from the fact, that affine normal ξ of such surfaces lies on a centro-affine quadric $\Sigma = \{X : L(X,X) = 1\}$, where L is a nondegenerate symmetric form on $\mathbb{R}^3$ (see [4]). Differentiating an equation $L(\xi,\xi) = 1$ we get for every $X \in TM$ $L(\xi_*(X),\xi) = 0$ and consequently $L(f_*(SX),\xi) = 0$. Hence ξ is a normal Riemanian field for (M,f) with respect to L and (M,f) is nondegenerate submanifold of $(\mathbb{R}^3, L)$. L is a scalar product with respect to which (M,f) has a constant nonzero Gauss curvature as the Riemannian structure coincides with the Blaschke structure, in particular is locally symmetric. On the other hand every surface with such a property is affine locally symmetric. In fact if g is the induced semi-Riemannian metric then for the second fundamental form we have $h(X,Y) = g(SX,Y)$ where S is the (Riemannian) shape operator. Hence, where by ν_h, ν_g we denote volume forms of metrics h and g, $\nu_h = \sqrt{|\det S|}\nu_g$ and if $\nabla g = 0$ then $\nabla \nu_h = \sqrt{|\det S|}\nabla \nu_g = 0$ as $\det S$ is constant. The last equation is equivalent to the apolarity condition (5). Hence the induced semi-Riemannian connection coincides with the induced Blaschke connection.

3 Affine locally strongly convex spheres

A surface (M, f) is called an *affine sphere* if $S = HI$ for some $H \in \mathbb{R}$. A surface (M, f) is an affine sphere if and only if $\hat{\nabla} K$ is a symmetric tensor. The other condition characterizing uniquely an affine sphere is the equation

$$R(X,Y)Z = \hat{R}(X,Y)Z + [K_X, K_Y]Z. \tag{28}$$

Let (M, f) be an affine locally strongly convex surface in $\mathbb{R}^3$ with a Blaschke structure (∇, h, S). Then the following lemmas hold.

LEMMA A. *Let $x_0 \in M$ satisfies the condition $K_{x_0} \neq 0$. Then there exists an open neighborhood V of x_0 and a local orthonormal frame $\{X, Y\}$ defined on V and satisfying the equations:*

$$K(X,X) = -\lambda Y, \quad K(X,Y) = -\lambda X, \quad K(Y,Y) = \lambda Y$$

with $\lambda = \frac{1}{2}|K| = \frac{1}{4}\sqrt{h(C,C)}$. In the basis $\{X, Y\}$ the endomorphisms K_X, K_Y are represented by the following matrices :

$$K_X = \begin{pmatrix} 0 & -\lambda \\ -\lambda & 0 \end{pmatrix}, \quad K_Y = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

An acute angle between any two null directions of the cubic form is $\frac{1}{3}\pi$ and for any vectors U, V, W the endomorphisms K_U, K_V satisfy the equation

$$[K_U, K_V](W) = -J(h(V,W)U - h(U,W)V). \tag{29}$$

Proof: Let $\{E_1, E_2\}$ be an orthonormal local frame defined on an open set $V \subseteq M$ and $x_0 \in$ V. From (4) it follows that there exist smooth functions $a, b \in C^\infty(V)$ such that

$$K(E_1, E_1) = aE_1 + bE_2, K(E_1, E_2) = bE_1 - aE_2.$$

Let us take $X = \sin(\phi)E_1 - \cos(\phi)E_2$, $Y = \cos(\phi)E_1 + \sin(\phi)E_2$ where ϕ is a smooth function. It is clear that $a^2 + b^2 = \frac{1}{16}h(C,C)$. Let us take $\lambda = \frac{1}{4}\sqrt{h(C,C)}$. Then $J = 2\lambda^2$. Locally there exists a function $\psi \in C^\infty(V)$ such that $a = \lambda\cos\psi, b = \lambda\sin\psi$.

Notice that:

$$\begin{aligned} K(Y,Y) &= (\cos(\phi)^2-\sin(\phi)^2)K(E_1,E_1)+2\sin(\phi)\cos(\phi)K(E_1,E_2)\\ &= \cos(2\phi)K(E_1,E_1)+\sin(2\phi)K(E_1,E_2)\\ &= (a\cos(2\phi)+b\sin(2\phi))E_1+(b\cos(2\phi)-a\sin(2\phi))E_2\\ &= \lambda(\cos(2\phi-\psi)E_1+\sin(\psi-2\phi)E_2)\\ &= \lambda(\cos(\psi-2\phi)E_1+\sin(\psi-2\phi)E_2). \end{aligned}$$

Hence $K(Y,Y)=\lambda Y$ if and only if $\phi=\frac{1}{3}\psi+\frac{2k\pi}{3}, k\in\mathbb{Z}$. It is easy to check using (4) that with such a choice of ϕ the other equations are satisfied. Let us note that X lies on the null direction of C (see [11]). As the angles ϕ and $\phi+\pi$ give the same null direction of C the lemma is proved. □

LEMMA B. *Let (M,f) be an affine locally strongly convex surface with induced Blaschke structure and $\{E_1,E_2\}$ be a local orthonormal frame satisfying the conditions stated in Lemma A . Then the following equations are satisfied :*

$$\begin{aligned} \hat{\nabla}K(X,E_1,E_1) &= -3\lambda\omega(X)E_1-(X\lambda)E_2,\\ \hat{\nabla}K(X,E_1,E_2) &= -(X\lambda)E_1+3\lambda\omega(X)E_2,\\ \hat{\nabla}K(X,E_2,E_2) &= 3\lambda\omega(X)E_1+(X\lambda)E_2, \end{aligned} \tag{30}$$

with $\omega=\omega_2^1$ the connection form defined by $\hat{\nabla}_X E_i=\omega_i^j(X)E_j$.

