Symmetries and recursion operators for soliton equations

I. S. Krasil'shchik

1st Tverskoy-Yamskoy per. 14, Apt.45, 125047 Moscow, Russia
e-mail: josephk@glasnet.ru

Abstract: The theory of symmetries for nonlinear differential equations is exposed using naïve coordinate language. As an example, the Burgers equation is considered. Methods for computing recursion operators are given and nonlocal problems arising in this context are discussed.

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Introduction

It is known that all soliton equations possess several structures which make of them very interesting objects for mathematical investigations. These structures are:

- infinite series of commuting symmetries and conservation laws,
- Hamiltonian (and, usually, bi-Hamiltonian) maps,
- recursion operators.

The first equation which was found to possess all these properties was the celebrated Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x. \tag{1}$$

It was shown, in particular, that soliton solutions of (1) were invariant (with respect to higher symmetries) solutions. Later similar results were obtained for other soliton equations.

In fact, geometrical theory of differential equations[1] originates from classical works by S. Lie and A. Bäcklund and much later was revised and put into the modern differential geometry framework in [3, 5, 6].

In this paper we do not intend to expose the general theory of symmetries for differential equations, but try to clarify practical aspects of computations. As an example, we choose the Burgers equation

$$u_t = u_{xx} + uu_x \tag{2}$$

for which the whole conceptual scheme looks quite transparent. Besides symmetries, we explain how to compute *recursion operators* using the methods developed in [1, 2]. Recursion operators are a very powerful instrument for symmetry computation, but as a rule they need the use of *nonlocal variables*. Hence, we discuss the latter ones as well.

[1] Of which the theory of symmetries is a natural component.

1 Symmetries of differential equations

Let us analyze the notion of a symmetry and deduce efficient formulae for computations.

What is a symmetry? Consider a differential equation[2] $\mathcal{E}$ given by

$$F\left(x, u, \dots, \frac{\partial^{|\sigma|}u}{\partial x^\sigma}, \dots\right) = 0, \tag{3}$$

where $x = (x_1, \dots, x_n)$ are independent variables, $u = (u^1, \dots, u^m)$ are unknown functions (dependent variables) and $\frac{\partial^{|\sigma|}u}{\partial x^\sigma} = \frac{\partial^{|\sigma|}u}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$ for $\sigma = (i_1, \dots, i_n)$, $|\sigma| = i_1 + \dots + i_n \le k$, k being the order of $\mathcal{E}$.

Let

$$\frac{\partial u}{\partial \tau} = \Phi\left(x, u, \dots, \frac{\partial^{|\sigma|}u}{\partial x^\sigma}, \dots\right) \tag{4}$$

be an evolution equation and suppose (3) and (4) to be compatible. Let $u = u(x)$ be a solution of (3) and $u_\tau = u(x, \tau)$ be a solution of (4) and (3) with initial data $u_0 = u$. Then $\{u_\tau\}$ is a curve in the space of solutions of equation (3). Taking an arbitrary solution[3] of (3), we get a correspondence $\Phi_\tau : u \mapsto u_\tau$. Hence, evolution equation (4) determines a one-parameter family of transformations on the solution space of (3). Thus we can give a preliminary

DEFINITION 1. *Symmetry* is an evolution equation which preserves solutions.

Of course, this is not an operational definition: the concept of a solution space, as well as of an action on this space, remains rather vague. To make them exact, we should answer the next question:

What is a differential equation? Groups of transformations arise in differential geometry as maps of main objects (curves, surfaces, etc.) preserving basic structures (e.g., Riemannian metrics). Thus, if one is going to use differential geometry techniques in application to differential equations, one should put these equations into appropriate conceptual framework. This is being done as follows.

Consider the space $J^k(n, m)$ with coordinates

$$(x_1, \dots, x_n, u^1, \dots, u^m, \dots, u^1_\sigma, \dots, u^m_\sigma, \dots),$$

where, as above, $\sigma = (i_1, \dots, i_n)$ and $|\sigma| \le k$. This space is called *the space of jets* of the order k. Then an equation $\mathcal{E}$ of the form (3) can be understood as a subset in $J^k(n, m)$ defined by relations

$$F^1(x, u, \dots, u_\sigma, \dots) = 0, \dots, F^r(x, u, \dots, u_\sigma, \dots) = 0.$$

[2]When we say *a differential equation* we mean in fact *a system* of equations and thus F in (3) is a vector function $F = (F^1, \dots, F^r)$.

