Exact solutions describing interaction of solitary waves of non-integrable equations

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Abstract: An effective method for constructing solutions describing interaction of solitary waves in the form of exponential series is developed. This method can be considered as the analogue of Poincaré's theory of normal forms for ordinary differential equations and can be applied to non-integrable partial differential equations as well as to integrable ones. As in Poincaré's theory it is important to investigate the resonances. The test for integrability based on investigation of resonances is suggested: for integrable equations the resonances cancel. Exponential series are constructed for non-integrable Klein–Gordon equation and Klein–Gordon type coupled non-integrable equations. Possibility to consider in the similar way multi-dimensional and linearizable equations is an interesting feature of the approach.

1 Introduction

In recent years there has been an interest in constructing solutions of nonlinear partial differential equations in the form of infinite series [1–15]. Exponential series were used for constructing solutions of the Boltzmann equations [4–9].nonlinear elliptic equations [10], and nonlinear wave equations [11, 12] with the emphasis on the availability of symbolic manipulation programs. In [5, 10, 13] it was also pointed out that such series can be useful for investigation of nonlinear wave equations. Some other references can be found in the cited papers.

In [14] we considered a class of one-dimensional nonlinear partial differential equations with constant coefficients and developed a simple technique for constructing particular solutions in the form of exponential series, based on formal linearization of nonlinear partial differential equation near its constant solutions to the system of linear ordinary differential equations, describing some finite-dimensional subspace of the space of solutions of the corresponding linearized equation. A similar approach was independently developed in [15] for wide class of evolution equations. Such exponential series converge in some domains of the coordinate space [10, 15].

In this paper we present some properties of the method, concerning distinctions between integrable and non-integrable equations (test for integrability is suggested) and possibilities to consider in the similar way multi-dimensional and linearizable equations. We also construct solutions in the form of exponential series for a number of equations (among them, for non-integrable Klein–Gordon equation and Klein–Gordon type coupled non-integrable equations, being a model for describing wave processes in two-component media [16]).

2 The method of formal linearization

Let us consider equations of the following form

$$\hat{L}(D_t, D_x)u(t,x) = N[u], \tag{1}$$

where

$$\hat{L}(D_t, D_x) \equiv \sum_{k=0}^{K}\sum_{m=0}^{M} l_{km} D_t^k D_x^m \tag{2}$$

is a linear differential operator with constant coefficients and

$$N[u] \equiv N(u, u_1, u_2, \ldots, u_p), \quad u_p \equiv \left\{ \frac{\partial^{p_1+p_2} u}{\partial t^{p_1} \partial x^{p_2}} \;\middle|\; p_1 + p_2 = p \right\},$$

is an arbitrary analytic function of u and its derivatives up to some finite order p. We suppose that (1) possesses a constant solution. Without loss of generality we assume that

$$N[0] = 0, \quad \frac{\partial N[0]}{\partial u} = 0, \quad \frac{\partial N[0]}{\partial u_1} = 0, \quad \ldots, \quad \frac{\partial N[0]}{\partial u_p} = 0.$$

We consider (1) in connection with the equation (linearized near zero solution)

$$\hat{L}(D_t, D_x)w(t,x) = 0. \tag{3}$$

Let L be the vector space of solutions of Eq. (3) and $P^N \subset L$ be the N-dimensional subspace with the basis

$$w_i = W_i \exp(\alpha_i \xi_i), \quad \xi_i = x - s_i t, \quad i = \overline{1, N}.$$

Here s_i and W_i are some constants. The constants $\alpha_i = \alpha_i(s_i)$ are assumed to satisfy the dispersion relation

$$\hat{L}(-\alpha_i s_i, \alpha_i) = 0.$$

We use the following notation:

$$w_{(N)}^{\delta} \equiv w_1^{\delta_1} w_2^{\delta_2} \ldots w_N^{\delta_N}, \quad \delta = (\delta_1, \delta_2, \ldots, \delta_N), \quad |\delta| = \sum_{i=1}^{N} \delta_i.$$

It is obvious that the monomials $w_{(N)}^{\delta}$ are eigenfunctions of the operator (2):

$$\hat{L}(D_t, D_x) w_{(N)}^{\delta} = \lambda_\delta w_{(N)}^{\delta}$$

with the eigenvalues

$$\lambda_\delta = \sum_{k=0}^{K}\sum_{m=0}^{M} l_{km} \left(-\sum_{i=1}^{N} \alpha_i s_i \delta_i \right)^k \left(\sum_{i=1}^{N} \alpha_i \delta_i \right)^m .$$

PROPOSITION 1. *If $\lambda_\delta \neq 0$ for every multi-index δ with nonnegative integer components $\delta_i \in Z_+$, $i = \overline{1, N}$, satisfying the condition $|\delta| \neq 0, 1$, then Eq. (1) possesses solutions, connected with the solutions from P^N by the formal transformation*

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2, \ldots, w_N), \tag{4}$$

where

$$\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(N)}^\delta \tag{5}$$

are homogeneous polynomials of degree n in the variables w_i. This transformation is unique (for the first term $\phi_1 \in P^N$ fixed).

