Introduction to the functions on compact Riemann surfaces and Theta-functions

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Abstract: We collect here some classical facts related to analysis on the Riemann surfaces. The notes may serve as an introduction to this field; we suppose that the reader is familiar only with the basic facts from topology and complex analysis. The treatment is organised to give the background for the further application to nonlinear differential equations.

Notations: $\mathbb{C}$ — complex plane; $\bar{\mathbb{C}}$ — compactified complex plane (Riemann sphere); $\mathbb{CP}^n$ — the complex projective space of dimension n.

1 Compact Riemann surfaces

1.1 Algebraic curves (compact Riemann surfaces)

Definition 1. *Algebraic curve* $\mathcal{L}$ is a submanifold in projective space $\mathbb{CP}^2$ defined by the equation $R(\omega,\lambda)=0$, where (ω,λ) are local coordinates in $\mathbb{CP}^2$ and R is an irreducible polynomial in ω and λ. Curve $\mathcal{L}$ is called *nonsingular* if the complex gradient of the function $R(\omega,\lambda)$ does not vanish, i.e.

$$\nabla R \equiv \left(\frac{\partial R}{\partial\omega},\frac{\partial R}{\partial\lambda}\right)\neq 0. \tag{1}$$

Further in this chapter we shall consider only nonsingular algebraic curves.

According to the classical point of view an algebraic curve is represented as a covering of the complex plane. Let polynomial R have degree m in ω. Then the equation of the curve $\mathcal{L}$ may be written as follows:

$$a_0(\lambda)\omega^m + a_1(\lambda)\omega^{m-1}+\cdots+a_m(\lambda)=0, \tag{2}$$

where $a_0(\lambda),\ldots,a_m(\lambda)$ are some polynomials and $a_0(\lambda)$ is not identically zero. So for every λ in general case we have m corresponding values of ω; therefore, curve $\mathcal{L}$ may be considered as m-sheeted covering of λ-plane. However, for some values of λ equation (2) may have less then m different roots; the points of this kind are called the "branch points". Apparently this definition is equivalent to the following more exact

Definition 2. Point P is called *a branch point* (in λ-plane) iff $\frac{\partial R}{\partial\omega}(P)=0$.

System of two equations $R(\omega, \lambda) = 0$ and $\frac{\partial R}{\partial \omega}(\omega, \lambda) = 0$ has in nontrivial situation a finite number of solutions, and, therefore, there is a finite set of the branch points $P_1, \dots, P_N$.

Of course, the definition of the branch point is noninvariant under the interchange $\omega \leftrightarrow \lambda$. Namely, if we want to realize $\mathcal{L}$ as a covering of ω-plane (having, in general, another number of sheets) then the branch points are defined by the equation $R_\lambda(\omega, \lambda) = 0$. From the nonsingularity condition (1) we see that the system $R_\lambda(\omega, \lambda) = R_\omega(\omega, \lambda) = 0$ is incompatible hence the branch points in λ-plane can not coincide with the branch points in ω-plane.

It is convenient to draw $\mathcal{L}$ as a covering of λ-plane in the form of the Hurwitz diagram (Fig. 1) where the horizontal lines denote the sheets (copies of λ-plane) and the vertical lines denote the positions of the branch points where some sheets are glued together.

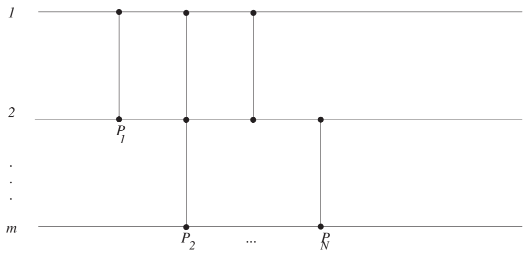


Fig. 1: Hurwitz diagram of curve L.

Another point of view on the algebraic curves is provided by the following

THEOREM 1. *An arbitrary nonsingular algebraic curve $\mathcal{L}$ is a compact complex-analytic manifold of dimension 1.*

Proof: We have to establish the map of the neighbourhood of the every point on $\mathcal{L}$ on some domain of the complex plane and to show that the resulting transition functions from $\mathbb{C}$ to $\mathbb{C}$ are complex-analytic. It is convenient to find for every point $P_0 \in \mathcal{L}$ the so-called *local parameter* at P_0, i.e. the function $\tau[P_0](P)$ mapping the neighbourhood of P_0 into the complex plane such that $\tau[P_0](P_0) = 0$.

Let's use the realization of $\mathcal{L}$ as a covering of λ-plane. If point $P_0 = (\omega_0, \lambda_0)$ doesn't coincide with the branch points i.e. $\frac{\partial R}{\partial \omega}(\omega_0, \lambda_0) \neq 0$ then we can simply take $\tau[P_0](\omega, \lambda) = \lambda - \lambda_0$. In the neighbourhood of some branch point $P_j = (\omega_j, \lambda_j)$ the situation is slightly more complicated. As we discussed above, the nonsingularity condition (1) implies that the point P_j can not be a branch point in λ-plane and in ω-plane realizations simultaneously. Therefore, in the neighbourhood of P_j we can define the local parameter as follows:

$$\tilde{\tau}[P_j](\omega, \lambda) = \omega - \omega_j. \tag{3}$$

To find the local parameter at P_j in terms of variable λ we can notice that condition $\frac{\partial R}{\partial \omega}(\omega_j, \lambda_j) = 0$ together with equation $R(\omega, \lambda) = 0$ immediately implies that $\lambda - \lambda_j = (\omega - \omega_j)^{k_j+1}(C_j + o(1))$, for $\lambda \to \lambda_j$ for some $k_j > 0$ and constant C_j. Power $k_j + 1$ is obviously equal to the number of the sheets of $\mathcal{L}$ glued at P_j. Number k_j is called the degree of the branch point P_j.

So instead of (3) we can choose in the neighbourhood of P_j the alternative local parameter $\tau[P_j](\omega, \lambda) = (\lambda - \lambda_j)^{\frac{1}{k_j+1}}$.

To provide compactness we have to find the local parameter in the neighbourhood of the infinite point in λ-plane. Obviously, if ∞ is not a branch point, we can choose $\tau[\infty](\omega, \lambda) = \lambda^{-1}$. If $\lambda = \infty$ is a branch point of degree k then $\tau[\infty](\omega, \lambda) = \lambda^{-\frac{1}{k+1}}$.

Finally, it is easy to see that all coordinate maps described above are compatible one to another i.e. related transition functions are holomorphic. □

The converse statement is also true, i.e. an arbitrary compact one-dimensional complex manifold is conformally equivalent to some algebraic curve [5].

As a corollary of the Theorem 1 we obtain that $\mathcal{L}$ is a compact oriented topological manifold of real dimension 2. So applying the following

THEOREM 2. *Any compact oriented manifold of real dimension 2 is topologically equivalent to sphere with a finite number of handles.*

(for proof cf. [1]) one obtains that topologically the algebraic curve represents the sphere with handles. The number of handles is called the genus; in the sequel we denote it by g.

Curves with $g = 1$, called elliptic, are topologically equivalent to torus.

The above treatment may be conveniently illustrated for hyperelliptic curves defined by

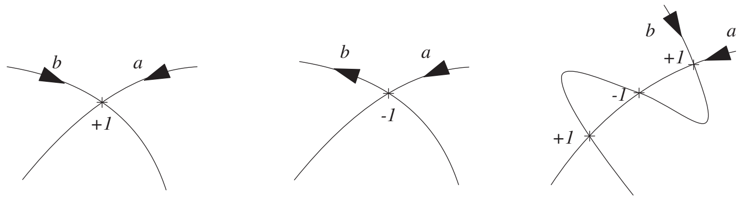
$$\omega^2 - \prod_{j=1}^{n}(\lambda - E_j) = 0, \quad E_i \neq E_j, \quad i \neq j. \tag{4}$$

Curve $\mathcal{L}$ considered as the covering of λ-plane consists of two sheets; points (ω, λ) and $(-\omega, \lambda)$ have the same projection on λ-plane and lie on the different sheets of $\mathcal{L}$. Points E_j, $j = 1, \ldots, n$ are obviously the branch points. If n is odd $(n = 2g + 1)$ then the infinite point $(\lambda = \infty)$ is the branch point on $\mathcal{L}$ too; if n is even $(n = 2g + 2)$ then $\lambda = \infty$ is an ordinary point and we have on $\mathcal{L}$ two infinities: ∞^1 and ∞^2. As we shall see below, the genus of curve $\mathcal{L}$ in both cases $(n = 2g + 1$ and $n = 2g + 2)$ is equal to g.

The local parameter on $\mathcal{L}$ in the neighbourhood of every point $P_0(\omega_0, \lambda_0)$, $\lambda_0 \neq E_i, \infty$, may simply be chosen as $\tau[P_0] = \lambda - \lambda_0$; when $\lambda_0 = E_i$ we can take $\tau[P_0] = \sqrt{\lambda - E_i}$. Finally, if ∞ is not a branch point $(n = 2g + 2)$ then $\tau[\infty] = \frac{1}{\lambda}$ and if ∞ is a branch point $(n = 2g + 1)$ then $\tau[\infty] = \frac{1}{\sqrt{\lambda}}$.

1.2 Canonical basis of cycles

Consider the first homology group $H_1(\mathcal{L}, \mathbb{Z})$. For arbitrary two elements of this group — two closed cycles a and b — we can define an integer number — the intersection index $a \circ b$. If a and b cross each other in one point as shown in Fig. 2a then $a \circ b = -b \circ a = 1$; in Fig. 2b $a \circ b = -1$.



a) $a \circ b = 1$ b) $a \circ b = -1$ c) $a \circ b = 1 - 1 + 1 = 1$

Fig. 2: Intersection index for contours a and b.

More rigorously, if contours a and b cross in one point P and the basis in the tangent plane at P consisting from their tangent vectors is positively oriented then $a \circ b = 1$ and $a \circ b = -1$ in the opposite case (here we fix the positive orientation of curve $\mathcal{L}$). If contours a and b cross each other in the several points $P_1, \dots, P_N$ then $a \circ b$ is defined as a sum of the intersection indexes at P_i (see, for example, Fig. 2c). As a result we obtain a bilinear map $H^1(\mathcal{L}, \mathbb{Z}) \times H^1(\mathcal{L}, \mathbb{Z}) \to \mathbb{Z}$.