[3]We assume the initial value problem for (4) to be uniquely solvable.

If all F^i are smooth functions and the system $F^1, \dots, F^r$ is of the maximal rank, then this subset is a smooth[4] submanifold of the codimension r. Thus, *differential equations are smooth submanifolds in jet spaces.*

Let $f = (f^1(x), \dots, f^m(x))$ be smooth vector function. Then f determines an n-dimensional submanifold in $J^k(n,m)$ defined by

$$u^j_\sigma = \frac{\partial^{|\sigma|} f^j}{\partial x^\sigma}, \quad j = 1, \dots, m, \quad |\sigma| \le k,$$

(here and below we formally assume $u_{(0,\dots,0)} = u$). This submanifold is called the k-jet of f and is denoted by $j_k(f)$. It easily seen that f satisfies equation (3) if and only if $j_k(f)$ lies in corresponding submanifold in $J^k(n,m)$. In other words, *solutions of $\mathcal{E}$ are identified with jets lying in the equation manifold.* Hence, the following definition is justified:

DEFINITION 2. Let $\mathcal{E} \subset J^k(n,m)$ be a differential equation. A *symmetry of $\mathcal{E}$* is a diffeomorphism $\varphi : J^k(n,m) \to J^k(n,m)$, such that (i) $\varphi(\mathcal{E}) = \mathcal{E}$ and (ii) $\varphi(j_k(f))$ is of the form $j_k(f_\varphi)$ for any smooth function f.

This definition, being perfectly correct, can not be considered as a working one though: to compute symmetries one needs to know solutions, and it does not make sense!

Infinite prolongations and the Cartan distribution. To overcome this problem, first note the following. Let $f(x)$ be a solution of (3). Differentiate $F^j(f(x))$ along x_α:

$$\begin{aligned}\frac{\partial F^j(f(x))}{\partial x_\alpha} \quad = \quad & \frac{\partial F^j}{\partial x_\alpha}(f(x)) + \frac{\partial F^j}{\partial u^1}(f(x))\frac{\partial f^1}{\partial x_\alpha} + \dots + \frac{\partial F^j}{\partial u^m}(f(x))\frac{\partial f^m}{\partial x_\alpha} + \dots \\ & \dots + \frac{\partial F^j}{\partial u^1_\sigma}(f(x))\frac{\partial^{|\sigma|+1} f^1}{\partial x^{\sigma+1_\alpha}} + \dots + \frac{\partial F^j}{\partial u^m_\sigma}(f(x))\frac{\partial^{|\sigma|+1} f^m}{\partial x^{\sigma+1_\alpha}} + \dots\end{aligned}$$

where $\sigma + 1_\alpha \stackrel{\text{def}}{=} (i_1, \dots, i_\alpha + 1, \dots, i_n)$. Define operators

$$D^{[k]}_\alpha = \frac{\partial}{\partial x_\alpha} + \sum_{j=1}^m \sum_{|\sigma| \le k} u^j_{\sigma+1_\alpha} \frac{\partial}{\partial u^j_\sigma} \tag{5}$$

Then one can easily see that the system

$$F = 0, \quad D^{[k]}_1(F) = 0, \quad \dots, \quad D^{[k]}_n(F) = 0 \tag{6}$$

possesses the same solutions as the initial equation (3). If $\mathcal{E} \subset J^k(n,m)$ is the submanifold corresponding to (3), then a submanifold in $J^{k+1}(n,m)$ corresponds to (6). We denote it by $\mathcal{E}^1$ and call *the first prolongation* of $\mathcal{E}$.

Now, we define by induction $\mathcal{E}^{l+1} = (\mathcal{E}^l)^1$ and see that the set $\mathcal{E}^l$ lies in $J^{k+l}(n,m)$; we call $\mathcal{E}^l$ the l-th prolongation of $\mathcal{E}$. It is reasonable to consider all prolongations of $\mathcal{E}$ simultaneously. What geometrical object does correspond to this construction?