The proof of the proposition is constructive. Substituting (4) into (1), expanding $N[u]$ into the power series in ε and then collecting equal powers of ε, we obtain the determining equations for the functions ϕ_n and show that if $\lambda_\delta \neq 0$, then these equations possess the solution (5) with the coefficients $(A_n)_\delta$, uniquely determined through the coefficients $(A_1)_\delta$ by the recursion relation. Thus, the proposition gives us the method for constructing particular solutions of Eq. (1).

In this approach ε is the grading parameter, finally we can put $\varepsilon = 1$.

Numerical computations for non-integrable Klein–Gordon equation show that these series converge in different domains of the coordinate plane despite the small denominators arising in some domains. Therefore it is natural to try to construct the "global" solutions with the help of such series, matching them on the boundaries of these domains.

This procedure can be generalized for the systems of the form

$$\sum_{j=1}^{n} \hat{L}_j^i(D_t, D_x) u^j(t, x) = N^i[u], \quad i = \overline{1, n}, \tag{6}$$

where

$$\hat{L}_j^i(D_t, D_x) \equiv \sum_{k=0}^{K} \sum_{m=0}^{M} (l_j^i)_{km} D_t^k D_x^m \tag{7}$$

are linear differential operators with constant coefficients and $N^i[u]$ are arbitrary analytic functions of u^j, $j = \overline{1, n}$, and their derivatives up to some finite order p. We suppose again that the system (6) possesses the constant solution. There is no loss of generality to assume that

$$N^i[0] = 0, \quad \frac{\partial N^i[0]}{\partial u^j} = 0, \quad \frac{\partial N^i[0]}{\partial u_1^j} = 0, \quad \ldots, \quad \frac{\partial N^i[0]}{\partial u_p^j} = 0, \quad i, j = \overline{1, n}.$$

The system, linearized near zero solution and corresponding to (6) has the form

$$\sum_{j=1}^{n} \hat{L}_j^i(D_t, D_x) w^j(t, x) = 0. \tag{8}$$

Now the subspace P^N of solutions of (8) is generated by the vector functions

$$w_l = (W_l^1, W_l^2, \dots, W_l^n)\exp(\alpha_l \xi_l), \quad \xi_l = x - s_l t, \quad l = \overline{1, N}$$

with some constants s_l and W_l^j. Here the constants $\alpha_l = \alpha_l(s_l)$ are assumed to satisfy the dispersion relation

$$\det[\hat{L}_j^i(-\alpha_l s_l, \alpha_l)] = 0$$

and the constants $W_l^j = W_l^j(s_l)$ are assumed to satisfy the system of linear algebraic equations

$$\sum_{j=1}^{n} \hat{L}_j^i(-\alpha_l s_l, \alpha_l) W_l^j = 0, \quad i = \overline{1, n}.$$

All constructions can be repeated. Let us use the notation $(\lambda_j^i)_\delta$ for the eigenvalues of the monomials

$$(w_{(N)}^1)^\delta \equiv (w_1^1)^{\delta_1}(w_2^1)^{\delta_2} \dots (w_N^1)^{\delta_N}$$

under the action of the operators (7):

$$\hat{L}_j^i(D_t, D_x)(w_{(N)}^1)^\delta = (\lambda_j^i)_\delta (w_{(N)}^1)^\delta.$$

Here

$$(\lambda_j^i)_\delta = \sum_{k=0}^{K}\sum_{m=0}^{M} (l_j^i)_{km} \left(-\sum_{l=1}^{N} \alpha_l s_l \delta_l\right)^k \left(\sum_{l=1}^{N} \alpha_l \delta_l\right)^m .$$

The generalization of the Proposition 1 is the following

PROPOSITION 2. *If* $\det[(\lambda_j^i)_\delta] \neq 0$ *for every multi-index* δ *with nonnegative integer components* $\delta_l \in Z_+$, $l = \overline{1, N}$, *satisfying the condition* $|\delta| \neq 0, 1$, *then the system (6) possesses solutions, connected with the solutions from* P^N *by the formal transformation*

$$u^j = \sum_{k=1}^{\infty} \varepsilon^k \phi_k^j(w_1^1, w_2^1, \dots, w_N^1), \quad j = \overline{1, n},$$

where

$$\phi_k^j = \sum_{|\delta|=k} (A_k^j)_\delta (w_{(N)}^1)^\delta$$

are homogeneous polynomials of degree k *in the variables* w_l^1. *This transformation is unique (for the first term* $\phi_1 \in P^N$ *fixed).*

3 Examples

3.1 The KdV equation

Let us consider the KdV equation

$$\hat{L}(D_t, D_x)u(t,x) = -6uu_x, \quad \hat{L}(D_t, D_x) = D_t + D_x^3. \tag{9}$$

For simplicity we look for solution of (9) in the form

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2), \tag{10}$$

where

$$w_i = W_i \exp[\sqrt{s_i}(x - s_i t)], \quad i = 1, 2$$

is the basis of the subspace $P^2 \subset L$ (let s_i and W_i be some real constants).