COROLLARY. *Let (M,f) be an affine locally strongly convex sphere in $\mathbb{R}^3$. Let us define $U:=\{x\in M : h(C,C)>0\}$. Then for every $x_0\in U$ there exists a local coordinate system (V,x_1,x_2)such that V is a neighborhood of $x_0, V\subseteq U$, and equations*

$$E_1=e^u\partial_1,\quad E_2=e^u\partial_2 \tag{31}$$

hold, where $u=\frac{1}{3}\ln\lambda$, $\lambda=\frac{1}{4}\sqrt{h(C,C)}$ and $\{E_1,E_2\}$ is a local frame satisfying equations (31). The coordinates (V,x_1,x_2) are isothermal coordinates for (M,h) and the following equations are satisfied:

$$h(\partial_1,\partial_1)=h(\partial_2,\partial_2)=e^{-2u},\quad h=e^{-2u}((dx_1)^2+(dx_2)^2) \tag{32}$$

$$K(\partial_1,\partial_1)=-e^{2u}\partial_2,\quad K(\partial_2,\partial_2)=e^{2u}\partial_2,\quad K(\partial_1,\partial_2)=-e^{2u}\partial_1 \tag{33}$$

and

$$\begin{aligned} \nabla_{\partial_1}\partial_1 &= -\partial_1u\partial_1+(\partial_2u-e^{2u})\partial_2\\ \nabla_{\partial_1}\partial_2 &= -(\partial_2u+e^{2u})\partial_1-\partial_1u\partial_2\\ \nabla_{\partial_2}\partial_2 &= \partial_1u\partial_1-(\partial_2u-e^{2u})\partial_2 \end{aligned} \tag{34}$$

$$\Delta_0u=e^{-2u}(H+2e^{6u}),$$

or equivalently

$$\Delta_0 \ln \lambda = 3\lambda^{-\frac{2}{3}}(H + 2\lambda^2),$$

where Δ_0 *is the standard Laplacian on* $\mathbb{R}^2$. *The condition* $h(C,C) = \mathrm{const}$ (*or equivalently* $K_h = \mathrm{const.}$) *is equivalent to* $\hat{\nabla}K = 0$ (*or equivalently* $\hat{\nabla}C = 0$).

Proof: From (30) we obtain that $\hat{\nabla}K$ is symmetric if and only if equations (35) below are satisfied.

$$\omega(E_1) = -du(E_2), \quad \omega(E_2) = du(E_1). \tag{35}$$

Let (W, y_1, y_2) be a coordinate system such that

$$E_1 = \phi\partial_1, \; E_2 = \psi\partial_2$$

for some smooth functions ϕ, ψ. We can assume that ϕ, ψ are positive in an opposite case changing coordinates as follows: $(y_1, y_2) \to (\epsilon y_1, \eta y_2)$ where $\epsilon, \eta \in \{-1, 1\}$. The equations

$$\hat{\nabla}_{E_2}E_1 = -\omega(E_2)E_2, \; \hat{\nabla}_{E_1}E_2 = \omega(E_1)E_1$$

hold on U. Hence we get

$$\partial_1\psi\partial_2 + \psi\hat{\nabla}_{\partial_1}\partial_2 = -\psi\partial_2 u\partial_1,$$
$$\partial_2\phi\partial_1 + \phi\hat{\nabla}_{\partial_2}\partial_1 = -\phi\partial_1 u\partial_2.$$

As $\hat{\nabla}$ is without torsion we obtain

$$\partial_1 \ln\psi = \partial_1 u, \qquad \partial_2 \ln\phi = \partial_2 u. \tag{36}$$

From (36) it follows that there exist smooth positive functions Φ, Ψ such that

$$\phi = e^u\Phi(y_1), \qquad \psi = e^u\Psi(y_2).$$

Let us change coordinates in the following way

$$x_1 = \int\left(\frac{1}{\Phi}\right)(y_1), \qquad x_2 = \int\left(\frac{1}{\Psi}\right)(y_2).$$

It is clear that in new coordinates equations (31) hold. Let us note also that from (31), (28) and equations

$$R(X,Y)Z = H(h(Y,Z)X - h(X,Z)Y), \tag{37}$$
$$\hat{R}(X,Y)Z = K_h(h(Y,Z)X - h(X,Z)Y), \tag{38}$$

where H is an affine mean curvature of an affine sphere (M, f) and K_h is the Gauss curvature of (M, h), we get the Affine Theorema Egregium

$$K_h = H + 2e^{6u} = H + \frac{1}{8}h(C,C). \tag{39}$$

Equation (39) is valid for any affine surface as one can check computing the Ricci tensor of the affine metric. From (39) it is clear that $K_h = \text{const}$ iff $h(C, C) = \text{const}$. It also follows from (30) and (35) that $h(C, C)$ is constant iff $\hat{\nabla} K = 0$. In a case $h(C, C) = 0$ it is obvious as then $K = 0$. Let us note also that in new coordinates

$$h = e^{-2u}((dx_1)^2 + (dx_2)^2). \tag{40}$$

Hence

$$\Delta u = K_h = H + 2e^{6u}$$

where Δ is the Laplacian for (M, h). This equation is equivalent to

$$\Delta_0 u = e^{-2u}(H + 2e^{6u}), H \in \mathbb{R} \tag{41}$$

where $\Delta_0 = \partial_1^2 + \partial_2^2$ is the standard Laplacian on $\mathbb{R}^2$. □

THEOREM 2. *Let u satisfy equation (41) on an open simply connected set $\Omega \subseteq \mathbb{R}^2$. Then there exists an immersion $f_u : \Omega \to \mathbb{R}^3$ such that (Ω, f_u) is a locally strongly convex affine sphere with an affine mean curvature H and Fubini–Pick invariant $16e^{6u}$. In the standard coordinates on Ω equations (32), (33), (34) hold where h is an affine metric, K is the difference tensor and $\nabla = \hat{\nabla} + K$ is the Blaschke connection for (Ω, f_u). An immersion f is unique up to an affine transformation.*

Let (M, f) be a locally strongly convex affine surface with induced Blaschke structure which is an affine sphere with affine mean curvature H. Then $M = U \cup \bar{V} = \bar{U} \cup V$ where U, V are open subsets of M such that $U \cap V = \emptyset$, (U, f) is a locally strongly convex quadric and $V = \{x : h(C, C) > 0\}$. The function $\lambda := \frac{1}{4}\sqrt{h(C, C)}$ satisfies on V the equation

$$\Delta \ln \lambda = 3(H + 2\lambda^2)$$

and around any point $x_0 \in V$ there exist local isothermal coordinates (W, x_1, x_2) such that the function $u = \frac{1}{3}\ln \lambda \circ \gamma^{-1}$ satisfies on $\gamma(W)$ equation (41) where $\gamma(p) = (x_1(p), x_2(p))$ and (W, f) is equiaffinely equivalent to an affine sphere in R^3 given by $(\gamma(W), f_u)$.