[4] By *smooth* we always mean of class C^∞.

Let $J^\infty(n,m)$ be the space with coordinates

$$(x_1,\dots,x_n,u^1,\dots,u^m,\dots,u^1_\sigma,\dots,u^m_\sigma,\dots),$$

where $|\sigma|$ is unlimited. This space is called the space of infinite jets and the whole series of prolongations of the equation $\mathcal{E}$ determines a submanifold $\mathcal{E}^\infty$ in $J^\infty(n,m)$ which is called *the infinite prolongation* of the equation $\mathcal{E}$. Infinite prolongations are expressed by relations

$$D_\sigma(F)=0,\qquad |\sigma|\geq 0,$$

where $D_\sigma = D_1^{i_1}\circ\dots\circ D_n^{i_n}$ and

$$D_\alpha = \frac{\partial}{\partial x_\alpha}+\sum_{j,\sigma} u^j_{\sigma+1_\alpha}\frac{\partial}{\partial u^j_\sigma} \tag{7}$$

are infinite counterparts of (5) and are called *total derivatives.*

Expressions of the form (7) may be viewed at as vector fields on the space $J^\infty(n,m)$. Thus, at any point $\theta\in J^\infty(n,m)$ an n-dimensional plane $\mathcal{C}_\theta$ arises spanned by vectors of these fields. It means that there exists a distribution $\mathcal{C}:\theta\mapsto\mathcal{C}_\theta$ on $J^\infty(n,m)$. We call it *the Cartan distribution.* Its importance to the theory of differential equations is explained by the following

THEOREM 1. [3] (i) *If $\mathcal{E}^\infty\subset J^\infty(n,m)$ is an infinite prolongation, then $\mathcal{C}_\theta$ is tangent to $\mathcal{E}^\infty$ at any point $\theta\in\mathcal{E}^\infty$.*
(ii) *A submanifold in $\mathcal{E}^\infty$ is a solution of $\mathcal{E}$ if and only if it is an integrable manifold of the Cartan distribution.*

Hence, $\mathcal{E}^\infty$ carries the structure which completely determines solutions of $\mathcal{E}$, and this structure is the Cartan distribution.

DEFINITION 3. **(final)** A diffeomorphism $\varphi:\mathcal{E}^\infty\to\mathcal{E}^\infty$ is called *a (higher) symmetry of $\mathcal{E}$,* if it preserves the Cartan distribution, i.e. if $\varphi_*(\mathcal{C}_\theta)=\mathcal{C}_{\varphi(\theta)}$ for any $\theta\in\mathcal{E}^\infty$.

This definition is in a complete agreement with the general geometrical approach. The only problem consists in efficient computation of symmetries. And in fact, everything becomes efficient, if one chooses

Infinitesimal approach. Let M be a smooth manifold and $\mathcal{D}$ be distribution on M. Suppose that $\mathcal{D}$ is spanned by vector fields $X_1,\dots,X_n$. Assume that a one-parameter group of symmetries $\{A_\tau\}$ of the distribution $\mathcal{D}$ is given, i.e. $(A_\tau)_*\mathcal{D}_\theta=\mathcal{D}_{A_\tau(\theta)}$ for any point $\theta\in M$. It is equivalent to the following system of equations

$$(A_\tau)_*(X_\alpha)=\lambda^1_\alpha X_1+\dots+\lambda^n_\alpha X_n,\quad \alpha=1,\dots,n, \tag{8}$$

where λ^β_α are depending on τ smooth functions on M with $\det\|\lambda^\beta_\alpha\|\neq 0$. Let X be the vector field corresponding to $\{A_\tau\}$, i.e. $\frac{dA_\tau}{d\tau}\big|_{\tau=0}=X$. Differentiating (8) with respect to τ at $\tau=0$ one obtains

$$[X,X_\alpha]=\mu^1_\alpha X_1+\dots+\mu^n_\alpha X_n,\quad \alpha=1,\dots,n, \tag{9}$$

where μ^β_α are smooth functions on M.

DEFINITION 4. A vector field X satisfying (9) is called *an infinitesimal symmetry of the distribution* $\mathcal{D}$.