Substituting (10) into (9) and collecting equal powers of ε, we obtain the determining equations for the functions ϕ_n as follows

$$\hat{L}\phi_1 = 0, \quad \hat{L}\phi_n = -6\sum_{k=1}^{n-1} \phi_k D_x \phi_{n-k}, \quad n \geq 2.$$

These equations possess solution $\phi_n = \sum_{|\delta|=n} (A_n)_\delta w_{(2)}^\delta, \delta = (\delta_1, \delta_2)$, which can be rewritten in this case in the following form

$$\phi_n = \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k}, \quad (\phi_1 \in P^2).$$

The coefficients A_k^n can be found through A_0^1 and A_1^1 by the recursion relation

$$\begin{aligned}
A_k^n &= -\frac{6}{\lambda_{(k,n-k)}} \sum_{l=1}^{n-1}\sum_{m=0}^{n-l} [\sqrt{s_1}m + \sqrt{s_2}(n-l-m)] A_{k-m}^l A_m^{n-l}, \\
& \quad n \geq 2, \quad 0 \leq k \leq n \\
A_k^n &= 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n, \\
\lambda_{(k,n-k)} &= s_1\sqrt{s_1}k(k^2-1) + s_2\sqrt{s_2}(n-k)[(n-k)^2 - 1] + \\
& \quad +3\sqrt{s_1 s_2}k(n-k)[\sqrt{s_1}k + \sqrt{s_2}(n-k)].
\end{aligned} \tag{11}$$

If $s_1 > 0$ and $s_2 > 0$, then $\lambda_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $k, n \in Z_+, \quad n \geq 2, \quad 0 \leq k \leq n$.

The constructed solution is the expansion of 2-soliton solution, which can be written as

$$u = 2D_x^2 \ \ln\left(1 + w_1 + w_2 + \left(\frac{\sqrt{s_1} - \sqrt{s_2}}{\sqrt{s_1} + \sqrt{s_2}}\right)^2 w_1 w_2\right). \tag{12}$$

Indeed, expanding (12) into the power series in w_1 and w_2 and substituting this expansion into (9), we obtain for the coefficients the recursion relation (11).

If $A_0^1 = 0$ (or $A_1^1 = 0$), then $\phi_1 \in P^1$ and we get from (10) the expansion for 1-soliton solution. For obtaining the N-soliton solution we must take $\phi_1 \in P^N$.

3.2 Non-integrable Klein–Gordon equation

Let us consider the equation

$$\hat{L}(D_t, D_x)u(t,x) = u^3, \quad \hat{L}(D_t, D_x) = D_t^2 - D_x^2 + 1. \tag{13}$$

It is known [17], that this equation is non-integrable and its solitary wave solutions are not solitons. We can construct solutions, describing interaction of these waves with the help of the method of formal linearization.

The constant solutions of (13) are $u = 0$ and $u = \pm 1$.

The equation, linearized near zero solution has the form $\hat{L}w = 0$ and the space of its solutions contains the subspace P^2 with the basis

$$w_i = W_i \exp \frac{x - s_i t}{\sqrt{1 - s_i^2}}, \quad i = 1, 2.$$

We look for solutions of (13) in the form (10) and obtain the determining equations as follows

$$\hat{L}\phi_1 = 0, \quad \hat{L}\phi_2 = 0, \quad \hat{L}\phi_n = \sum_{k=2}^{n-1} \phi_{n-k} \sum_{l=1}^{k-1} \phi_l \phi_{k-l}, \quad n \geq 3.$$

These equations possess the solution

$$\phi_{2p+1} = \sum_{n=0}^{2p+1} A_n^p w_1^n w_2^{2p+1-n}, \quad \phi_{2p+2} = 0, \quad p \geq 0,$$

where

$$\begin{aligned}
A_n^p &= \frac{1}{\lambda_{(n,2p+1-n)}} \sum_{m=0}^{p-1} \sum_{r=0}^{p-m-1} \sum_{k=0}^{2m+1} \sum_{l=0}^{2r+1} A_k^m A_l^r A_{n-k-l}^{p-m-r-1}, \\
& \quad p \geq 1, \quad 0 \leq n \leq 2p+1, \\
A_n^p &= 0 \quad \text{if} \quad n < 0 \quad \text{or} \quad n > 2p+1, \\
\lambda_{(n,2p+1-n)} &= 1 - n^2 - (2p+1-n)^2 - 2\frac{1 - s_1 s_2}{\sqrt{(1-s_1^2)(1-s_2^2)}} n(2p+1-n).
\end{aligned}$$

If $|s_1| < 1$ and $|s_2| < 1$, then $\lambda_{(n,2p+1-n)} \neq 0$ for every pair $(n, 2p+1-n)$ with $n, p \in Z_+, \quad p \geq 1, \quad 0 \leq n \leq 2p+1$.