If the genus of $\mathcal{L}$ is equal to g then it is always possible [1, 5] to choose in $H_1(\mathcal{L}, \mathbb{Z})$ the canonical basis of $2g$ cycles $a_1, \dots, a_g, b_1, \dots, b_g$ having the following matrix of intersections

$$a_i \circ a_j = b_i \circ b_j = 0, \qquad a_i \circ b_j = \delta_{ij}. \tag{5}$$

Examples of the canonical basis on an abstract curve of genus 2 and on the hyperelliptic curve

$$\omega^2 = \prod_{i=1}^{2g+2} (\lambda - E_i) \tag{6}$$

are presented in Fig. 3a, b.

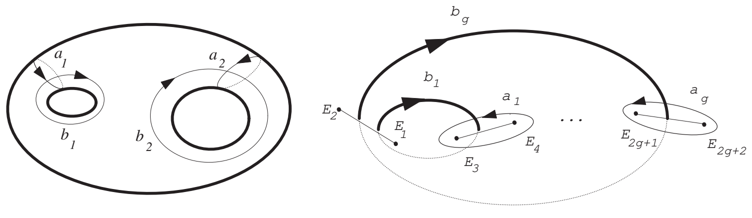


Fig. 3: Canonical basis of cycles. a) $g = 2$. b) Hyperelliptic curve: continuous contours lie on the 1-st sheet, dotted on the 2-nd.

The choice of the canonical basis is not unique. Using the definition (5) it is easy to verify that if the basis (a_j, b_j) is canonical then the basis

$$\tilde{a}_i = \sum_{j=1}^{g}(P_{ij}a_j + S_{ij}b_j), \qquad \tilde{b}_i = \sum_{j=1}^{g}(Q_{ij}a_j + T_{ij}b_j) \tag{7}$$

is also canonical iff the $2g \times 2g$ integer-valued matrix

$$M = \begin{pmatrix} P & S \\ Q & T \end{pmatrix}$$

is symplectic, i.e.

$$MJM^T = J, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is $g \times g$ unit matrix, M^T denotes the matrix transposed to M.

2 Abelian differentials

2.1 Holomorphic differentials

DEFINITION 3. Differential 1-form $dU(P)$ on algebraic curve $\mathcal{L}$ is called *the holomorphic differential* (or *abelian differential of the first kind*) if in the neighbourhood of every point $Q \in \mathcal{L}$ it may be represented as $dU = f(\tau)d\tau$, where τ is the local parameter at Q, $f(\tau)$ is a holomorphic function.

Obviously all holomorphic differentials on $\mathcal{L}$ constitute the linear vector space.

THEOREM 3. *The dimension of the vector space of the holomorphic differentials on $\mathcal{L}$ coincides with its genus g.*

As we shall prove below by the analysis of the cyclic periods of the holomorphic differentials, this dimension is always less than or equal to g. For the proof of existence of g linearly independent holomorphic differentials on an arbitrary genus g algebraic curve see [1, 2]. In the case of hyperelliptic Riemann surface (4) (both for $n = 2g + 1$ and $n = 2g + 2$ cases) these differentials may be written out explicitly:

$$dU_k(P) = \frac{\lambda^{k-1}d\lambda}{\omega}, \quad k = 1, \dots, g. \tag{8}$$

The holomorphicity of the differentials (8) on $\mathcal{L}$ may be easily verified using the explicit form of the local parameter at the different points of $\mathcal{L}$.

It is easy to see that an arbitrary holomorphic differential $dU(P)$ is closed: in the neighbourhood of the every point on $\mathcal{L}$ we have $d(f(\tau)d\tau) = \frac{\partial f(\tau)}{\partial \bar{\tau}} d\bar{\tau} \wedge d\tau - \frac{\partial f(\tau)}{\partial \tau} d\tau \wedge d\tau = 0$.

An arbitrary anti-holomorphic differential is, of course, closed too.

Taking into account the obvious relation $\oint_C dW = 0$, where C is an arbitrary closed cycle homological to zero, we can correctly define the cyclic periods of an arbitrary closed differential $dW(P)$ by

$$A_j \equiv \oint_{a_j} dW(P), \qquad B_j \equiv \oint_{b_j} dW(P).$$

To define correctly the function $\int_{P_0}^P dW$, where P_0 is some fixed point, we can proceed as follows. Let all basic cycles $a_j, b_j,\ j = 1, \dots, g$ pass through some fixed point $Q \in \mathcal{L}$ (see Fig. 4a for $g = 2$). Then, cutting $\mathcal{L}$ along all these cycles we obtain the simply-connected domain $\tilde{\mathcal{L}}$ — the so-called fundamental polygon having $4g$ sides (Fig. 4b) (this is in fact the fundamental domain of the Fuchsian group related to $\mathcal{L}$ [12]).

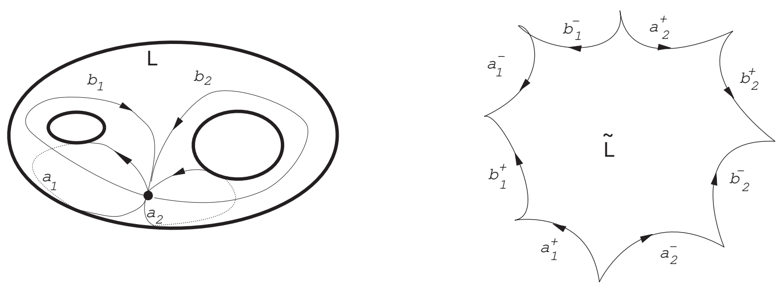


Fig. 4: Curve L with $g = 2$ and its fundamental polygon $\widetilde{L}$.

Function $F(P) = \int_{P_0}^P dW$ is obviously single-valued on $\tilde{\mathcal{L}}$. The boundary of $\tilde{\mathcal{L}}$ may be represented as follows

$$\partial\tilde{\mathcal{L}} = \sum_{j=1}^{g} (a_j^+ + b_j^+ - a_j^- - b_j^-). \tag{9}$$

Let's prove the following

Lemma 1. *Let $dW(P)$ and $dW'(P)$ be two closed differentials on $\mathcal{L}$ and A_j, B_j, A'_j, B'_j, $j = 1, \dots, g$ be sets of related cyclic periods; $W(P) = \int_{P_0}^P dW$ (where $P_0 \in \tilde{\mathcal{L}}$ is some fixed point) is a function on $\tilde{\mathcal{L}}$. Then*

$$\iint_{\mathcal{L}} dW \wedge dW' = \oint_{\partial\tilde{\mathcal{L}}} W(P) dW'(P) = \sum_{j=1}^{g} (A_j B'_j - A'_j B_j) \tag{10}$$

(we shall use this statement in two situations: when dW and dW' are both holomorphic or one is holomorphic while the other anti-holomorphic).

Proof: The first equality in (10) is a simple corollary of the Stokes formula and the requirement that differential dW' is closed. To prove the second one use the representation (9) of $\partial\tilde{\mathcal{L}}$:

$$\oint_{\partial\tilde{\mathcal{L}}} W(P)dW'(P) =$$

$$\sum_{j=1}^{g}\left(\int_{a_j^+}-\int_{a_j^-}\right)W(P)dW'(P)+\sum_{j=1}^{g}\left(\int_{b_j^+}-\int_{b_j^-}\right)W(P)dW'(P).$$

Now, for every j-th part of the boundary of the fundamental polygon consisting of four sides $a_j^+, b_j^+, a_j^-, b_j^-$ (Fig. 5)

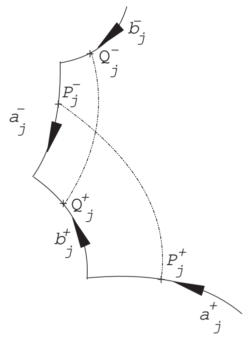


Fig. 5: j-th component of $\partial\widetilde{L}$.

we can easily see that for the coinciding on $\mathcal{L}$ points $P_j^\pm$ lying on the cycles $a_j^\pm$ respectively: $W(P_j^+)-W(P_j^-)=-B_j$ and for the points $Q_j^\pm$ lying on cycles $b_j^\pm$: $W(Q_j^+)-W(Q_j^-)=A_j$.

The differential $dW'(P)$ is single-valued on $\mathcal{L}$ and, therefore, is the same on the different banks of a_j and b_j. As a result we have

$$\begin{aligned}\oint_{\partial\tilde{\mathcal{L}}} W(P)dW'(P) &= \sum_{j=1}^{g}(-B_j)\oint_{a_j} dW'(P)+\sum_{j=1}^{g}A_j\oint_{b_j} dW'(P)\\ &= \sum_{j=1}^{g}(A_jB'_j-B_jA'_j)\end{aligned}$$

□

COROLLARY 1. *Let $dU(P)$ be a nonzero holomorphic differential on $\mathcal{L}$ with cyclic periods A_j, B_j, $j=1,\ldots,g$. Then*

$$\operatorname{Im}\sum_{j=1}^{g} A_j\bar{B}_j < 0. \tag{11}$$

Proof: Let's choose in Lemma 1 $dW(P) = dU(P)$, $dW'(P) = d\bar{U}(P)$. Then $A'_j = \bar{A}_j$, $B'_j = \bar{B}_j$; besides that, at every point of $\mathcal{L}$ we can choose the local parameter $\tau = x+iy$ in such a way that $dU = d\tau = dx+idy$ and $dU \wedge d\bar{U} = (dx+idy)\wedge(dx-idy) = -2idx\wedge dy$.

Taking into account that the curve $\mathcal{L}$ is positively oriented, we obtain

$$\operatorname{Im}\iint_{\mathcal{L}} dU \wedge d\bar{U} < 0,$$

which immediately entails (11). □

From (11) we obtain the following

COROLLARY 2. *If all the periods of the holomorphic differential $dU(P)$ are equal to zero then $dU(P) \equiv 0$.*

This corollary obviously implies that the dimension of the linear space of the holomorphic differentials on $\mathcal{L}$ is not greater than g (otherwise we would be able to construct a nontrivial holomorphic differential with zero periods). Together with the proof of existence of g linearly independent holomorphic differentials that may be found in [1, 2] (for hyperelliptic curve (6) they are given by the exact formulae (8)) this gives Theorem 3.