Proof : Let a function u satisfies on Ω equation (41). Let us define an affine metric h, a connection ∇ and a difference tensor K by formulae (32), (34) and (33). Let us define also a connection $\hat{\nabla}$ as $\nabla = \hat{\nabla} + K$. It is clear that $\hat{\nabla}$ is the Levi-Civitá connection for h. If we define an orthonormal frame $\{E_1, E_2\}$ by equations (31) and by (35) the differential form ω then it easy to check that equations (28') hold for $\{E_1, E_2\}$ and ω is the connection form ω_2^1 with respect to $\{E_1, E_2\}$. From (30) it follows that $\hat{\nabla} K$ is a symmetric tensor. Hence equation (28) is satisfied where R is the curvature tensor of ∇. Let us define a shape operator by $S = H\ \mathrm{Id}_{T\Omega}$. From (28), (38) and (29) it follows that the Gauss equation (37) is equivalent to (39) which in turn is equivalent to (41). From (33) and an equation $\nabla = \hat{\nabla} + K$ it easily follows that $C = \nabla h$ is a symmetric tensor. Hence equations of Gauss, Codazzi and Ricci are satisfied for the structure (∇, h, S). From Radon's theorem (see [3]) it follows that there exists a nondegenerate immersion $f : \Omega \to \mathbb{R}^3$ with (∇, h, S) as the induced Blaschke structure. The uniqueness up to an equiaffine transformation follows from [2].

The second part of the theorem follows from lemmas and Corollary. □

Remark. Let us note that the equation (41) is equivalent to the equation

$$\Delta_0 \Psi = e^{2\Psi} + \epsilon e^{-\Psi}$$

where $\epsilon = \operatorname{sgn} H \in \{-1, 0, 1\}$ and $\Psi(x_1, x_2) = 2u(\frac{x_1}{a}, \frac{x_2}{a}) - \ln b$ where $a = (4|H|)^{\frac{1}{3}}$, $b = \left(\frac{|H|}{2}\right)^{\frac{1}{3}}$ if $H \neq 0$ and $\Psi = 2u(\frac{x_1}{2}, \frac{x_2}{2})$ if $H = 0$.

It is interesting to know to what extent an affine sphere is determined by its affine metric and the Fubini–Pick invariant. The following corollary gives an answer to this question.

Corollary. *Let (M, f) be an affine sphere in $\mathbb{R}^3$ with a definite affine metric h and Fubini-Pick invariant $h(C, C)$. Let us assume that $h(C, C)$ is positive on M. Then for every $x_0 \in M$ there exists a neighborhood V of x_0 and a one parameter family of affine immersions $\{f_a : a \in O(2)\}$ defined on V, such that (V, f_a) is an affine sphere and each immersion f_a has the same induced affine metric h and the same Fubini–Pick invariant $h(C, C)$. Every affine sphere immersion whose induced Blaschke structure has an affine metric h and the Fubini–Pick invariant $h(C, C)$ is locally equiaffinely equivalent to one of immersions f_a.*

Proof: Let $(M, f), (M, f')$ be two affine spheres with the same affine metrics and Fubini–Pick invariants. From the theorem for every point $x_0 \in M$ there exist local charts $(U, x_1, x_2), (U', x'_1, x'_2)$ around x_0 such that equations (33), (31) and (34) hold respectively for $u = \frac{1}{3} \ln \lambda$, $u' = \frac{1}{3} \ln \lambda'$ where $\lambda = \frac{1}{4}\sqrt{h(C, C)} \circ \gamma^{-1}$, $\lambda' = \frac{1}{4}\sqrt{h(C, C)} \circ \gamma'^{-1}$. Let us define $\phi = \gamma' \circ \gamma^{-1} = (x'(x_1, x_2), x'(x_1, x_2))$. Then $\lambda = \lambda' \circ \phi$ and $h_{ij} = h'_{ij} \circ \phi$. If we denote $A^i_j = \frac{\partial x'_i}{\partial x_j}$ then the transformation rules for h give us the following equations: $(A^1_1)^2 + (A^2_1)^2 = 1$, $(A^2_2)^2 + (A^1_2)^2 = 1$, $A^1_1 A^1_2 + A^2_1 A^2_2 = 0$. Hence $A^1_1 = \cos\alpha, A^2_1 = \sin\alpha, A^2_2 = \epsilon\cos\alpha, A^1_2 = -\epsilon\sin\alpha$ for $\epsilon \in \{-1, 1\}$ and α a smooth function. As $\partial_i A^k_j = \partial_j A^k_i$ it is easy to check that α is constant. We conclude that $\phi = g$ where

$$g = \begin{pmatrix} \cos\alpha & -\epsilon\sin\alpha & a \\ \sin\alpha & \epsilon\cos\alpha & b \\ 0 & 0 & 1 \end{pmatrix} \in AO(2), \quad \alpha, a, b \in \mathbb{R}, \tag{42}$$

is an affine orthogonal transformation of $\mathbb{R}^2$. The group $AO(2) = E(2)$ is a symmetry group of equation (41). Let us note that the difference tensor K' of (M, f') in the chart $(V, x_1, x_2), V = \operatorname{dom}\gamma \cap \operatorname{dom}\gamma'$, is represented as follows (see Lemma A)

$$\begin{aligned} K'(\partial_1, \partial_1) &= -e^{2u}(\sin 3\alpha\partial_1 + \epsilon\cos 3\alpha\partial_2), \\ K'(\partial_2, \partial_2) &= e^{2u}(\sin 3\alpha\partial_1 + \epsilon\cos 3\alpha\partial_2), \\ K'(\partial_1, \partial_2) &= -e^{2u}(\epsilon\cos 3\alpha\partial_1 - \sin 3\alpha\partial_2), \end{aligned} \tag{43}$$

and $h, \hat\nabla$ are the same for f and f'. h is given by (40) and $\hat\nabla$ is determined by h. On the other hand let us assume that u satisfies equation (41) on a simply connected subset Ω of