If $A_0^0 = 0$, then we obtain

$$u = \sum_{p=0}^{\infty} \left(-\frac{1}{8}\right)^p (\varepsilon w_1)^{2p+1} = \frac{\varepsilon w_1}{1 + \frac{1}{8}(\varepsilon w_1)^2} = \frac{2\sqrt{2}w}{1 + w^2},$$

where $w = \varepsilon w_1/2\sqrt{2}$. In (t,x)-variables we have

$$u = \pm\sqrt{2}\ \mathrm{sech}\ \frac{x - st + x_0}{\sqrt{1-s^2}}.$$

These are the solitary wave solutions. Solitary waves of this type can move with velocities satisfying the condition $|s| < 1$.

Another possibility to construct solutions is to linearize (13) near the solution $u = 1$. The change of variables $u = 1 + v$ leads to equation

$$\hat{M}(D_t, D_x)v(t,x) = 3v^2 + v^3, \quad \hat{M}(D_t, D_x) = D_t^2 - D_x^2 - 2,$$

which should be linearized near zero solution. In this case the subspace P^2 is generated by the functions

$$w_i = W_i \exp \frac{\sqrt{2}(x - s_i t)}{\sqrt{s_i^2 - 1}}, \quad i = 1, 2.$$

Our procedure gives the solution

$$\begin{aligned}
v &= \sum_{n=1}^{\infty}\sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k}, \\
A_k^n &= \frac{1}{\tilde{\lambda}_{(k,n-k)}}\left\{3\sum_{l=1}^{n-1}\sum_{m=0}^{l} A_m^l A_{k-m}^{n-l} + \sum_{l=2}^{n-1}\sum_{m=1}^{l-1}\sum_{p=0}^{n-l}\sum_{q=0}^{m} A_p^{n-l} A_q^m A_{k-p-q}^{l-m}\right\}, \\
& n \geq 2, \quad 0 \leq k \leq n, \\
A_k^n &= 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n,
\end{aligned}$$

$$\tilde{\lambda}_{(k,n-k)} = 2[k^2 + (n-k)^2 + 2\frac{s_1 s_2 - 1}{\sqrt{(s_1^2 - 1)(s_2^2 - 1)}}k(n-k) - 1].$$

If $|s_1| > 1$, $|s_2| > 1$, and $s_1 s_2 - 1 > 0$, then $\tilde{\lambda}_{(k,n-k)} \neq 0$ for every pair $(k, n-k)$ with $n, k \in Z_+, n \geq 2, 0 \leq k \leq n$.

If $A_0^1 = 0$, then we get

$$v = \sum_{n=1}^{\infty}(\frac{1}{2})^{n-1}(\varepsilon w_1)^n = \frac{\varepsilon w_1}{1 - \frac{1}{2}\varepsilon w_1} = \frac{2w}{1-w},$$

where $w = \varepsilon w_1/2$. Thus, if $w < 0$ (i.e. $W < 0$), then in (t,x)-variables we have

$$u = 1 + v = -\tanh \frac{x - st + x_0}{\sqrt{2(s^2 - 1)}}.$$

This formula represents kink for negative value of the square root and anti-kink for positive one. Both waves can move with velocities, satisfying the condition $|s| > 1$.

Linearization near the solution $u = -1$ gives expansions for the same solutions.

3.3 Klein–Gordon type coupled non-integrable equations

Let us consider the system of equations

$$\begin{aligned} u_{tt} - u_{xx} &= -\delta^2[u - w - (u - w)^3], \\ w_{tt} - c^2 w_{xx} &= u - w - (u - w)^3, \end{aligned} \tag{14}$$

Here δ^2 and c^2 are positive constants. Equations of this type describe some wave processes in two-component media [16].

Let us rewrite the system (14) in the form

$$\begin{aligned} \hat{L}_1^1 u^1 + \hat{L}_2^1 u^2 &= \delta^2 (u^2)^3, \\ \hat{L}_1^2 u^1 + \hat{L}_2^2 u^2 &= 0, \end{aligned} \tag{15}$$

where

$$\begin{aligned} u^1 &= u, & u^2 &= u - w, \\ \hat{L}_1^1 &= D_t^2 - D_x^2, & \hat{L}_2^1 &= \delta^2, \\ \hat{L}_2^2 &= -\delta^2 (D_t^2 - c^2 D_x^2), & \hat{L}_1^2 &= \hat{L}_1^1 - \hat{L}_2^2. \end{aligned}$$