Let $dW_1, \ldots, dW_g$ be linearly independent holomorphic differentials on $\mathcal{L}$. Then their matrix of a-periods $A_{ij} = \oint_{a_i} dW_j$ is obviously nondegenerate (otherwise some nontrivial linear combination of them has all zero a-periods that contradicts Corollary 2). So it is possible to choose another basis of holomorphic differentials $dU_j(P)$, $j = 1, \ldots, g$ normalized by the condition

$$\oint_{a_k} dU_j(P) = \delta_{kj}. \tag{12}$$

Differentials dU_j constitute the *canonical basis of holomorphic 1-forms* dual to the canonical basis of cycles (a_j, b_j). Now introduce the new fundamental object associated to the curve $\mathcal{L}$ — the *matrix of b-periods*

$$B_{jk} = \oint_{b_j} dU_k. \tag{13}$$

THEOREM 4. *The matrix B is symmetric and has positively-defined imaginary part.*

Proof: To prove that B is symmetric let's put in Lemma 1 $dW = dU_j$, $dW' = dU_k$; then obviously $dU_j \wedge dU_k = 0$ and the r.h.s. of (10) gives $B_{kj} = B_{jk}$.

To prove the second part of the theorem consider for an arbitrary $x \in \mathbb{R}^g$ the linear combination $dU = \sum_{j=1}^g x_j dU_j$. It has the following a- and b-periods: $A_k = x_k$, $B_k = \sum_{j=1}^g x_j B_{kj}$. Applying the Corollary 1 to the differential dU we obtain $\operatorname{Im}\sum_{j,k=1}^g x_j \bar{B}_{kj} x_k = \operatorname{Im}\sum_{k=1}^g A_k \bar{B}_k < 0$ and, therefore, $\operatorname{Im}\langle Bx, x\rangle$ is always positive ($\langle ., . \rangle$ is the ordinary scalar product). □

Matrix B is a very important object characterizing the curve $\mathcal{L}$. The famous Schottky problem is isolating the matrices of b-periods of algebraic curves among the whole family of symmetric matrices with positively-defined imaginary part. The essential progress in this field was recently achieved in the framework of algebra-geometrical approach to the integrable equations [7].

It is easy to verify that if we choose the new basis of cycles $(\tilde{a}_j, \tilde{b}_j)$ related to basis (a_j, b_j) by (7) then the corresponding matrices of b-periods $\tilde{B}$ and B are related as follows:

$$\tilde{B} = (TB + Q)(SB + P)^{-1}. \tag{14}$$

Now we can construct the so-called Jacobi manifold of curve $\mathcal{L}$.

Definition 4. The complex torus of dimension g $J(\mathcal{L})$ defined as the following quotient

$$J(\mathcal{L}) = \mathbf{C}^g / \{N + BM\},$$

where $N, M \in \mathbb{Z}^g$, is called *the Jacobi manifold* (or *Jacobian*) of the curve $\mathcal{L}$.

The transformation of the matrix B (14) with respect to the change of the canonical basis of cycles leads to some nonessential transformations of $J(\mathcal{L})$ (see [11, 3]).

2.2 Meromorphic differentials

Definition 5. Meromorphic 1-form having on $\mathcal{L}$ the unique pole at point Q with the following local expansion:

$$\left(\frac{1}{\tau^{n+1}} + O(1)\right) d\tau,$$

where $n \in \mathbf{N}$ and τ is the local parameter in the neighbourhood of Q is called *the Abelian differential of the 2nd kind.*

Definition 6. Meromorphic 1-form having on $\mathcal{L}$ the simple poles at $P = Q$ and $P = R$ with the residues $+1$ and -1 respectively is called *the Abelian differential of the 3rd kind.*

For an arbitrary Abelian differential dW on $\mathcal{L}$ (of 1st, 2nd, or 3rd kind) the multi-valued (in general) function $W(P) = \int_{P_0}^{P} dW$, where $P_0 \in \mathcal{L}$ is some fixed point, is called the *Abelian integral* (of 1st, 2nd or 3rd kind, respectively).

Notice that the Abelian integrals (or simply integrals) of the 1st kind are everywhere holomorphic; integrals of the 2nd kind are meromorphic and the integrals of the 3rd kind have logarithmic singularities at Q and R. The integrals of the 1st and 2nd kind are single-valued on the fundamental polygon $\tilde{\mathcal{L}}$; integral W_{QR} is single-valued on $\tilde{\mathcal{L}}$ with the cut $[Q, R]$.

It appears possible to prove that the meromorphic differentials of the 2nd and 3rd kind exist for arbitrary positions of the poles and arbitrary related singular parts [1, 2] (for differentials of the third kind the only restriction is the vanishing of the sum of all the residues).

For the hyperelliptic curve (6) these integrals may be easily written out explicitly. Namely, if $Q = (\omega_1, \lambda_1)$ and $R = (\omega_2, \lambda_2)$ do not coincide with the branch points, we can express the related differentials of the 3rd kind as follows:

$$dW = \frac{d\lambda}{2\omega}\left(\frac{\omega+\omega_1}{\lambda-\lambda_1} - \frac{\omega+\omega_2}{\lambda-\lambda_2}\right). \tag{15}$$

For the differential of the 2nd kind having pole at Q with $n = 1$ we have

$$dW = \frac{d\lambda}{2\omega}\left(\frac{\omega+\omega_1}{(\lambda-\lambda_1)^2} + \frac{\omega'(\lambda_1)}{\lambda-\lambda_1}\right) \tag{16}$$

(it is not difficult to write analogous expressions if λ_1 or λ_2 coincide with the branch points).

Of course, the meromorphic differential having prescribed poles and the singular parts is not unique: it is defined up to an arbitrary holomorphic differential. To get rid of this nonuniqueness we can normalize meromorphic differentials as follows

Definition 7. Abelian differential of the 2nd or the 3rd kind is called *normalized* if all its a-periods are equal to zero.

An arbitrary meromorphic differential may obviously be normalized by adding some linear combination of basic holomorphic differentials. The normalization together with the positions of the poles and values of related singular parts defines the meromorphic differential uniquely. Indeed, if dW_1 and dW_2 are two normalized holomorphic differentials with the same poles and the same singular parts then $dW_1 - dW_2$ is a holomorphic differential with vanishing a-periods, and, therefore, $dW_1 = dW_2$.

The b-periods of normalized meromorphic differentials may be easily expressed in terms of holomorphic differentials. Namely,

Lemma 2. *The b-periods of the normalized abelian differentials of the 2nd and the 3rd kind may be represented in the following form:*

$$(B_Q^{(n)})_j \equiv \oint_{b_j} dW_Q^{(n)} = \frac{2\pi i}{n!}\frac{d^{(n-1)} f_j}{d\tau^{(n-1)}}\bigg|_{\tau=0}, \tag{17}$$

where function f_j describes the local behaviour of holomorphic differential dU_j ($dU_j(P) = f_j(\tau)d\tau$, where τ is the local parameter at Q)

$$(B_{QR})_j \equiv \oint_{b_j} dW_{QR} = 2\pi i(U_j(Q) - U_j(R)). \tag{18}$$

Proof: Consider (18) ((17) may be proved in the same way). In analogy to the proof of Lemma 1 we have

$$(*) \equiv \oint_{\partial\tilde{\mathcal{L}}} U_j(P)dW_{QR}(P) = \sum_{k=1}^{g}(A_{jk}(B_{QR})_k - (A_{QR})_k B_{jk}) = (B_{QR})_j,$$

where $A_{jk} = \delta_{jk}$ and B_{jk} are a- and b-periods of basic holomorphic differential $dU_j(P)$, $(A_{QR})_k = 0$ are a-periods of $dW_{QR}^{(n)}$. Calculating $(*)$ in another way according to residue theorem, we obtain (18). $\square$

3 Meromorphic functions

3.1 Meromorphic functions. The Abel theorem

Now we are in position to start the treatment of meromorphic functions on $\mathcal{L}$. Any meromorphic function may be considered as a linear combination of the abelian integrals of the 2nd kind which has vanishing cyclic periods. This condition appears crucial: the space of meromorphic functions has much more complicated structure then the space of the abelian integrals. This structure is simplest for $\mathcal{L} = \bar{\mathbb{C}}$ (Riemann sphere). Here every meromorphic function may be represented as the ratio of two polynomials:

$$f(\lambda) = C\frac{(\lambda - a_1)\dots(\lambda - a_n)}{(\lambda - b_1)\dots(\lambda - b_m)}. \tag{19}$$

Note the difference between the theory of functions on the ordinary noncompact complex plane $\mathbb{C}$ and its compactification $\bar{\mathbb{C}}$. We know, for example, that every holomorphic function on $\bar{\mathbb{C}}$ is a constant; the same statement for $\mathbb{C}$ is not true (counterexample is the function $\exp\lambda$). Moreover, the function (19) has the same number of poles and zeros (taking into account their order) on $\bar{\mathbb{C}}$. This is not true (in general) for $\mathbb{C}$.

Some features of the space of meromorphic functions on compact Riemann surface of nonzero genus appear very similar to $\bar{\mathbb{C}}$ case (for example, the statement that every holomorphic function on algebraic curve is a constant, or the fact that the number of poles is always equal to the number of zeros). However, in other aspects the genus plays a crucial role. For example, if $g > 0$, then the positions of zeros and poles of the meromorphic function should obey some additional restrictions (in contrast to $g = 0$, where the poles and zeros in (19) may be defined arbitrarily).

Now let's prove some facts about the meromorphic functions on compact Riemann surfaces.

THEOREM 5. *The number of poles of a nonconstant meromorphic function $f(P)$ on $\mathcal{L}$ is always finite and equal to the number of zeros (we calculate the number of poles and zeros taking into account their order).*

Proof: The finiteness of the number of zeros obviously follows from the compactness of $\mathcal{L}$ and the existence of nontrivial Taylor expansion of $f(P)$ at every point.

To prove that the number of zeros of $f(P)$ is equal to the number of poles it is sufficient to integrate expression $d\log f(P)$ along the boundary of fundamental polygon. On one hand, it vanishes since the cyclic periods along the basic cycles of this differential are equal to zero. On the other hand, it is equal to the difference between the number of zeros and the number of poles of $f(P)$ (up to a factor $2\pi i$). □

Now it is natural to ask when we can construct function $f(P)$ having prescribed set of zeros and poles. To answer this question define the *Abel map* U from the curve $\mathcal{L}$ into its Jacobi manifold: $U : \mathcal{L} \to J(\mathcal{L})$ in the following way:

$$U_j(P) = \int_{P_0}^{P} dU_j, \qquad j = 1, \dots, g\,, \tag{20}$$

where $dU_j(P)$ are normalized basic holomorphic differentials; $P_0 \in \mathcal{L}$ is some fixed point and the path from P_0 to P is chosen in the same way for all j. To prove that definition (20) is correct we have to check its independence of the choice of the path between P_0 and P. Indeed, if we add to this path some linear combination of basic cycles, $\gamma = \sum_k n_k a_k + \sum_k m_k b_k$, then in the r.h.s. of (20) we get the additional term $\oint_\gamma dU_j = n_j + \sum_k B_{jk} m_k$, i.e. the vector of the lattice defining $J(\mathcal{L})$; so the change of the path in (20) doesn't change the position of the point on $J(\mathcal{L})$ and (20) correctly defines the map from $\mathcal{L}$ to $J(\mathcal{L})$.