$\mathbb{R}^2$, $\Omega = \gamma(U)$ and h is given by (40). Let us denote by $\hat{\nabla}$ the Levi-Civitá connection for h. It is easy to show that tensor $\hat{\nabla} K_\alpha$ where K_α is defined by formula (α) is symmetric. It is also clear that K_α satisfies (4). Hence one can show as in the proof of the Theorem that there exists an immersion f_a with an induced affine metric h and a difference tensor K_α , where $\nabla = \hat{\nabla} + K_\alpha$ and

$$a = \begin{pmatrix} \cos 3\alpha & -\epsilon \sin 3\alpha \\ \sin 3\alpha & \epsilon \cos 3\alpha \end{pmatrix} \tag{44}$$

is an element of the group $O(2)$. Let us note that $K_\alpha = K_{\alpha + \frac{2\pi}{3}}$. It is obvious from the construction of coordinates as the angle between two null directions is $\frac{2\pi}{3}$. □

Remark. Let us take $M = \Omega$ and $f = f_u$ where u satisfies equation (41). We can construct as above a family of immersions f_a. Let us note that f_a are not equiaffinely equivalent as affine immersions $f_a : \Omega \to \mathbb{R}^3$. It is interesting to know the conditions under which (Ω, f_a) is equivalent to (Ω, f_{id}) as an affine hypersurface. We have the following :

Proposition. *Let a function u satisfies on Ω equation* (41) *and let* (Ω, f_u) *be an affine sphere with affine metric and Blaschke connection given by formulae* (40) *and* (34). *Let* (Ω, f_a) *be as above,* $f_{id} = f_u$. *Then* (Ω, f_a) *is affinely equivalent to* (Ω, f_u) *iff* $u = u \circ g$, *where a is given by the formula* (44) *and g is given by* (42) *for some* $a, b \in R$.

Proof: Let us assume that there exists a diffeomorphism $\phi : \Omega \to \Omega$ such that $f_u \circ \phi = A \circ f_a$ for an equiaffine transformation A. Then ϕ is an isometry and an affine diffeomorphism with respect to ∇ and $\nabla_\alpha = \hat{\nabla} + K_\alpha$, which means $\nabla_{\phi_*(X)} \phi_*(Y) = \phi_*(\nabla_{\alpha X} Y)$ for every vector fields X, Y on Ω. Hence

$$\phi_*(K_\alpha(X,Y)) = K(\phi_*(X), \phi_*(Y)).$$

It follows $h_\alpha(C,C) \circ \phi^{-1} = h_{id}(C,C)$. Hence $\lambda \circ \phi^{-1} = \lambda$ and $u \circ \phi^{-1} =$ u. As ϕ is an isometry and $u \circ \phi^{-1} = u$ it is easy to show as above that ϕ has a form of (42). On the other hand if $u \circ g = u$ then one can check exactly as in the Corollary above that (Ω, f_a) is affinely equivalent to (Ω, f_u) which concludes the proof. □

Remark. Let us note that if u is constant then all hypersurfaces (Ω, f_a) are equiaffinely equivalent. In particular a locally strongly convex affine sphere with constant Fubini–Pick invariant $h(C,C) \neq 0$ is characterized uniquely by $h(C,C)$. In fact every affine sphere with constant nonzero Fubini–Pick invariant and definite affine metric h is equiaffinely equivalent to an open part of the surface $(X^2 - Y^2)Z = \pm\frac{1}{c}$ where $c \neq 0$ depends only on $h(C,C)$. The fundamental system of equations for f is very simple for constant λ (see (34)) and it is easy to see that if we take a chart for which $E_i = \partial_i$ then

$$f(x_1, x_2) = (e^{-\lambda x_2} \cosh(\sqrt{3}\lambda x_1), e^{-\lambda x_2} \sinh(\sqrt{3}\lambda x_1), \pm\frac{1}{c} e^{2\lambda x_2})$$

where $c = \frac{3\sqrt{3}}{128} h(C,C)^2$ (see also [6])). From our Proposition it follows that the surface $M = \{(X,Y,Z) : (X^2 - Y^2)Z = \pm\frac{1}{c}\}$ is equiaffinely homogeneous. It is easy to check that $(\mathbb{R}^2, f)$ is an orbit of the point $(1, 0, \pm\frac{1}{c})$ by the group $G = \mathbb{R} \oplus \mathbb{R}$ of equiaffine transformations

$$\begin{pmatrix} e^{-a}\cosh b & e^{-a}\sinh b & 0 \\ e^{-a}\sinh b & e^{-a}\cosh b & 0 \\ 0 & 0 & e^{2a} \end{pmatrix} \qquad a, b \in \mathbb{R} \oplus \mathbb{R}.$$

4 Affine spheres with an indefinite affine metric

Let (M, f) be an affine surface with an induced Blaschke structure whose affine metric is indefinite. Let $x_0 \in M$ be any point of M and $\{E_1, E_2\}$ be a local orthonormal frame defined on a neighborhood U of the point x_0, such that $h(E_1, E_1) = 1, h(E_2, E_2) = -1$.