Looking for solutions of (15) in the form of travelling waves we find in particular two types of solitary wave solutions:

$$u^1 = \kappa^{-1} u^2, \quad u^2 = \pm\sqrt{2}\ \mathrm{sech}\ \sqrt{|\lambda|}\ (x - vt), \tag{16}$$

$$u^1 = \kappa^{-1} u^2, \quad u^2 = \pm \mathrm{th}\ \sqrt{\frac{\lambda}{2}}\ (x - vt). \tag{17}$$

Here we use the following notation:

$$\lambda = \frac{\delta^2 (v^2 - c^2) + v^2 - 1}{(v^2 - 1)(v^2 - c^2)}, \quad \kappa = \frac{\lambda (v^2 - 1)}{\delta^2}.$$

Solitary waves of the type (16) can propagate with the velocities

$$v^2 \in]0, S[\cup]M, L[,$$

the waves of the type (17) propagate with the velocities

$$v^2 \in]S, M[\cup]L, +\infty[.$$

Here

$$S = \min\{1, c^2\}, \quad M = \frac{1 + \delta^2 c^2}{1 + \delta^2}, \quad L = \max\{1, c^2\}.$$

Let us point out, that if $c^2 \neq 1$, then there is a gap in the velocity spectrum of the solitons, i.e. the system can be considered as a kind of soliton filter (physical applications of the equations are discussed in [16]).

We can construct solutions, describing interaction of solitary waves in the form of formal series with the help of Proposition 2.

Essentially different constant solutions of the system (15) are the solutions $u^1 = u^2 = 0$ and $u^1 = 0,\ u^2 = \pm 1$. (These solutions can not be obtained one from another using translation along u^1).

The system, linearized near zero solution has the form

$$\sum_{j=1}^{2} \hat{L}^i_j w^j = 0, \quad i = 1,2$$

and the space of its solutions contains the subspace P^2 with the basis

$$\begin{aligned} (w^1_i, w^2_i) &= (W^1_i, W^2_i) \exp\left[\alpha_i(x - v_i t)\right], \quad i = 1,2, \\ \alpha_i &= \left[\frac{\delta^2(v_i^2 - c^2) + v_i^2 - 1}{(1 - v_i^2)(v_i^2 - c^2)}\right]^{1/2}, \\ W^2_i &= \kappa_i W^1_i, \quad \kappa_i = \frac{\alpha_i^2(1 - v_i^2)}{\delta^2} \end{aligned}$$

(we consider for simplicity 2-wave case).

We look for solution of (15) in the form

$$u^1 = \sum_{n=1}^{\infty} \varepsilon^n \phi^1_n(w^1_1, w^1_2), \quad u^2 = \sum_{n=1}^{\infty} \varepsilon^n \phi^2_n(w^1_1, w^1_2). \tag{18}$$

Substituting (18) in (15) and collecting equal powers of ε, we obtain the determining equations as follows:

$$\begin{aligned} &\hat{L}^1_1 \phi^1_n + \hat{L}^1_2 \phi^2_n = \gamma \delta^2 \sum_{k=2}^{n-1} \phi^2_{n-k} \sum_{l=1}^{k-1} \phi^2_l \phi^2_{k-l}, \\ &\hat{L}^2_1 \phi^1_n + \hat{L}^2_2 \phi^2_n = 0. \end{aligned}$$

Here $\gamma = 0$ for $n = 1,2$ and $\gamma = 1$ for $n \geq 3$.

These equations possess solutions

$$\begin{aligned} \phi^1_{2p+1} &= \sum_{n=0}^{2p+1} A^p_n (w^1_1)^n (w^1_2)^{2p+1-n}, \quad \phi^1_{2p+2} = 0, \\ \phi^2_{2p+1} &= \sum_{n=0}^{2p+1} B^p_n (w^1_1)^n (w^1_2)^{2p+1-n}, \quad \phi^2_{2p+2} = 0, \quad p \geq 0, \end{aligned} \tag{19}$$

where the coefficients A^p_n and B^p_n can be found recurrently through A^0_0, A^0_1,

$B_0^0 = \kappa_2 A_0^0,\ B_1^0 = \kappa_1 A_1^0$ by the system of linear algebraic equations

$$
\begin{array}{rcl}
\lambda_1^1 A_n^p + \lambda_2^1 B_n^p & = & \delta^2 \sum\limits_{m=0}^{p-1} \sum\limits_{r=0}^{p-m-1} \sum\limits_{k=0}^{2m+1} \sum\limits_{l=0}^{2r+1} B_k^m B_l^r B_{n-k-l}^{p-m-r-1}, \\
\lambda_1^2 A_n^p + \lambda_2^2 B_n^p & = & 0, \quad p \geq 1, \quad 0 \leq n \leq 2p+1 \\
B_n^p & = & 0 \quad \text{if} \quad n < 0 \quad \text{or} \quad n > 2p+1, \\
\lambda_1^1 & = & \left(\sum\limits_{l=1}^{2} \alpha_l v_l \sigma_l \right)^2 - \left(\sum\limits_{l=1}^{2} \alpha_l \sigma_l \right)^2, \\
\lambda_2^2 & = & -\delta^2 \left[\left(\sum\limits_{l=1}^{2} \alpha_l v_l \sigma_l \right)^2 - c^2 \left(\sum\limits_{l=1}^{2} \alpha_l \sigma_l \right)^2 \right],
\end{array}
\tag{20}
$$

$$
\begin{array}{rl}
\lambda_1^2 = \lambda_1^1 - \lambda_2^2, & \lambda_2^1 = \delta^2, \\
\sigma_1 = n, & \sigma_2 = 2p+1-n.
\end{array}
$$