Now we can formulate the following

THEOREM 6. (ABEL) *Two sets of points: $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ on $\mathcal{L}$ are the sets of poles and zeros (respectively) of some meromorphic function iff*

$$\sum_{j=1}^{g} (U(P_j) - U(Q_j)) \equiv 0 \tag{21}$$

(symbol $\equiv$ here and below means that the left-hand side and the right-hand side define the same point on $J(\mathcal{L})$, i.e. they differ by some vector of the lattice $\{m + Bn\}$).

Proof: Let meromorphic function $f(P)$ have poles and zeros at $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ respectively. Then the meromorphic differential

$$d\Omega(P) = d \log f(P)$$

has simple poles at the points $P_1, \dots, P_n$ with the residues equal to -1 and at the points $Q_1, \dots, Q_n$ with the residues equal to $+1$; therefore, it may be represented as follows:

$$d\Omega(P) = \sum_{j=k}^{n} dW_{Q_k P_k}(P) + \sum_{k=1}^{n} C_k dU_k(P)$$

where $dW_{Q_k P_k}(P)$ are normalized abelian differentials of the 3rd kind, $dU_k(P)$ are basic holomorphic differentials and C_k are some constants.

From the single-valuedness of function $f(P)$ on $\mathcal{L}$ we conclude that the integral of $d\Omega$ along an arbitrary closed contour should be equal to $2\pi i n$ for some integer n. In particular,

$$\oint_{a_j} d\Omega = 2\pi i n_j, \qquad \oint_{b_j} d\Omega = 2\pi i m_j,$$

where $n_j, m_j \in \mathbb{Z}$. So, taking into account the normalization condition (12) and formulae for the b-periods of differentials $dW_{Q_j P_j}$ (18), we obtain $2\pi i n_j = C_j$ and

$$2\pi i m_j = 2\pi i \sum_{k=1}^{g} (U_j(P_k) - U_j(Q_k)) + 2\pi i \sum_{k=1}^{g} B_{jk} n_k.$$

Therefore

$$\sum_{k=1}^{g}(U_j(P_k) - U_j(Q_k)) = m_j - \sum_{k=1}^{g} B_{jk} n_k$$

that yields (21).

Conversely, having a set of the points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ obeying (21), we can construct the meromorphic differential

$$d\Omega = \sum_{k=1}^{g} dW_{P_k Q_k} + 2\pi i \sum_{k=1}^{g} n_k dU_k.$$

Then the function $f(P) = \exp \int_{P_0}^{P} d\Omega$ has poles at $P_1, \ldots, P_n$, zeros at $Q_1, \ldots, Q_n$ and is single-valued on $\mathcal{L}$ for some integer n_k, $k = 1, \ldots, g$ due to the normalization conditions of $dW_{P_k Q_k}$ and relations (21). □

Note that the Abel theorem doesn't give us the effective criterion for the existence of meromorphic functions with prescribed zeros and poles. To formulate the further results in this direction it is convenient to define some new objects.

3.2 Divisors on algebraic curves

Definition 8. The formal linear combination $D = \sum_{j=1}^{N} n_j P_j$, where $P_j \in \mathcal{L}$, $n_j \in \mathbb{Z}$, $j = 1, \ldots, N$, is called *a divisor* on the curve $\mathcal{L}$.

All divisors on $\mathcal{L}$ constitute the abelian group with respect to the obvious summation operation. For $D = \sum_{j=1}^{N} n_j P_j$ and $D' = \sum_{j=1}^{N'} n'_j P'_j$ we define

$$D + D' = \sum_{j=1}^{N} n_j P_j + \sum_{j=1}^{N'} n'_j P'_j.$$

The number

$$\deg D = \sum_{j=1}^{N} n_j$$

is called the *degree* of the divisor D.

For an arbitrary meromorphic function $f(P)$ having zeros at the points $Q_1, \ldots, Q_n$ of the order $q_1, \ldots, q_n$ and poles at the points $P_1, \ldots, P_m$ of the order $p_1, \ldots, p_m$ respectively, the divisor

$$(f) = p_1 P_1 + \cdots + p_m P_m - q_1 Q_1 - \cdots - q_n Q_n$$

is called the divisor of the function f. Theorem 5 shows that $\deg(f) = 0$.

Two divisors D and D' are called linearly equivalent if divisor $D - D'$ is a divisor of some meromorphic function. Obviously in this case $\deg D = \deg D'$. The divisor of arbitrary meromorphic function is linearly equivalent to zero.

We can associate the divisor of zeros and poles (dW) to any meromorphic differential dW (the zeros and poles of dW are defined as zeros and poles of related local function $h(\tau)$ in the local representation $dW = h(\tau)d\tau$ at the every point of $\mathcal{L}$).

It is easy to prove that the divisors (dW) and (dW') of arbitrary two meromorphic differentials dW and dW' are linearly equivalent:

$$(dW) - (dW') = \left(\frac{dW}{dW'}\right)$$

where $\frac{dW}{dW'} = \frac{dW}{dW'}(P)$ is a meromorphic function on $\mathcal{L}$.

The equivalence class of the divisors of meromorphic differentials is called the *canonical class* of of curve $\mathcal{L}$.

Now define the Abel map on the divisors by linearity. For $D = \sum_{j=1}^{N} n_i P_i$ we put

$$U(D) = \sum_{j=1}^{N} n_j U(P_j).$$

The language of divisors is very convenient to formulate the theorems related to meromorphic functions on $\mathcal{L}$. For example, the Abel theorem may be formulated as follows:

Divisors D and D' are linearly equivalent iff $\deg D = \deg D'$ and $U(D) \equiv U(D')$ ($U(D) \equiv U(D')$ means that $U(D)$ and $U(D')$ define the same point of $J(\mathcal{L})$)

Divisor $D = \sum_{j=1}^{N} n_j P_j$ is called *positive* if $n_j > 0$ for all $j = 1, \ldots, N$.

Now we can define the partial ordering on the set of the divisors assuming $D \geq D'$ iff $D - D'$ is a positive divisor. With every divisor D we can associate the linear vector space $L(D)$ of meromorphic functions $f(P)$ on $\mathcal{L}$ obeying the condition

$$(f) \geq -D. \tag{22}$$

In less abstract language it means that given any divisor $D = n_1 P_1 + \cdots + n_k P_k - m_1 Q_1 - \cdots - m_l Q_l$ $(n_j, m_j > 0)$, function $f(P)$ belongs to the space $L(D)$ iff

1. It has no poles on $\mathcal{L}$ outside $P_1, \ldots, P_k$ and the order of its pole at P_j is $\leq n_j$, $j = 1, \ldots, k$.

2. It has zeros at Q_j of the order at least m_j, $j = 1, \ldots, l$.

The dimension of the linear space $L(D)$ is denoted by $l(D)$. If divisors D and D' are linearly equivalent then apparently $l(D) = l(D')$ (if D is linearly equivalent to D' then the isomorphism between the spaces $L(D)$ and $L(D')$ may be established by the multiplication of any function from $L(D)$ on the function $h(P)$ obeying condition $(h) = D - D'$).

Below we present some facts about the number $l(D)$. The central result here is the classical version of the Riemann–Roch theorem establishing the link between $l(D)$ and $l(C - D)$ where C is an arbitrary divisor from the canonical class (divisor of arbitrary meromorphic differential on $\mathcal{L}$). Our treatment here mainly follows [2].

3.3 Riemann–Roch theorem

Let's first prove the following

THEOREM 7. *For any positive divisor D on curve $\mathcal{L}$ of genus g*

$$l(D) = \deg D - g + 1 + l(C - D), \tag{23}$$

where C is an arbitrary divisor from the canonical class.

Proof: For the sake of transparency we restrict ourselves to divisors D having the form $D = P_1 + \cdots + P_n$ where $P_i \neq P_j$. Then $L(D)$ is the linear space of the functions that have no singularities on $\mathcal{L}$ except, probably, the simple poles at $P_1, \ldots, P_n$ (the generalization to higher order poles is straightforward). Let also for definiteness $n \geq g$ (the opposite case may be considered analogously).

Consider the set of normalized (all a-periods are zero) meromorphic differentials of the 2nd kind $dW^{(1)}_{P_j}(P)$, $\;j = 1, \ldots, n$. (Existence of differentials $dW^{(n)}_Q$ on $\mathcal{L}$ for an arbitrary $Q \in \mathcal{L}$ and $n \geq 1$ is proved in [1, 2]. Formula (16) gives this differential for $n = 1$ in the case of hyperelliptic curve (6)).

Define by $W_j(P)$ the related integrals:

$$W_j(P) = \int_{P_0}^{P} dW^{(1)}_{P_j}, \quad j = 1, \ldots, n$$

and consider the linear combination

$$f(P) = \sum_{j=1}^{n} \alpha_j W_j(P) \tag{24}$$

for some $\alpha_j \in \mathbb{C}$. Integral $f(P)$ has simple poles at $P_1, \ldots, P_n$ and no other singularities. Besides that, all its a-periods vanish. So this is a meromorphic function from the space $L(D)$ if all its b-periods also vanish. Moreover, it is obvious that any function $f(P) \in L(D)$ may be represented in the form (24): α_j are equal the residue at P_j.