Let K be a difference tensor of the induced structure.Then the equations

$$K(E_1, E_1) = aE_1 + bE_2, \qquad K(E_1, E_2) = -(bE_1 + aE_2)$$

hold for some smooth functions $a, b \in C^\infty(U)$. The above equations are a consequence of (4). Let us note that $h(K(E_1, E_1), K(E_1, E_1)) = a^2 - b^2$ and $h(C, C) = 16(a^2 - b^2)$. Let us denote $\epsilon := \text{sign } (h(C, C))$ and $\lambda := \sqrt{\epsilon(a^2 - b^2)}$. Let us assume that $\lambda(x_0) \neq 0$ and define $V := \{x : \epsilon(x) = \epsilon(x_0)\} \cap U$. V is an open neighborhood of x_0. Now we consider three cases :

i) $\epsilon(x_0) = 1$ which means $(a^2 - b^2) > 0$. Then there exists a function $\psi \in C^\infty(V)$ such that $a = \lambda\cosh(\psi), b = \lambda\sinh(\psi)$. Let us define a new orthonormal frame $\{X, Y\}$ by the equations

$$X = \cosh(\phi)E_1 + \sinh(\phi)E_2, \qquad Y = \sinh(\phi)E_1 + \cosh(\phi)E_2, \tag{45}$$

where ϕ is a smooth function. Then

$$\begin{aligned} K(X, X) &= (\cosh(\phi)^2 + \sinh(\phi)^2)K(E_1, E_1) \\ &\quad +2\sinh(\phi)\cosh(\phi)K(E_1, E_2) \\ &= \cosh(2\phi)K(E_1, E_1) + \sinh(2\phi)K(E_1, E_2) \\ &= (a\cosh(2\phi) - b\sinh(2\phi))E_1 + (b\cosh(2\phi) - a\sinh(2\phi))E_2 \\ &= \lambda(\cosh(2\phi - \psi)E_1 + \sinh(\psi - 2\phi)E_2) \\ &= \lambda(\cosh(\psi - 2\phi)E_1 + \sinh(\psi - 2\phi)E_2). \end{aligned}$$

Hence $K(X, X) = \lambda X$ if and only if $\phi = \frac{1}{3}\psi$. From (45) it is clear that $K(X, Y) = -\lambda Y$ if we choose ϕ as above.

ii) $\epsilon(x_0) = -1$. Then there exists a function $\psi \in C^\infty(V)$ such that $a = \lambda\sinh(\psi)$, $b = \lambda\cosh(\psi)$. Let us define a new orthonormal frame $\{X, Y\}$ by equations (45). It is easy to show as above that

$$K(X, X) = \lambda(\sinh(\psi - 2\phi)E_1 + \cosh(\psi - 2\phi)E_2).$$

Hence in that case $K(X, X) = \lambda Y$, $K(X, Y) = -\lambda X$ for $\phi = \frac{1}{3}\psi$.

iii) Now we consider the case $\epsilon(x_0) = 0$. Then, in general, we can not expect that $\epsilon(x) = 0$ in a certain neighborhood of x_0. Let us assume that there exists a neighborhood V of x_0 such that $\epsilon(x) = 0$ for $x \in$ V. Then in V an equality $a^2 = b^2$ holds and:

$$K(E_1, E_1) = aE_1 \pm aE_2, \qquad K(E_1, E_2) = -(\pm aE_1 + aE_2)$$

and we can assume $a = b$ in an opposite case changing E_1 by $-E_1$. Let us define local fields X, Y by $X = (E_1 + E_2)$, $Y = \frac{1}{2}(E_1 - E_2)$. Then it is easy to check that

$$K(X,X) = 0, \quad K(X,Y) = 0, \quad K(Y,Y) = aX$$

and

$$h(X,X) = h(Y,Y) = 0, \qquad h(X,Y) = 1.$$

Hence we have proved:

Lemma A1. *Let (M, f) be as above and $x_0 \in$ M. Let us define $\alpha := \epsilon(x_0)$ and take $M_\alpha := \mathrm{int}(\{x : \epsilon(x) = \alpha\} \cap M)$. If $x_0 \in M_\alpha$ then there exists a local frame $\{E_1, E_2\}$ defined on a neighborhood $V_\alpha \subseteq M_\alpha$ of x_0 and satisfying respectively for $\alpha = 1, -1, 0$ the equations :*

$$\begin{array}{llll}
\alpha = 1 & K(E_1,E_1) = \lambda E_1, & K(E_1,E_2) = -\lambda E_2, & K(E_2,E_2) = \lambda E_1. \quad (46)\\
& h(E_1,E_1) = 1, & h(E_1,E_2) = 0, & h(E_2,E_2) = -1;\\
\alpha = -1 & K(E_1,E_1) = \lambda E_1, & K(E_1,E_2) = -\lambda E_2, & K(E_2,E_2) = \lambda E_1. \quad (47)\\
& h(E_1,E_1) = -1, & h(E_1,E_2) = 0, & h(E_2,E_2) = 1;\\
\alpha = 0 & K(E_1,E_1) = 0, & K(E_1,E_2) = 0, & K(E_2,E_2) = aE_1, \quad (48)\\
& h(E_1,E_1) = h(E_2, & E_2) = 0, & h(E_1,E_2) = 1,
\end{array}$$

where $\lambda = \frac{1}{4}\sqrt{\alpha h(C,C)}$ and $a \in C^\infty(V_0)$.

Corollary. *For a difference tensor K of a Blaschke structure (∇, h, S) the following equation holds:*

$$[K_X, K_Y]Z = -J(h(Y,Z)X - h(X,Z)Y).$$

Proof: It follows from Lemmas A and A1. □

Remark. It is clear that $\{x : \epsilon(x) = \alpha\} \cap M$ is an open set for $\alpha \in \{-1, 1\}$. Let us note also that relations (47) could be obtained from (46) by replacing h by $-h$, which corresponds to a change of orientation of M. Hence we can restrict ourselves to the investigation of surfaces (M, f) satisfying (46) or (48).