The condition

$$\lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2 \neq 0 \tag{21}$$

provides compatibility and definiteness of this system. The condition

$$v_i^2 \in]0, S[\cup]M, L[$$

selects the domain in the parameter space, where $\alpha_i \in R,\ i = 1, 2$. Here the subdomain exists, where the condition (21) is satisfied for any pair $(n, 2p+1-n)$ with $n, p \in Z_+, \quad p \geq 1, \quad 0 \leq n \leq 2p+1$. For example, this condition is satisfied if $v_i^2 \in]0, S[,\ i = 1, 2$.

To obtain the series for 1-wave solution (16) one should assume in (20) $A_0^0 = 0$ (or $A_1^0 = 0$). The solution (18)–(20), constructed in the form of formal series describes the interaction of two solitary waves of the type (16).

Another possibility to construct solutions is to linearize the system (14) near the solution $u^1 = 0,\ u^2 = 1$. The change of variables

$$u^1 = v^1, \quad u^2 = 1 + v^2 \tag{22}$$

leads to the system

$$
\begin{array}{rcl}
\hat{M}_1^1 v^1 + \hat{M}_2^1 v^2 & = & \delta^2 [3(v^2)^2 + (v^2)^3], \\
\hat{M}_1^2 v^1 + \hat{M}_2^2 v^2 & = & 0,
\end{array}
\tag{23}
$$

where

$$
\begin{array}{ll}
\hat{M}_1^1 = \hat{L}_1^1, & \hat{M}_2^1 = -2\hat{L}_2^1, \\
\hat{M}_1^2 = \hat{L}_1^2, & \hat{M}_2^2 = \hat{L}_2^2.
\end{array}
$$

The system (23) should be linearized near zero solution. In this case the linearized system has the form

$$\sum_{j=1}^{2} \hat{M}_j^i w^j = 0, \quad i = 1, 2$$

and the subspace P^2 of its solutions is generated by the vector-functions

$$\begin{aligned}
(w_i^1, w_i^2) &= (W_i^1, W_i^2) \exp\left[\beta_i(x - v_i t)\right], \quad i = 1, 2, \\
\alpha_i &= \left[\frac{2[\delta^2(v_i^2 - c^2) + v_i^2 - 1]}{(v_i^2 - 1)(v_i^2 - c^2)}\right]^{1/2}, \\
W_i^2 &= \nu_i W_i^1, \quad \nu_i = \frac{\beta_i^2(v_i^2 - 1)}{2\delta^2}.
\end{aligned}$$

The linearization procedure gives the solution

$$\begin{aligned}
v^1 &= \sum_{n=1}^{\infty}\sum_{k=0}^{n} C_k^n (w_1^1)^k (w_2^1)^{n-k}, \\
v^2 &= \sum_{n=1}^{\infty}\sum_{k=0}^{n} D_k^n (w_1^1)^k (w_2^1)^{n-k},
\end{aligned} \tag{24}$$

where the coefficients C_k^n and D_k^n can be found recurrently through C_0^1, C_1^1, $D_0^1 = \nu_2 C_0^1$, $D_1^1 = \nu_1 C_1^1$ by the system of linear algebraic equations

$$\begin{aligned}
\mu_1^1 C_k^n + \mu_2^1 D_k^n &= \delta^2 \left\{ 3\sum_{l=1}^{n-1}\sum_{m=0}^{l} D_m^l D_{k-m}^{n-l} + \right. \\
&\quad \left. + \sum_{l=2}^{n-1}\sum_{m=1}^{l-1}\sum_{p=0}^{n-l}\sum_{q=0}^{m} D_p^{n-l} D_q^m D_{k-p-q}^{l-m} \right\}, \\
\mu_1^2 C_k^n + \mu_2^2 D_k^n &= 0, \quad n \geq 2, \quad 0 \leq k \leq n, \\
D_k^n &= 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n, \\
\mu_1^1 &= \left(\sum_{l=1}^{2} \beta_l v_l \tilde{\sigma}_l\right)^2 - \left(\sum_{l=1}^{2} \beta_l \tilde{\sigma}_l\right)^2, \\
\mu_2^2 &= -\delta^2 \left[\left(\sum_{l=1}^{2} \beta_l v_l \tilde{\sigma}_l\right)^2 - c^2 \left(\sum_{l=1}^{2} \beta_l \tilde{\sigma}_l\right)^2\right], \\
\mu_2^1 = -2\delta^2, &\qquad \mu_1^2 = \mu_1^1 - \mu_2^2, \\
\tilde{\sigma}_1 = k, &\qquad \tilde{\sigma}_2 = n - k.
\end{aligned} \tag{25}$$