Defining n vectors of b-periods

$$(B_j)_k = \oint_{b_k} dW^{(1)}_{P_j}, \quad j = 1, \ldots, n, \; k = 1, \ldots, g$$

and considering matrix $n \times g$ $M_{jk} = (B_j)_k$ we see that the vanishing of all b-periods of the integral (24) is equivalent to the following linear system

$$\sum_{j=1}^{n} M_{jk} \alpha_j = 0\,, \quad k = 1, \ldots, g. \tag{25}$$

This system has $n - \mathrm{rank}M$ linearly independent solutions, therefore, in $L(D)$ we obtain $n - \mathrm{rank}M$ linearly independent nontrivial functions (with the different sets of residue at P_j). Taking into account the constant function we obtain

$$l(D) = \deg D - \mathrm{rank}M + 1. \tag{26}$$

Now let's establish the link between $\mathrm{rank}M$ and $l(C - D)$. It remains to prove that

$$g - \mathrm{rank}M = l(C - D) \tag{27}$$

Instead of C we can substitute a divisor of an arbitrary meromorphic differential. We shall prove (27) in two steps. First, show that

$$g - \mathrm{rank}M = \dim H(D), \tag{28}$$

where by $H(D)$ we denote the linear space of holomorphic differentials having zeros at $P_1, \dots, P_n$. An arbitrary holomorphic differential $dU(P)$ on $\mathcal{L}$ can be represented as a linear combination of the basic normalized differentials

$$dU(P) = \sum_{k=1}^{g} \beta_k dU_k(P). \tag{29}$$

Now consider the following integral:

$$\oint_{\partial\tilde{\mathcal{L}}} W_k(P) dU(P) = (*), \qquad k = 1, \dots, g. \tag{30}$$

It is equal to zero iff $dU(P)$ vanishes at the pole of $W_k(P)$ i.e. at P_k. So $dU(P) \in H(D)$ iff all the integrals (30) vanish. On the other hand, calculating integrals (30) as in the proof of Lemma 2.1, using normalization conditions for dU_k, W_k and representation (29) we see that

$$(*) = -\sum_{j=1}^{g} \beta_j (B_k)_j,$$

where as before B_k, $k = 1, \dots, n$ is the vector of b-periods of the integral W_k. So $dU(P) \in H(D)$ iff coefficients β_k obey the following linear system:

$$\sum_{k=1}^{g} M_{jk}\beta_k = 0, \quad j = 1, \dots, n \tag{31}$$

where $n \times g$ matrix M_{jk} consisting of vectors B_k is the same as in (25) (notice, however, that the linear systems (25) and (31) are different). Since we consider the case $g \le n$, the system (31) has $g - \mathrm{rank}M$ linearly independent solutions, that implies (28).

To verify (27) it remains to prove that

$$\dim H(D) = l(C - D). \tag{32}$$

Let's take $C = (dW)$ where dW is some meromorphic differential ($l(C-D)$ is invariant with respect to this choice). The one-to-one linear correspondence between the linear spaces $H(D)$ and $\mathcal{L}((dW)-D)$ may be established by means of differential dW as follows:

$$dU(P) = f(P)dW(P)\,, \quad dU \in H(D), \quad f \in L((dW)-D).$$

So (32) is proved and, combining it with (28) and (26) we obtain (23). □

Relation (23) allows to get information about the existence and the number of linearly independent functions having prescribed set of the poles. In particular, we have the following

Corollary 3. (Riemann inequality) *For any positive divisor D*

$$l(D) \geq \deg D - g + 1 \tag{33}$$

So for arbitrary $g+1$ points $P_1, \dots, P_{g+1}$ on $\mathcal{L}$ we have $l(D) \geq 2$ and, therefore, we can always find a nontrivial meromorphic function f on $\mathcal{L}$ which has no singularities except, probably, the simple poles at $P_1, \dots, P_{g+1}$. For the set of g poles such a function is generically absent. The divisors $D_1 + \dots + D_g$ for which such a function exists are called special; they constitute the subset of complex codimension 1 in the space of all divisors. The generic divisors are described by the following

Definition 9. Positive divisor is called *nonspecial* if $l(D) = \deg D - g + 1$, otherwise it is called *special*.

Here it is relevant to prove the following simple statement which allows to classify the positive divisors of degree g on hyperelliptic curves:

Statement 1. *Divisor $D = P_1 + \dots + P_g$ with $P_j \neq P_k$ on the hyperelliptic algebraic curve $\mathcal{L}$ of genus g (6) is nonspecial iff points $P_1, \dots, P_g$ have different projections on λ-plane.*

This fact illustrates the general situation that divisors in general position are nonspecial and, therefore, an arbitrary special divisor may be made nonspecial by a "little stirring".

Proof: According to the proof of Theorem 7 it is enough to show that $\dim H(D) = 0$ iff points $P_j \equiv (\omega_j, \lambda_j)$ have different projections on λ-plane (here $H(D)$ is the linear space of holomorphic differentials having poles at $P_1, \dots, P_g$). Since an arbitrary holomorphic differential on the curve (6) is a linear combination of basic differentials (8), it may be represented in the form

$$dU(P) = \frac{P_{g-1}(\lambda)}{\omega},$$

where $P_{g-1}(\lambda)$ is some polynomial of degree $g-1$. Then conditions $dU(P_j) = 0,\ j = 1, \dots, g$ are equivalent to the linear system $P_{g-1}(\lambda_j) = 0, (j = 1, \dots, g)$ for g coefficients of polynomial P_{g-1}. This system has nontrivial solutions iff its determinant $\Delta \equiv \prod_{j \neq k} (\lambda_j -$

λ_k) (Vandermond determinant of values $\lambda_1, \ldots, \lambda_g$) vanishes, i.e. $\lambda_j = \lambda_k$ for some j, k. □

Another result that can be easily deduced from Theorem 7 is about the degree of an arbitrary divisor from the canonical class.

COROLLARY 4. *For an arbitrary meromorphic differential dW on $\mathcal{L}$ we have*

$$\deg(dW) = 2g - 2.$$

Proof: As we have shown before, $\deg(dW)$ is the same for all meromorphic differentials since $(dW) - (dW')$ is the divisor of the meromorphic function $\frac{dW}{dW'}$. Now let's put in (23) $D = (dU_0)$ where dU_0 is an arbitrary holomorphic differential ($D > 0$ since dU_0 is holomorphic). Linear space $L(D)$ consists of the functions having poles at the points of D; every function $f \in L(D)$ may obviously be represented in the form

$$f(P) = \frac{dU(P)}{dU_0(P)} \tag{34}$$

where dU is an arbitrary holomorphic differential.

The linear space of meromorphic differentials on $\mathcal{L}$ (and, therefore, the linear space of functions that may be represented as (34) for fixed dU_0) has dimension g; therefore, $l(D) = g$. Dimension $l(C - D)$ may be calculated taking C to be an arbitrary divisor from the canonical class, for example, $C = (dU_0)$); then $l(C - D) = l(0)$ (0 is the zero divisor). So $l(C - D) = 1$ (space $L(0)$ includes only the constant functions). As a result we can write (23) as $g = \deg D - g + 1 + 1$. Therefore, $\deg D = 2g - 2$ gives the degree of an arbitrary divisor from the canonical class. □

Now we can prove the following *Riemann–Roch* theorem (more exactly, its "classical" version related to meromorphic functions on algebraic curves).

THEOREM 8. (RIEMANN–ROCH) *The following relation is true for an arbitrary divisor D on algebraic curve $\mathcal{L}$ of genus g*

$$l(D) = \deg D - g + 1 + l(C - D). \tag{35}$$

Proof: As we have proved above, (35) is true for an arbitrary positive divisor $D_0 > 0$. Let's consider the divisor

$$\tilde{D} = D_0 + (f) \tag{36}$$

where f is some meromorphic function. Divisors $\tilde{D}$ and D are linearly equivalent, and, therefore,

$$\deg D_0 = \deg \tilde{D}, \quad l(D_0) = l(\tilde{D}), \quad l(C - D_0) = l(C - \tilde{D}).$$

So (35) is valid for an arbitrary divisor which may be represented as (36), where D_0 is a positive divisor.

Now consider two cases:

1. $l(D) > 0$. Take some function $f_0 \in (D)$; by definition it means that $D_0 \equiv (f_0) + D > 0$; so we can claim that

$$l((f_0) + D) = \deg((f_0) + D) - g + 1 + l(C - (f_0) - D)$$

or, equivalently,

$$l(D) = \deg D - g + 1 + l(C - D)$$

which coincides with (35)

2. $l(D) = 0$. Again consider two cases:

   (a) $l(C - D) \neq 0$. Then, taking into account item 1. and Corollary 4, we can claim that

$$\begin{aligned} l(C-D) &= \deg(C-D) - g + 1 + l(D) \\ &= 2g - 2 - \deg D - g + 1 + l(D) \end{aligned}$$

   that again gives (35).

   (b) $l(C - D) = l(D) = 0$. Here we have to prove that $\deg D = g - 1$.
   Let $D = D_1 - D_2$ where $D_1 = P_1 + \cdots + P_k$, $D_2 = Q_1 + \cdots + Q_l$ are two positive divisors. Using condition $l(D) = 0$ we can claim that $\deg D_2 \geq l(D_1)$ since otherwise we could find in $L(D_1)$ the function having zeros at $Q_1, \ldots, Q_l$ and $l(D) \neq 0$.
   Moreover, by Riemann–Roch inequality we have

$$l(D_1) \geq \deg D_1 - g + 1 = \deg D + \deg D_2 - g + 1.$$

   Combining this with the previous relation we have $\deg D \leq g - 1$. In full analogy using relation $l(C - D) = 0$ we obtain $\deg(C - D) \leq g - 1$ or, using Corollary 4, $\deg D \geq g - 1$.

So $\deg D = g - 1$ and the Riemann–Roch theorem is completely proved. □

3.4 Riemann–Hurwitz formula

Here we shall prove another useful fact — the *Riemann–Hurwitz formula* that allows to calculate the genus of algebraic curves. For this purpose we shall use Corollary 4 which shows that degree of an arbitrary divisor from canonical class is a purely topological thing: it depends only on the genus of algebraic curve.

THEOREM 9. (RIEMANN–HURWITZ) *Let an algebraic curve $\hat{\mathcal{L}}$ of genus $\hat{g}$ be an N-sheeted covering with the branch points $P_1, \ldots, P_n \in \mathcal{L}$ of degree $k_1, \ldots, k_n$ respectively of the curve $\mathcal{L}$ of genus g. Then*

$$2\hat{g} - 2 = N(2g - 2) + \sum_{j=1}^{n} k_j. \tag{37}$$

(The order k_j of the branch point P_j is defined as the number of the copies of the curve $\mathcal{L}$ which coalesce at this point minus 1).

Proof: The main idea of the proof is the following: starting with some holomorphic differential on $\mathcal{L}$ one can construct explicitly some holomorphic differential on $\hat{\mathcal{L}}$ and calculate the degree of its divisor.