Lemma B1. *Let (M, f) be an affine nonconvex surface with induced Blaschke structure and $\{E_1, E_2\}$ be a local frame on M. If $\{E_1, E_2\}$ satisfies relations (46) then*

$$\hat\nabla K(X, E_1, E_1) = (X\lambda)E_1 + 3\lambda\omega(X)E_2,$$
$$\hat\nabla K(X, E_1, E_2) = -3\lambda\omega(X)E_1 - (X\lambda)E_2,$$
$$\hat\nabla K(X, E_2, E_2) = (X\lambda)E_1 + 3\lambda\omega(X)E_2,$$

with $\omega = \omega_2^1 = \omega_1^2$ *the connection form defined by* $\hat{\nabla}_X E_i = \omega_i^j(X)E_j$. *If* $\{E_1, E_2\}$ *satisfies relations* (48) *then the equations below hold*

$$\hat{\nabla}K(X, E_1, E_1) = 0, \quad \hat{\nabla}K(X, E_1, E_2) = 0, \quad \hat{\nabla}K(X, E_2, E_2) = ((Xa) - 3a\omega(X))E_1$$

with $\omega = \omega_1^1 = -\omega_2^2$

Corollary 1. *Let* (M, f) *be an affine sphere and* $U = \{h(C, C) \neq 0\}$. *Then for every point* $x_0 \in U$ *there exists a chart* (W, x_1, x_2) *with an associated local frame* $\{\partial_1, \partial_2\}(\partial_i = \frac{\partial}{\partial x_i})$ *such that* $x_0 \in W$, $\alpha = \operatorname{sgn} h(C, C)$ *is constant on* W *and the equations below hold:*

$$E_1 = e^u \partial_1, \qquad E_2 = e^u \partial_2,$$

$$h = \alpha e^{-2u}((dx_1)^2 - (dx_2)^2), \tag{49}$$

$$K(\partial_1, \partial_1) = e^{2u}\partial_1, \quad K(\partial_2, \partial_2) = e^{2u}\partial_1, \quad K(\partial_1, \partial_2) = -e^{2u}\partial_2 \tag{50}$$

and

$$\begin{aligned} \nabla_{\partial_1}\partial_1 &= (-\partial_1 u + e^{2u})\partial_1 - \partial_2 u \partial_2, \\ \nabla_{\partial_1}\partial_2 &= -\partial_2 u \partial_1 - (\partial_1 u + e^{2u})\partial_2, \\ \nabla_{\partial_2}\partial_2 &= (-\partial_1 u + e^{2u})\partial_1 - \partial_2 u \partial_2, \end{aligned} \tag{51}$$

$$\Delta_0 u = e^{-2u}(\alpha H + 2e^{6u}), \tag{52}$$

or equivalently

$$\Delta_0 \ln \lambda = 3\lambda^{-\frac{2}{3}}(\alpha H + 2\lambda^2),$$

where $\Delta_0 = \partial_1^2 - \partial_2^2$ *is the standard Laplacian on* $(R^2, dx_1^2 - dx_2^2)$, $u = \frac{1}{3}\ln\lambda$ *and* $\lambda = \frac{1}{4}\sqrt{\alpha h(C, C)}$.

Proof: We omit the proof as it just the same as the proof of Corollary below Lemma B. □

Corollary 2. *Let us assume that* (M, f) *is an affine sphere, with indefinite affine metric* h *and affine mean curvature* H, *for which* $h(C, C) = 0$ *on* M. *Then the Gauss curvature* K_h *of* (M, h) *is constant and equals* H. *Let us define* $U = \operatorname{int}\{x \in M : K_x = 0\}$ *and* $V = \{x \in M : K_x \neq 0\}$ *where* K *is the difference tensor of* (M, f). *An affine surface* (U, f) *is a quadric and for every point of* V *there exist a local frame* $\{E_1, E_2\}$ *and local coordinates* (W, x_1, x_2) *such that the following equations hold:*

$$\begin{array}{ccc} E_1 = e^u\partial_1, & & E_2 = \partial_2, \\ K(\partial_1, \partial_1) = 0, & K(\partial_1, \partial_2) = 0, & K(\partial_2, \partial_2) = e^u\partial_1, \\ h(\partial_1, \partial_1) = h(\partial_2, \partial_2) = 0, & & h(\partial_1, \partial_2) = e^{-u}, \end{array}$$

$$\partial_1(\partial_2 u) = H\exp(-u). \tag{53}$$

In the introduced coordinates the immersion f in a case $H \neq 0$ has the form:

$$f(z_1, z_2) = \frac{1}{H}(\partial_2 u(z_1, z_2)\xi(z_2) - \xi^{'}(z_2)) \tag{54}$$

where $\xi : \mathbb{R} \to \mathbb{R}^3$ is a smooth curve in $\mathbb{R}^3$ defined by the differential equation

$$\xi^{'''} = a\xi' + b\xi \tag{55}$$

with the initial condition $\det(\xi, \xi', \xi'')_0 = H$, where a, b are some smooth functions such that $a' - 2b = 2H$. In a case $H = 0$ (V, f) is equiaffinely equivalent to the graph of a function $z(x, y) = xy + \phi(y)$ where ϕ is *any smooth function such that $\phi''' \neq 0$.*

Proof: We consider only the case $H \neq 0$ as the case $H = 0$ is well known (see [7]). From Lemma A1 there exists local frame $\{\bar{E}_1, \bar{E}_2\}$ such that equations (48) hold for some function $a \in C^\infty(M)$. Let us denote by V the set $\{x : a(x) \neq 0\}$. Let us define on V a local frame $\{E_1, E_2\}$ as follows $E_1 = a^{-\frac{1}{3}}\bar{E}_1, E_2 = a^{\frac{1}{3}}\bar{E}_2$. Equations (48) hold for $\{E_1, E_2\}$ with $a = 1$. We also have:

$$\hat{\nabla}K(X, E_1, E_1) = 0, \quad \hat{\nabla}K(X, E_1, E_2) = 0, \quad \hat{\nabla}K(X, E_2, E_2) = -3\omega(X)E_1 \tag{56}$$

where $\hat{\nabla}_X E_1 = \omega(X)E_1$, $\hat{\nabla}_X E_2 = -\omega(X)E_2$. From (56) it follows that $\hat{\nabla}K$ is symmetric if and only if $\omega(E_1) = 0$. Let us denote $\alpha := \omega(E_2)$. The equations