The condition of compatibility and definiteness of this system has the form

$$\mu_1^1 \mu_2^2 - \mu_2^1 \mu_1^2 \neq 0. \tag{26}$$

The condition

$$v_i^2 \in]S, M[\cup]L, +\infty[$$

provides $\beta_i \in R,\ i = 1, 2$. Here again the subdomain exists in the parameter space, where the condition (26) is satisfied for any pair $(k, n-k)$ with $k, n \in Z_+, n \geq 2, \quad 0 \leq k \leq n$. For example, this condition is satisfied if $v_i^2 \in]L, +\infty[,\ i = 1, 2$.

Assuming in (25) $C_0^1 = 0$ (or $C_1^1 = 0$) one can obtain the series for 1-wave solution (17). The solution, determined by (22), (24), (25) describes the interaction of two solitary waves of the form (17).

The series, obtained by the linearization near $u^1 = 0,\ u^2 = -1$ represent the same solutions.

4 Integrability and cancellation of resonances

The method of formal linearization can be considered as an analogue of the theory of normal forms for ordinary differential equations ([18, 5]) and can be applied to non-integrable partial differential equations as well as to integrable ones. But we know from the theory of integrable systems, that integrable equations possess a number of remarkable mathematical properties, distinguishing them from non-integrable cases. Therefore the following question naturally arises: what is the difference between integrable and non-integrable cases in our approach?

As in Poincaré's theory it is important to investigate the resonances (relations, preventing one from carrying out the formal procedure). In our procedure resonances are determined by the relations $\lambda_\delta = 0$ in scalar case and $\det\,[\lambda_j^i]_\delta = 0$ in the case of systems.

It is well known, that in the theory of normal forms two essentially different cases exist, depending on the distribution of eigenvalues in the complex plane, the so called Poincaré and Siegel domains (every point of the Poincaré domain satisfies no more than a finite number of resonance relations and has a neighborhood, where no new resonances can appear, while in the Siegel domain resonances are everywhere dense, see e.g. [18]). It turns out, that in our theory the situation resembles that of the theory of normal forms. Moreover, investigation of resonances demonstrates the difference between integrable and non-integrable equations.

Indeed, let us return to the examples of 2-wave interaction. In the case of KdV equation the exponential expansions exist if $\lambda_{(k,n)} \neq 0$, where

$$\begin{aligned} \lambda_{(k,n)} &= \alpha_1^3 k(k^2-1) + \alpha_2^3 n(n^2-1) + 3\alpha_1\alpha_2 kn[\alpha_1 k + \alpha_2 n], \\ \alpha_i &= \pm\sqrt{s_i}, \qquad s_i > 0, \qquad s_1 \neq s_2. \end{aligned}$$

(Here it is more convenient to fix the positive value of the square root and to write the signs explicitly).

If both α_i are positive (negative) and $k + n \neq 1$, then $\lambda_{(k,n)} \neq 0$ and no resonances appear. But the situation is more complicated, when α_1 and α_2 have opposite signs: for instance, α_1 negative, α_2 positive. The first nontrivial case arises for $k = 2,\ n = 2$, when we get

$$\lambda_{(2,2)} = -6(\sqrt{s_1} - \sqrt{s_2})[s_1 + s_2 - 3\sqrt{s_1 s_2}].$$

If $\sqrt{s_2} = \frac{3+\sqrt{5}}{2}\sqrt{s_1}$, then $\lambda_{(2,2)} = 0$ and formally it is a resonance, but calculating the coefficient A_2^4 we obtain

$$\lambda_{(2,2)} A_2^4 = -12 \frac{(s_1 + s_2 - 3\sqrt{s_1 s_2})(s_1 + s_2 - 4\sqrt{s_1 s_2})}{(s_1 s_2)^{3/2}(\sqrt{s_1} - \sqrt{s_2})},$$

so that the resonance cancel. It is clear, that in the case of KdV equation the same cancellation will happen with all other formal resonances.

In the case of bell type solitary waves of non-integrable Klein–Gordon equation resonances are determined by the equality

$$\begin{aligned} \lambda_{(k,n)} &= 1 - k^2 - n^2 - 2(1 - s_1 s_2)\alpha_1 \alpha_2 kn = 0, \\ \alpha_i &= \pm \frac{1}{\sqrt{1 - s_i^2}}, \ |s_i| < 1, \ s_1 \neq s_2. \end{aligned}$$

The situation for α_1, α_2 positive (or negative) is similar to that of the KdV equation (there exists an open domain in the parameter space such that resonances do not appear). But when α_1 and α_2 have the opposite signs the situation is quite different: there are infinitely many resonances, which do not cancel. Even if we choose parameters s_1 and s_2 in such a way that we do not meet exact resonances (for instance, $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{4}$, when $\sqrt{(1 - s_1^2)(1 - s_2^2)}$ is irrational number and therefore $\lambda_{(k,n)} \neq 0$), the small denominators appear. But numerical experiments show convergence of the series despite the small denominators. This situation resembles again that of the theory of normal forms, where it is proved, that Poincaré's series are convergent for almost all (in the sense of measure theory) linear parts of a vector field at a singular point [18].