So take on $\mathcal{L}$ some holomorphic differential $dU(P)$ (existence of dU implies that $g \geq 1$; case $g = 0$ may be considered analogously taking an arbitrary meromorphic differential on $\mathcal{L} = \bar{\mathbb{C}}$). The number of zeros of dU on $\mathcal{L}$ is equal to $2g - 2$. Now let's define the differential $d\hat{U}(P)$, $P \in \hat{\mathcal{L}}$ by

$$d\hat{U}(P) = dU(\pi(P))$$

where $\pi : \hat{\mathcal{L}} \to \mathcal{L}$ is the natural projection. So we simply choose $d\hat{U}(P)$ taking the same value on all the "sheets" (i.e. copies of $\mathcal{L}$) of $\hat{\mathcal{L}}$. Obviously $d\hat{U}(P)$ has on $\hat{\mathcal{L}}$ $N(2g-2)$ zeros corresponding to the zeros of $dU(P)$ on $\mathcal{L}$. Besides that, it has some additional zeros due to the different local parameters on $\mathcal{L}$ and $\hat{\mathcal{L}}$ in the neighbourhoods of the branch points $P_1, \ldots, P_n$.

Namely, if we define the local parameter on $\mathcal{L}$ in the neighbourhood of the point P_j by τ, then the local parameter at the same point on $\hat{\mathcal{L}}$ is equal to $\hat{\tau} = \sqrt[k_j+1]{\tau}$ ($k_j + 1$ is the number of copies of $\mathcal{L}$ glued at P_j) or, equivalently, $\tau = \hat{\tau}^{k_j+1}$. So if $dU(P)$ has in the neighbourhood of P_j on $\mathcal{L}$ the local representation $dU(\tau) = f(\tau)d\tau$ then for the differential $d\hat{U}(P)$ on $\hat{\mathcal{L}}$ we have $d\hat{U}(\hat{\tau}) = dU(\tau) = f(\tau)d\tau = f(\hat{\tau}^{k_j+1})(k_j+1)\hat{\tau}^{k_j}d\hat{\tau}$, i.e. P_j is the zero of $d\hat{U}$ of degree k_j. As a result, the number of zeros of differential $d\hat{U}$ on $\hat{\mathcal{L}}$ is equal to $N(2g-2) + \sum_{j=1}^{n} k_j = 2\hat{g} - 2$ where $\hat{g}$ is the genus of the curve $\hat{\mathcal{L}}$. □

For $g = 0$ we get the following

COROLLARY 5. *If curve $\mathcal{L}$ is N-sheeted covering of the Riemann sphere having n branch points of the order $k_1, \ldots, k_n$ then the genus of $\mathcal{L}$ is equal to*

$$g = \sum_{j=1}^{n} \frac{k_j}{2} - N + 1. \tag{38}$$

For example, we can calculate the genus of the hyperelliptic curve (6): in this case $N = 2$, $k_j = 1$, $j = 1, \ldots, 2g+2$. Substituting this into (38) we see that the genus is equal to g.

4 Theta-functions on Riemann surfaces

Here we present some facts about the one- and multi-dimensional theta functions on Riemann surfaces that provide the universal tool to explicitly construct the meromorphic functions and the functions with exponential essential singularities.

4.1 Definition and simplest properties

DEFINITION 10. Let B be symmetric $g \times g$ matrix with positively-defined imaginary part. Then the function

$$\Theta(z|B) = \sum_{m\in\mathbb{Z}^g} \exp\{\pi i\langle Bm, m\rangle + 2\pi i\langle z, m\rangle\} \tag{39}$$

where $z \in \mathbb{C}^g$ and $\langle .,. \rangle$ is the ordinary scalar product, is called *the g-dimensional theta-function.*

The convergence of (39) for every $z \in \mathbb{C}^g$ immediately follows from the condition that the matrix $\operatorname{Im} B$ is positively defined. The convergence is absolute and uniform on every compact domain in $\mathbb{C}^g$ and, therefore, $\Theta(z|B)$ considered as a function of z is holomorphic everywhere on $\mathbb{C}^g$.

If matrix B is fixed, we shall often use the brief notation $\Theta(z) \equiv \Theta(z|B)$.

Function (39) possesses important periodicity properties. To formulate them denote by $e_1, \ldots, e_g$ the standard basis in $\mathbb{C}^g$: $(e_j)_k = \delta_{jk}$ and by $f_1, \ldots, f_g$ the columns of matrix B: $f_j = Be_j, j = 1, \ldots, g$.

STATEMENT 2. *Theta-function (39) satisfies the following relations:*

$$\Theta(z + e_j) = \Theta(z) \tag{40}$$
$$\Theta(z + f_j) = e^{-\pi i B_{jj} - 2\pi i z_j}\Theta(z) \tag{41}$$

Proof: The relation (40) is obvious. Let's prove (41):

$$\begin{aligned}
\Theta(z+f_k) &= \sum_{m\in\mathbb{Z}^g} \exp\{\pi i\langle Bm, m\rangle + 2\pi i\langle m, z+f_k\rangle\} \\
&\overset{m=n-e_k}{=} \sum_{n\in\mathbb{Z}^g} \exp\{\pi i\langle B(n-e_k), n-e_k\rangle + 2\pi i\langle n-e_k, z+f_k\rangle\} \\
&= \sum_{n\in\mathbb{Z}^g} \exp\{\pi i\langle Bn, n\rangle - 2\pi i\langle Be_k, n\rangle + \pi i\langle Be_k, e_k\rangle \\
&\quad +2\pi i\langle n, z\rangle + 2\pi i\langle n, f_k\rangle - 2\pi i\langle e_k, z\rangle - 2\pi i\langle e_k, f_k\rangle\} \\
&= \exp(-\pi i\langle Be_k, e_k\rangle - 2\pi i\langle e_k, z\rangle)\Theta(z) \\
&= \exp(-\pi i B_{kk} - 2\pi i z_k)\Theta(z).
\end{aligned}$$

□

So the vectors e_k are the periods of function $\Theta(z)$; the vectors f_k are called the quasi-periods.

Now we immediately see that for arbitrary $N, M \in \mathbb{Z}^g$

$$\Theta(z + N + BM) = \exp\{-\pi i\langle BM, M\rangle - 2\pi i\langle M, z\rangle\}\Theta(z).$$

In fact properties (40), (41) almost completely define $\Theta(z)$ up to some nonessential transformations [3].

The theta-function (39) admits natural generalization to the theta-function with characteristics:

$$\Theta[\alpha,\beta](z|B) = \sum_{m\in\mathbb{Z}^g} \exp\{\pi i\langle B(m+\alpha),(m+\alpha)\rangle + 2\pi i\langle z+\beta, m+\alpha\rangle\} \tag{42}$$

or, equivalently,

$$\Theta[\alpha,\beta](z|B) = \exp\{\pi i\langle B\alpha,\alpha\rangle + 2\pi i\langle z+\beta,\alpha\rangle\}\Theta(z+\beta+B\alpha),$$

where $\alpha,\beta\in\mathbb{R}^g$.

The periodicity property of the theta-functions with characteristics is the following:

$$\begin{aligned}\Theta[\alpha,\beta](z+N+BM) &= \\ = \exp\{-\pi i\langle BM,M\rangle - 2\pi i\langle z,M\rangle &+ 2\pi i[\langle\alpha,N\rangle - \langle\beta,M\rangle]\}\Theta[\alpha,\beta](z).\end{aligned} \tag{43}$$

If vectors α and β consist of 0 and $\frac{1}{2}$ then the set $[\alpha,\beta]$ is called the half-period. Half-period $[\alpha,\beta]$ is called *even* if $4\langle\alpha,\beta\rangle \equiv 0 (\mathrm{mod}\ 2)$ and *odd* in the opposite case. For related theta-function we have the following

STATEMENT 3. *Function $\Theta[\alpha,\beta](z)$ with half-integer characteristics is even if half-period $[\alpha,\beta]$ is even and odd if half-period $[\alpha,\beta]$ is odd.*

Proof: Substituting in (42) $-z$ instead of z and changing the summation variable as $m\to -m-2\alpha$, we see that (42) multiplies by the factor $\exp\{4\pi i\langle\alpha,\beta\rangle\}$. □

As the simple corollary we see that $\Theta(z)=\Theta(-z)$.

Before to demonstrate how the meromorphic functions on the Riemann surfaces of an arbitrary genus can be constructed in terms of the multi-dimensional theta-functions let's consider the simplest case of genus 1.

4.2 Meromorphic functions on elliptic curves in terms of theta-functions

Four well-known elliptic Jacobi theta-functions have all possible sets of half-integer characteristics $[\alpha,\beta]$:

$$\begin{aligned} i\Theta_1(z) &\equiv \Theta\left[\frac{1}{2},\frac{1}{2}\right](z), \\ \Theta_2(z) &\equiv \Theta\left[\frac{1}{2},0\right](z), \\ \Theta_3(z) &\equiv \Theta\left[0,0\right](z) \equiv \Theta(z), \\ \Theta_4(z) &\equiv \Theta\left[0,\frac{1}{2}\right](z). \end{aligned}$$

All functions Θ_i are holomorphic on $\mathbb{C}$ (they are not on $\bar{\mathbb{C}}$, of course). Numbers 1 and B define the so-called parallelogram of the periods. Using the transformation property (43), it is easy to see that the set of zeros of every function Θ_i is invariant with respect to the shift along every period. Thus it is sufficient to find the zeros of Θ_i in the parallelogram of the periods Ω (Fig. 6).

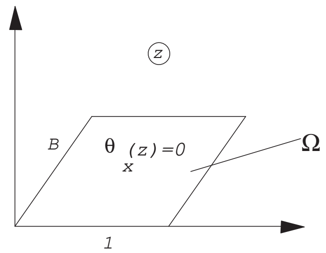


Fig. 6: Fundamental parallelogram Ω and position of the zero of $\theta(z|B)$ inside Ω.