$$\hat{\nabla}_{E_1}E_1 = \hat{\nabla}_{E_1}E_2 = 0, \quad \hat{\nabla}_{E_2}E_1 = \alpha E_1, \quad \hat{\nabla}_{E_2}E_2 = -\alpha E_2 \tag{57}$$

hold on V. Let us introduce coordinates y_1, y_2 such that

$$E_1 = \phi\partial_1, \qquad E_2 = \psi\partial_2. \tag{58}$$

From (57) it is clear that $\partial_1\psi = 0$. Let us change coordinates taking $z_1 = y_1, z_2 = \Psi(y_2)$ where $\Psi = \int \psi^{-1}$. In new coordinates equations (58) hold with $\psi = 1$ and we also have:

$$\hat{\nabla}_{\partial_1}\partial_1 = -\partial_1 \ln|\phi|\partial_1, \qquad \hat{\nabla}_{\partial_1}\partial_2 = 0, \qquad \hat{\nabla}_{\partial_2}\partial_2 = -\partial_2 \ln|\phi|\partial_2$$

and $\alpha = \partial_2 \ln|\phi|$. Let us define $u := \ln|\phi|$. Notice that $[E_1, E_2] = -\alpha E_1$. Hence $d\omega(E_1, E_2) = E_1\omega(E_2) = \phi\partial_1\alpha = \phi\partial_{12}\ln|\phi|$. On the other hand $d\omega(E_1, E_2) = K_h = H$. We get the equation $\partial_1(\partial_2 u) = H\exp(-u)$. Let us assume $H \neq 0$. We can assume without loss of generality that $-Hf$ is an affine normal field for (M, f). In the coordinates (W, z_1, z_2) the immersion f satisfies the following fundamental system of equations:

$$\begin{aligned} \partial_1^2 f &= -\partial_1 u\partial_1 f, \\ \partial_2^2 f &= e^u\partial_1 f - \partial_2 u\partial_2 f, \\ \partial_{12}^2 f &= -e^{-u}Hf. \end{aligned} \tag{59}$$

From the first of equations (59) it follows that $\partial_1 f = e^{-u}\xi(z_2)$ for some smooth function $\xi : \mathbb{R} \to \mathbb{R}^3$. From the third equation we get $f(z_1, z_2) = \frac{1}{H}(\partial_2 u\xi(z_2) - \xi'(z_2))$ and

$$\partial_2 f = \frac{1}{H}(\partial_2^2 u\xi + \partial_2 u\xi' - \xi'').$$

It is not difficult to see that the second equation is satisfied iff ξ satisfies the following equation:

$$\xi''' = (2\partial_2^2 u + (\partial_2 u)^2)\xi' + (\partial_2^3 u + (\partial_2 u)(\partial_2^2 u) - H)\xi.$$

Let us note that in fact it is an ordinary differential equation of the third order. If we denote by a, b the functions $(2\partial_2^2 u + (\partial_2 u)^2), (\partial_2^3 u + (\partial_2 u)(\partial_2^2 u) - H)$ respectively then $2b = \partial_2 a - 2H$ and a, b do not depend on z_1. The last statement follows from equation (53). If a mapping ξ satisfies equation (55) then it is obvious that $\det(\xi, \xi', \xi'')$ is constant. One can check that $\det(\xi, \xi', \xi'') = H$ as f is a Blaschke immersion. On the other hand if u satisfies the Liouville equation (53) on $\Omega \subseteq R^2$ and we choose a solution ξ of equation (55) such that $\det(\xi, \xi', \xi'') = H$ then an immersion f given by equation (55) is a Blaschke immersion and (Ω, f) is an affine sphere with an affine normal $-Hf$. □

Remark. If we change coordinates as follows:

$$x_1 = \frac{1}{H}\partial_2 u(z_1, z_2), \qquad x_2 = z_2,$$

then it is clear that in new coordinates equations (60)

$$f(x_1, x_2) = x_1\xi(x_2) - \frac{1}{H}\xi'(x_2) \tag{60}$$

and (55) hold. It is easy to check that if we define an immersion f by equation (60) where ξ satisfies equation (55) together with the initial condition and a, b are *any* smooth functions $a, b \in C^\infty(\mathbb{R})$ then $(\mathbb{R}^2, f)$ is an affine sphere with vanishing Fubini–Pick invariant and nonzero affine mean curvature H. $(\mathbb{R}^2, f)$ is a quadric iff $a' = 2b$.

THEOREM 3. *Let a function u satisfies equation (52) on an open simply connected set $\Omega \subseteq \mathbb{R}^2$. Then there exists an immersion $f_u : \Omega \to \mathbb{R}^3$ such that (Ω, f_u) is an affine sphere with affine mean curvature H and indefinite affine metric h. The Fubini–Pick invariant of (Ω, f_u) equals $16\alpha e^{6u}$. In the standard coordinates on Ω equations (49), (50), (51) hold where h is an affine metric, K is the difference tensor and ∇ is the Blaschke connection for (Ω, f_u).*

Let (M, f) be an affine surface with induced Blaschke structure which is an affine sphere with an affine mean curvature H. Then $M = U \cup \bar{V} = \bar{U} \cup V$ where U, V are open subsets of M such that $U \cap V = \emptyset$, $U \subseteq \{x : h(C, C) = 0\}$ and $V = \{x : h(C, C) \neq 0\}$. Around any point $x_0 \in V$ there exists a local chart (W, x_1, x_2) such that the function $u = \frac{1}{3}\ln\lambda \circ \gamma^{-1}$, where $\lambda = \frac{1}{4}\sqrt{\alpha h(C, C)}$, satisfies on $\gamma(W)$ equation (52) where $\gamma(p) = (x_1(p), x_2(p)), \alpha = \operatorname{sgn} h(C, C)$ and (W, f) is equiaffinely equivalent to an affine sphere in $\mathbb{R}^3$ given by $(\gamma(W), f_u)$. We also have $U = U_0 \cup U_1$, where $U_0 = \operatorname{int}\{x : K_x = 0\}$ and

$U_1 = \{x : K_x \neq 0\} \cap \{x : h(C,C) = 0\}$. *A surface* (U_1, f) *is locally equiaffinely equivalent to one of the surfaces described in the Corollary* 2 *and a surface* (U_0, f) *is a nonconvex quadric.*

Proof: We omit the proof as it is just the same as the proof of the Theorem 2.