Summarizing, in both cases, integrable and non-integrable, investigation of resonances demonstrates existence of two essentially different situations, analogous to the Poincaré and Siegel cases in the theory of normal forms. However, integrable equations in this theory also have the remarkable property, distinguishing them from non-integrable ones, namely, the resonances are removable. This property can be used as a very simple test for integrability.

5 Multi-dimensional equations

An attractive feature of the approach represented above is the possibility to consider in the similar way multi-dimensional equations. Let us consider as example the KP equation

$$\begin{aligned} \hat{L}(D_t, D_x, D_y)u(t,x,y) &= -6(u_x^2 + uu_x), \\ \hat{L}(D_t, D_x, D_y) &= D_t D_x + D_x^4 + 3D_y^2. \end{aligned} \tag{27}$$

The equation, linearized near zero solution, has the form $\hat{L}w = 0$ and possesses solutions

$$w_i = W_i \exp[\alpha_i(x + \beta_i y - s_i t)], \quad \alpha_i = \sqrt{s_i - 3\beta_i^2}.$$

(s_i, β_i and W_i are some real constants).

Looking for solution of (27) in the form

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2), \quad \phi_n = \sum_{k=0}^{n} A_k^n w_1^k w_2^{n-k}, \tag{28}$$

we find for coefficients A_k^n the recursion relation

$$\begin{aligned}
A_k^n &= -\tfrac{6}{\lambda_{(k,n-k)}} \sum_{l=1}^{n-1} \sum_{m=0}^{l} \rho A_m^l A_{k-m}^{n-l}, \qquad n \geq 2, \quad 0 \leq k \leq n, \\
A_k^n &= 0 \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n, \\
\lambda_{(k,n-k)} &= (\alpha_1 k + \alpha_2(n-k))^4 - k^2 - (n-k)^2 + \\
&\quad + \alpha_1\alpha_2 k(n-k)(6\beta_1\beta_2 - s_1 - s_2), \\
\rho &= (\alpha_1 k + \alpha_2(n-k))(\alpha_1(k-m) + \alpha_2(n-l-k+m)).
\end{aligned} \tag{29}$$

The series (28) with the coefficients determined by the recursion relation (29) represents the expansion of the well known 2-soliton solution, which can be written as follows

$$\begin{aligned}
u &= 2D_x^2 \ \ln \ (1 + w_1 + w_2 + \mu w_1 w_2), \\
\mu &= \frac{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2}{(\alpha_1 + \alpha_2)^2 - (\beta_1 - \beta_2)^2}.
\end{aligned}$$

So, in this approach multi-dimensional equations are considered similarly to one-dimensional ones.

6 Linearizable equations

We consider the Burgers equation

$$\hat{L}(D_t, D_x)u(t,x) = -uu_x, \quad \hat{L}(D_t, D_x) = D_t - D_x^2. \tag{30}$$

The linearized equation $\hat{L}w = 0$ possesses the solution

$$w = W \exp[-s(x - st)]$$

and we look for solution of (30) in the form

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w), \quad \phi_n = A_n w^n. \tag{31}$$

Substituting (31) into (30), we obtain the recursion relation

$$A_n = \frac{\sum_{k=1}^{n-1} (n-k) A_k A_{(n-k)}}{sn(1-n)}.$$

Calculating the first coefficients one can easily guess the general formula

$$A_n = \left(-\frac{1}{2s}\right)^{n-1} A_1^n.$$

Therefore, we get the solution

$$u = \sum_{n=1}^{\infty} \left(-\frac{1}{2s}\right)^{n-1} (A_1 w)^n = 2s\frac{\tilde{w}}{1+\tilde{w}} = -2\frac{\tilde{w}_x}{1+\tilde{w}}, \tag{32}$$

where $\tilde{w} = \dfrac{A_1 w}{2s}$.

In (32) $\tilde{w}$ is a particular exponential solution of the linearized equation. But if we now try to do the change of variables of the form (32) with $\tilde{w}$ being an arbitrary solution of the linearized equation, we get the Cole–Hopf transformation, linearizing Burgers equation. In this example the particular solution gives the form of the linearizing transformation.

We point out, that similar transformations, which can be obtained from the form of 1-soliton solution coincide in the case of integrable equations with Hirota transformation. It seems, that such transformations are also useful in non-integrable cases, accelerating the convergence of the series.

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