From statement 1 it follows that function $\Theta_1(z)$ is odd and other Θ_i are even. So $\Theta_1(0) = 0$; the zeros of other Θ_i may be easily found from (43). In particular, we see that $\Theta\left(\frac{B}{2} + \frac{1}{2}\right) = 0$. It is easy to verify that this zero of $\Theta(z)$ in Ω is unique. Namely, consider the following integral along boundary $\partial\Omega$:

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{\partial\Omega}\frac{d\Theta(z)}{\Theta(z)} &= \frac{1}{2\pi i}\oint_{\partial\Omega} d\log\Theta(z) = \frac{1}{2\pi i}\int_0^1 [d\log\Theta(z) - d\log\Theta(z+B)] \\
&+\frac{1}{2\pi i}\int_0^B [d\log\Theta(z+1) - d\log\Theta(z)] \\
&= \frac{1}{2\pi i}\int_0^1 \{d\log\Theta(z) - d\log[e^{-\pi i B - 2\pi i z}\Theta(z)]\} \\
&= \frac{1}{2\pi i}\int_0^1 2\pi i dz = 1.
\end{aligned}$$

We have proved the following

STATEMENT 4. *The elliptic function $\Theta(z|B)$ has in the parallelogram Ω with sides* 1 *and B only one zero at the point $z = \frac{1}{2} + \frac{B}{2}$.*

As we shall see below, this property together with the transformation properties allows to express the meromorphic functions on torus in terms of $\Theta(z|B)$.

We shall skip here numerous remarkable properties of the Jacobi theta-functions such as summation theorems, Jacobi formula for the derivative and so on (see [8, 3, 6]). Notice only that it obeys the heat conductivity equation as the function of z and B:

$$4\pi i\frac{\partial\Theta(z|B)}{\partial B} = \frac{\partial^2\Theta(z|B)}{\partial z^2} \tag{44}$$

which may be easily verified by direct differentiation.

Now we are going to show how to construct functions on algebraic curve of genus 1 in terms of elliptic function $\Theta(z|B)$.

First we construct meromorphic functions on the abstract Riemann surface - the torus $T = \mathbb{C}/\{1, B\}$ (T is in fact the fundamental parallelogram Ω with identified opposite sides); later we shall clarify the relationship between torus T and an arbitrary elliptic curve.

Following [3] we can propose several ways to express meromorphic functions on T in terms of $\Theta(z|B)$:

1. Let's construct function $f(z)$ on T having zeros at the points $a_1, \ldots, a_n$ and poles at the points $b_1, \ldots, b_n$. Due to the Abel theorem we can not, in contrast to the Riemann sphere, take arbitrary values of a_j and b_j: we get one additional restriction given by (21). The unique holomorphic differential $dU(z)$ may be written on T in the very simple form: $dU(z) = dz$, so restriction (21) gives the following condition of existence of function $f(z)$:
$$a_1 + \cdots + a_n = b_1 + \cdots + b_n. \tag{45}$$
We again see that it is impossible to construct nontrivial function $f(z)$ having one pole and one zero: in this case $a_1 = b_1$ and $f(z) = const$.
For $n \geq 2$ the function $f(z)$ may be expressed in terms of $\Theta(z|B)$ as follows:
$$f(z) = C \prod_{j=1}^{n} \frac{\Theta(z - a_j - \frac{B+1}{2})}{\Theta(z - b_j - \frac{B+1}{2})}. \tag{46}$$
To verify that $f(z)$ is indeed single-valued on T we have to check that $f(z+1) = f(z)$ and $f(z + B) = f(z)$. The first relation is trivial; using the periodicity property (41) of the theta-function, it is easy to see that the second of them reduces to (45).
Formula (46) may be considered as the straightforward generalization of the following representation for the meromorphic (rational) function on $\bar{\mathbb{C}} \cong \mathbb{CP}^1$:
$$f(z) = \prod_{j=1}^{n} \frac{z - a_j}{z - b_j}. \tag{47}$$
The most important difference is that in (47) the positions of the poles and zeros are arbitrary.

2. Another way to construct meromorphic functions on T in terms of the theta-functions is the following. Consider the function $\log \Theta(z)$. It is obviously equal to the sum of some periodic function with periods $(1, B)$ and some linear function. So function $\frac{d^2}{dz^2} \log \Theta(z)$ is meromorphic and periodic, and, therefore, is a meromorphic function on T. Besides that, it has the double pole at $z = \frac{1+B}{2}$. This is nothing but the Weierstrass $\mathcal{P}$-function up to some constant:
$$\mathcal{P}(z) = -\frac{d^2}{dz^2} \log \Theta(z) + \text{const},$$

where const is chosen to kill the term of zero degree in Laurent series of $\mathcal{P}(z)$ at $z = 0$.

3. Finally, by slight modification of the previous method we can construct an arbitrary meromorphic function on T in the following form

$$f(z) = \sum_j \lambda_j \frac{d}{dz} \log \Theta(z - b_j) + \text{const}. \tag{48}$$

For the function (48) be single-valued on T, we have to impose the condition $\sum_j \lambda_j = 0$ that provides the condition of the Abel theorem in this case. Notice that in (48) we have no apparent information about the zeros of $f(z)$; instead we know the residues λ_j at the points b_j.

Representation (48) is the analog of the representation of meromorphic function on $\mathbb{CP}^1$ as the sum of the simple fractions:

$$f(z) = \text{const} + \sum_j \frac{\lambda_j}{z - b_j}$$

Now to be able to construct meromorphic functions on an arbitrary elliptic curve one should be able to establish an isomorphism between the algebraic curve and the fundamental parallelogram with some B.

This isomorphism is defined by the Abel map

$$U(P) = \int_{P_0}^{P} dU, \tag{49}$$

where dU is unique normalized holomorphic differential on $\mathcal{L}$; the first period of the Jacobi torus is equal to $\oint_a dU = 1$ and the second period B should coincide with the second period of the torus T. It may be easily verified that in the case $g = 1$ the correspondence between $\mathcal{L}$ and $J(\mathcal{L})$ is an the isomorphism (for $g > 1$ this is, of course, not true, because $\dim J(\mathcal{L}) = g$). The direct correspondence $\mathcal{L} \to J(\mathcal{L})$ is given by the Abel map (49); the inverse is defined by the Weierstrass $\mathcal{P}$-function [8, 11].

Now being able to construct meromorphic functions on $J(\mathcal{L})$ by the methods 1–3 we can easily construct meromorphic functions on $\mathcal{L}$.

For example, according to the method **1.**, the function $f(P)$ on $\mathcal{L}$ having poles at points $P_1, \ldots, P_n$ and zeros at $Q_1, \ldots, Q_n$ related by the Abel theorem

$$\sum_{j=1}^{n} U(P_j) = \sum_{j=1}^{n} U(Q_j) \tag{50}$$

may be defined by the following expression:

$$f(P) = C \prod_{j=1}^{n} \frac{\Theta(U(P) - U(Q_j) - \frac{B+1}{2})}{\Theta(U(P) - U(P_j) - \frac{B+1}{2})}.$$

Certainly we can not usually explicitly resolve condition (50). However, it is not always necessary to know *all* the poles and zeros of $f(P)$ on $\mathcal{L}$.

For example, function having on $\mathcal{L}$ 2 poles at $P_{1,2}$, one zero at Q_1 and one zero somewhere also (we can not find it explicitly) may be expressed as follows:

$$f(P) = \frac{\Theta(U(P) - U(Q_1) - \frac{B+1}{2})\Theta(U(P) + U(Q_1) - U(P_1) - U(P_2) - \frac{B+1}{2})}{\Theta(U(P) - U(P_1) - \frac{B+1}{2})\Theta(U(P) - U(P_2) - \frac{B+1}{2})}.$$

The position of the second zero is given by the Abel theorem (50).

4.3 Meromorphic functions on algebraic curves of arbitrary genus in terms of theta-functions

To extend the results of the previous paragraph to the curves of an arbitrary genus we have to prove some facts about the zeros of the multidimensional theta-functions. Consider the function

$$F(P) = \Theta(U(P) - d|B), \tag{51}$$

where $U(P)$ is the Abel map on $\mathcal{L}$; B is the matrix of b-periods and $d \in \mathbb{C}^g$ is some fixed vector. Of course, $F(P)$ is nonsingle-valued on $\mathcal{L}$; so consider it on the fundamental polygon $\tilde{\mathcal{L}}$ where it is single-valued and holomorphic.

LEMMA 3. *If $F(P)$ is not identically zero then it has on $\tilde{\mathcal{L}}$ g zeros (taking into account their order).*

Proof: We have to calculate the integral $(*) = \frac{1}{2\pi i}\oint_{\partial\tilde{\mathcal{L}}} d\log F(P)$.

Let's use the representation (9) for $\partial\tilde{\mathcal{L}}$. Denote the value of $F(P)$ on a_j^- and b_j^- by F^- and on a_j^+ and b_j^+ - by F^+; the same for the Abel map $U(P)$. Then

$$(*) = \frac{1}{2\pi i}\sum_{k=1}^{g}\left(\oint_{a_k} + \oint_{b_k}\right)[d\log F^+ - d\log F^-]. \tag{52}$$

Notice that if $P \in a_k$ then $U_j^-(P) = U_j^+(P) + B_{jk}$ and if $P \in b_k$ then $U_j^+(P) = U_j^-(P) + \delta_{jk}$. Using the periodicity properties of theta-function (40), (41), we see that on a_k

$$d\log F^-(P) = d\log F^+(P) - 2\pi i dU_k(P)$$

and on b_k

$$d\log F^-(P) = d\log F^+(P).$$

Therefore integral (52) may be written as

$$(*) = \frac{1}{2\pi i}\sum_k \int_{a_k} 2\pi i dU_k(P) = g.$$

□

The information about the positions of the zeros of $F(P)$ on $\tilde{\mathcal{L}}$ is given by the following

LEMMA 4. *Assume that $F(P)$ is not identically zero on $\tilde{\mathcal{L}}$ and denote by $P_1, \ldots, P_g$ the positions of its zeros. Then on the Jacobi manifold $J(\mathcal{L})$ we have*

$$\sum_{k=1}^{g} U(P_k) \equiv d - K \pmod{I, B},$$

where K is the vector of the Riemann constants:

$$K_j = \frac{1+B_{jj}}{2} - \sum_{k \neq j} \oint_{a_k} U_j(P) dU_k(P), \qquad j = 1, \ldots, g. \tag{53}$$

Proof: Consider the integral $I_j = \oint_{\partial \tilde{\mathcal{L}}} U_j(P) d\log F(P)$, $j = 1, \ldots, g$.
From the residue theorem we have $I_j = \sum_{k=1}^{g} U_j(P_k)$.
Now calculate I_j in analogy to the previous lemma

$$\begin{aligned}
I_j &= \frac{1}{2\pi i} \sum_{k=1}^{g} \left(\oint_{a_k} + \oint_{b_k} \right) [U_j^+ d\log F^+ - U_j^- d\log F^-] \\
&= \frac{1}{2\pi i} \sum_{k=1}^{g} \oint_{a_k} [U_j^+ d\log F^+ - (U_j^+ + B_{jk})(d\log F^+ - 2\pi i dU_k)] \\
&\quad + \frac{1}{2\pi i} \sum_{k=1}^{g} \oint_{b_k} [U_j^+ d\log F^+ - (U_j^+ - \delta_{jk}) d\log F^+] \\
&= \sum_{k=1}^{g} \left(\oint_{a_k} U_j^+ dU_k - \frac{B_{jk}}{2\pi i} \oint_{a_k} d\log F^+ + B_{jk} \right) + \frac{1}{2\pi i} \oint_{b_j} d\log F^+.
\end{aligned}$$