Remark. Let us note that the equation (52) is equivalent to the equation

$$\Delta_0 \Psi = e^{2\Psi} + \epsilon e^{-\Psi},$$

where $\epsilon = \operatorname{sgn} \alpha H \in \{-1, 0, 1\}$ and $\Psi(x_1, x_2) = 2u(\frac{x_1}{a}, \frac{x_2}{a}) - \ln b$ where $a = (4|H|)^{\frac{1}{3}}$, $b = \frac{|H|}{2})^{\frac{1}{3}}$. if $H \neq 0$ and $\Psi = 2u(\frac{x_1}{2}, \frac{x_2}{2})$ if $H = 0$.

Corollary 3. *Let* (M, f) *be an affine sphere in* $\mathbb{R}^3$ *with an affine hyperbolic metric* h *and Fubini–Pick invariant* $h(C,C)$. *Let us assume that* $h(C,C)$ *is different from* 0 *on* M. *Then for every* $x_0 \in M$ *there exist a neighborhood* V *of* x_0 *and a one parameter family of affine immersions* $f_a, a \in O(1,1)$, *such that* (V, f_a) *is an affine sphere and each immersion* f_a *has the same induced affine metric* h *and the same Fubini–Pick invariant* $h(C,C)$. *Every affine sphere immersion whose induced Blaschke structure has the same as* (M, f) *affine metric* h *and Fubini–Pick invariant* $h(C,C)$ *is locally equiaffinely equivalent to one of immersions* f_a.

Proof: The symmetry group of equation (52) is an affine orthogonal group $AO(1,1)$ of all affine transformations of the form:

$$g = \begin{pmatrix} \cosh\alpha & \epsilon \sinh\alpha & a \\ \sinh\alpha & \epsilon\cosh\alpha & b \\ 0 & 0 & 1 \end{pmatrix} \in AO(1,1), \qquad \alpha, a, b \in \mathbb{R},$$

Let us define in the coordinates described in the Theorem 3 a difference tensor K_α by the formulae:

$$\begin{aligned} K_\alpha(\partial_1, \partial_1) &= e^{2u}(\sinh 3\alpha\partial_1 - \epsilon\cosh 3\alpha\partial_2), \\ K_\alpha(\partial_2, \partial_2) &= e^{2u}(\sinh 3\alpha\partial_1 - \epsilon\cosh 3\alpha\partial_2), \\ K_\alpha(\partial_1, \partial_2) &= e^{2u}(\epsilon\cosh 3\alpha\partial_1 - \sinh 3\alpha\partial_2). \end{aligned} \tag{61}$$

Let $\hat{\nabla}$ be the Levi-Civitá connection of an metric h defined by (32) and define a connection ∇_α as $\nabla_\alpha = \hat{\nabla} + K_\alpha$ where K_α is given by the formulae (61). Then as before one can prove the existence of an immersion f_a, where

$$a = \begin{pmatrix} \cosh\alpha & \epsilon\sinh\alpha \\ \sinh\alpha & \epsilon\cosh\alpha \end{pmatrix}$$

with a Blaschke connection ∇_α and an affine metric h. The rest of the proof is the same as the proof of Corollary following the Theorem 2. □

Proposition 1. *Let a function* u *satisfies on* Ω *equation* (52) *and let* (Ω, f_u) *be an affine sphere with induced affine metric and induced Blaschke connection given by formulae* (49) *and* (51). *Let* (Ω, f_a) *be as above,* $f_{id} = f_u$. *Then* (Ω, f_a) *is affinely* *equivalent to* (Ω, f_u) *iff* $u = u \circ g$, *where* a *is given by the formula* (44) *and* g *is given by* (42) *for some* $a, b \in \mathbb{R}$.

Proof: The proof is the same as the proof of the Proposition. □

Remark. Let us note that if $h(C,C)$ is constant and different from 0 then all hypersurfaces (Ω, f_a) are equiaffinely equivalent. In particular an affine sphere with constant Fubini–Pick invariant $h(C,C) \neq 0$ and indefinite affine metric is characterized uniquely by $h(C,C)$. Every nonconvex affine sphere with constant nonzero Fubini–Pick invariant is equiaffinely equivalent to an open part of the surface $Z(X^2+Y^2) = \pm\frac{1}{c}$ where $c \neq 0$ depends only on $h(C,C)$. The fundamental system of equations for f is very simple in the case of constant nonzero $h(C,C)$ and it is easy to see that (in a chart for which $\partial_i = E_i$)

$$f(x_1, x_2) = (e^{-\lambda x_2}\cos(\sqrt{3}\lambda x_1), -e^{-\lambda x_2}\sin(\sqrt{3}\lambda x_1), \pm\frac{1}{c}e^{2\lambda x_2})$$

where $c = \frac{3\sqrt{3}}{128}h(C,C)^2$ (see also [6]). The surface $M = \{(X,Y,Z) : Z(X^2+Y^2) = \pm\frac{1}{c}\}$ is affinely homogeneous. It is easy to check that $(\mathbb{R}^2, f)$ is an orbit of the point $(1, 0, \pm\frac{1}{c})$ by the group $G = \mathbb{R} \oplus S^1$ of equiaffine transformations

$$\begin{pmatrix} e^{-a}\cos b & e^{-a}\sin b & 0 \\ -e^{-a}\sin b & e^{-a}\cos b & 0 \\ 0 & 0 & e^{2a} \end{pmatrix}, \qquad (a,b) \in \mathbb{R} \oplus \frac{\mathbb{R}}{2\pi\mathbb{Z}}.$$

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