Consider these integrals separately:

$$\oint_{a_k} d\log F^+ = 2\pi i n_k \ , \quad n_k \in \mathbb{Z}$$

since function F^+ takes the same values at the different ends of a_k.
Let Q_j and $\tilde{Q}_j$ be the beginning and the end of b_j. Then

$$\begin{aligned}
\oint_{b_j} d\log F^+ &= \log F^+(\tilde{Q}_j) - \log F^+(Q_j) + 2\pi i m_j \\
&= \log \Theta(U(Q_j) + f_j - d) - \log \Theta(U(Q_j) - d) + 2\pi i m_j \\
&= -\pi i B_{jj} + 2\pi i d_j - 2\pi i U_j(Q_j) + 2\pi i m_j, \qquad m_j \in \mathbb{Z},
\end{aligned}$$

vector f_j is the same as in (41).
As a result we have (module the lattice periods)

$$I_j \equiv d_j - \frac{B_{jj}}{2} - U_j(Q_j) + \sum_{k=1}^{g} \oint_{a_k} U_j(P) dU_k(P).$$

Denoting the beginning of contour a_j by R_j (its end coincides with Q_j), we have

$$\begin{aligned}
\oint_{a_j} U_j(P)dU_j(P) - U_j(Q_j) &= \frac{1}{2}[U_j^2(Q_j) - U_j^2(R_j)] - U_j(Q_j) \\
&= \frac{1}{2}[(U_j(R_j)+1)^2 - U_j^2(R_j)] - U_j(R_j) - 1 \\
&= -\frac{1}{2}.
\end{aligned}$$

Finally,

$$I_j = \sum_{k=1}^{g} U_j(P_k) = d_j - \frac{B_{jj}+1}{2} + \sum_{k=1,\, k\neq j}^{g} \oint_{a_k} U_j(P)dU_k(P).$$

□

Notice that the vector K is in fact not a "constant" — it depends on the choice of the initial point of the Abel map P_0. This dependence disappears only for $g=1$. In this case we have

$$K = \frac{B+1}{2}.$$

It is possible to prove [5] the following simple link between vector K and the divisors from the canonical class C (i.e. the divisors of meromorphic differentials): $2K \equiv -U(C)$.

Expression (53) may be considerably simplified [4] in the case of the hyperelliptic curves if the point P_0 is chosen in some special way:

LEMMA 5. *Let $\mathcal{L}$ be hyperelliptic curve of genus g defined by the equation (6) and $\sigma : \mathcal{L} \to \mathcal{L}$ be the involution on $\mathcal{L}$ interchanging the sheets. Let also the canonical basis of cycles (a_j, b_j) be chosen in such a way that $\sigma(a_j) = -a_j$ and $\sigma(b_j) = -b_j$ (Fig. 7). Then choosing the point P_0 to coincide with E_1 we can express the vector of the Riemann constants as follows:*

$$K_j = \sum_{k=1}^{g} B_{jk} + \frac{j}{2}. \tag{54}$$

Proof: Existence of the involution σ allows to rewrite expression (53) as follows:

$$\begin{aligned}
K_j &= \frac{B_{jj}+1}{2} + \sum_{k\neq j} \oint_{a_k} \left[\int_{P_0}^{P} dU_j\right] dU_k \\
&= \frac{B_{jj}+1}{2} + \sum_{k\neq j} \oint_{a_k} \left[\int_{E_1}^{E_{2k+1}} dU_j + \int_{E_{2k+1}}^{P} dU_j\right] dU_k(P)
\end{aligned}$$

$$= \frac{B_{jj}+1}{2} + \sum_{k\neq j} \int_{E_1}^{E_{2k+1}} dU_j \oint_{a_k} dU_k$$
$$+ \sum_{k\neq j} \int_{E_{2k+1}}^{E_{2k+2}} \left(\left[\int_{E_{2k+1}}^{P} dU_j \right] dU_k(P) - \left[\int_{E_{2k+1}}^{\sigma P} \sigma^* dU_j \right] dU_k(\sigma P) \right).$$

From the relation $dU_j(\sigma P) = -dU_j(P)$ that immediately follows from the behaviour of basic cycles under the involution σ we see that the second sum is equal to zero; the first sum gives (54) module the linear combination $M + BN$. □

So we know the positions of the zeros of the theta-function on $\mathcal{L}$ if it is not identically zero. At this point we refer to the following theorem (see [3, 2]):

THEOREM 10. *Function $\Theta(U(P) - d)$ identically vanishes on $\mathcal{L}$ iff vector d may be represented in the form $d \equiv U(Q_1) + \cdots + U(Q_g) + K$, where $Q_1 + \cdots + Q_g$ is some special divisor.*

As a result we get from Lemma 4 and Theorem 10 the following

THEOREM 11. *If $D = P_1 + \cdots + P_g$ is a nonspecial divisor on algebraic curve $\mathcal{L}$ of genus g then the function $F(P) = \Theta(U(P) - U(D) - K)$ has on $\tilde{\mathcal{L}}$ g zeros at points $P_1, \ldots P_g$.*

Notice also the following important role of the zeros of theta-function. In the case $g = 1$ we have the one-to-one correspondence between the curve $\mathcal{L}$ and its Jacobi torus $J(\mathcal{L})$ given in different directions by the elliptic integral $U(P)$ and by Weierstrass $\mathcal{P}$-function. For $g > 1$ $\mathcal{L}$ and $J(\mathcal{L})$ have different complex dimensions and it is natural to consider a map between $\mathcal{L}^g$ and $J(\mathcal{L})$. In analogy to the elliptic case we can define the Abel map

$$U : \mathcal{L}^g \to J(\mathcal{L}) \tag{55}$$

that assigns to every positive divisor of degree g some point of $J(\mathcal{L})$. The reasonable question here is how to construct (if possible) the inverse map i.e. to solve the equation $U(D) \equiv \zeta$, where $\zeta \in \mathbb{C}^g$ is some point on $J(\mathcal{L})$; we want to find the divisor $D = P_1 + \cdots + P_g$ in terms of ζ. This is the formulation of the Abel inversion problem.

Theorems 10 and 11 provide the answer to this question: if $\Theta(U(P) - \zeta - K)$ does not vanish identically on $\mathcal{L}$ then its zeros coincide with the points $P_1, \ldots, P_g$ which solve the Abel inversion problem. We shall not consider in details the particular case $\Theta(U(P) - \zeta - K) \equiv 0$; in this case according to Theorem 11 $\zeta = U(\tilde{D})$ where divisor $\tilde{D}$ is special. It is easy to prove [11, 3] that all these points ζ may be represented as $\zeta = U(Q_1 + \cdots + Q_{g-1})$, where Q_i, $i = 1, \ldots, g-1$ are arbitrary points of $\mathcal{L}$. Taking into account that the special divisors constitute the submanifold in $\mathcal{L}^g$ of complex dimension $g - 1$, it is possible to claim that the Jacobi inversion problem may be solved for any $\zeta \in J(\mathcal{L})$ and the Abel map gives the one-to-one correspondence between $\mathcal{L}^g$ and $J(\mathcal{L})$ everywhere except the special divisors on $\mathcal{L}$.

Now we are in position to achieve our last goal — to demonstrate how we can construct the meromorphic functions on $\mathcal{L}$ in terms of the theta-functions. Here, in analogy to the elliptic case, there are many possibilities. For illustration let's construct function f having poles in prescribed $g+1$ points $P_1, \ldots, P_{g+1}$ (in general case, when divisor $P_1+\cdots+P_{g+1}$ is nonspecial, this function is unique up to a linear transformation).

LEMMA 6. *Let $P_1+\cdots+P_{g+1}$ be a nonspecial divisor on the curve $\mathcal{L}$ of genus g, $Q \in \mathcal{L}$ and W_{QP_j} be the normalized (all a-periods are zero) differential of the 3rd kind. Then the function*

$$f(P) = \frac{\Theta(U(P) - \sum_{j=1}^{g+1} U(P_j) + U(Q) - K)}{\Theta(U(P) - \sum_{j=1}^{g} U(P_j) - K)} \exp(W_{QP_{g+1}}), \tag{56}$$

is meromorphic on $\mathcal{L}$, has poles at $P_1, \ldots, P_{g+1}$ and zero at $P = Q$ (other zeros are the zeros of the theta-function in the numerator).

Proof: The fact that the expression (56) has poles at $P_1, \ldots P_g$ is obvious: g poles come from the denominator (here we use the assumption that the divisor D is nonspecial) and pole P_{g+1} comes from the exponential factor; this factor gives also the zero at Q. The only fact one should check is that function $f(P)$ is single-valued on $\mathcal{L}$ i.e. that it is invariant with respect to the tracing of the point P around any of the basic cycles. The invariance of f with respect to the tracing around a-cycles is obvious due to the normalization of the differential $dW_{QP_{g+1}}$ and the periodicity of the theta-function (40). The invariance with respect to the tracing around b-cycles follows from the expression (18) for the vector of b-periods of the integral of the third kind and the transformation property of the theta-function (41). □

Analogously we can construct meromorphic functions with more poles. It is also possible to express arbitrary differentials of the third and second kind in terms of the theta-functions through the so-called prime form — the special differential of the order $(-\frac{1}{2}, -\frac{1}{2})$ on $\mathcal{L} \times \mathcal{L}$ (see [3, 10])

Another opportunity is to insert in the exponent of (56) the normalized integral of the 2nd kind instead of $dW_{QP_{g+1}}$ (adding simultaneously its b-period vector divided by $2\pi i$ in the argument of the theta-function in the numerator). Then we get the function having on $\mathcal{L}$ g poles and an essential singularity of the exponential kind. The functions of this sort with one or more essential singularities (so-called Baker–Akhiezer functions) arise in applications to KdV-like soliton equations. The construction of the functions of this kind in terms of the theta-functions is considered in details in [10]. The pure meromorphic functions arise in the applications, for example, to the Ernst equations [13].

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