Symmetries of $\mathcal{D}$ form a Lie algebra with respect to the commutator of vector fields: if X and Y satisfy (9), then $[X, Y]$ satisfies it as well. Denote this algebra by $s(\mathcal{D})$.

Note now that any vector field X lying in $\mathcal{D}$ is a symmetry of $\mathcal{D}$ and, by definition, is tangent to any integral manifold of this distribution. It means, if A_τ is a one-parameter group corresponding to X, then any transformation A_τ moves an integrable manifold of $\mathcal{D}$ along itself and thus trivially acts on the space of integral manifolds as a whole. In this sense, such symmetries are trivial. Respectively, fields lying in $\mathcal{D}$ are trivial infinitesimal symmetries and we denote the set of these symmetries by $s_0(\mathcal{D})$.

Obviously, $s_0(\mathcal{D})$ is an ideal in the Lie algebra $s(\mathcal{D})$: if $X \in s(\mathcal{D})$ and $Y \in s_0(\mathcal{D})$, then $[X, Y] \in s_0(\mathcal{D})$. It means that one can consider the quotient Lie algebra

$$\operatorname{sym}(\mathcal{D}) \stackrel{\text{def}}{=} s(\mathcal{D})/s_0(\mathcal{D}),$$

and its elements *are naturally identified with nontrivial symmetries* of $\mathcal{D}$.

We apply now this construction to the Cartan distribution $\mathcal{C}$.

How to compute infinitesimal symmetries? Consider the space of infinite jets $J^\infty(n, m)$ and the infinite prolongation $\mathcal{E}^\infty$ of equation $\mathcal{E} \subset J^k(n, m)$. Let $\mathcal{C}$ be the Cartan distribution on $J^\infty(n, m)$ and $\mathcal{C}_\mathcal{E}$ be its restriction onto $\mathcal{E}^\infty$.

DEFINITION 5. Elements of $\operatorname{sym}(\mathcal{C})$ are called *infinitesimal symmetries of the infinite jet space* and their set is denoted by $\operatorname{sym}_{n,m}$. Elements of $\operatorname{sym}(\mathcal{C}_\mathcal{E})$ are called *infinitesimal symmetries of the equation* $\mathcal{E}$ and their set is denoted by $\operatorname{sym}(\mathcal{E})$.

Both $\operatorname{sym}(\mathcal{C}_\mathcal{E})$ and $\operatorname{sym}(\mathcal{E})$ are Lie algebras.

In the sequel we omit the adjective "infinitesimal", since from now on we work with infinitesimal symmetries only.

In principal, there exist two approaches to computation of symmetries of equation $\mathcal{E}$. The first, exterior, one consists of finding those symmetries $X \in \operatorname{sym}_{n,m}$ which are tangent to $\mathcal{E}^\infty$. The second, interior, is the direct computation of $\operatorname{sym}(\mathcal{E})$. While the former seems to be preferable from computational viewpoint, the latter is, at first glance, more justified. But, as it happens, these approaches are equivalent.

PROPOSITION 1. [3] *Any symmetry* $X \in \operatorname{sym}(\mathcal{E})$ *is a restriction onto* $\mathcal{E}$ *of some symmetry* $X' \in \operatorname{sym}_{n,m}$.

Thus we can start with description of $\operatorname{sym}_{n,m}$.

First note the following. Any vector field on $J^\infty(m, n)$ is represented as

$$X = \sum_\alpha A_\alpha \frac{\partial}{\partial x_\alpha} + \sum_{\beta,\sigma} B^\beta_\sigma \frac{\partial}{\partial u^\beta_\sigma}, \tag{10}$$

where A_α, B^β_σ are smooth functions on $J^\infty(n, m)$. We say that the field X is *vertical*, if it vanishes on functions depending on x only. Hence, vertical fields are of the form $\sum_{\beta,\sigma} B^\beta_\sigma \frac{\partial}{\partial u^\beta_\sigma}$. Consider an arbitrary field (10). Then the field $X - \sum_\alpha X_\alpha D_\alpha$, where D_α, $\alpha = 1, \ldots, n$ are total derivatives (7), is a vertical field. Hence, the algebra $D(J^\infty(n,m))$ of all vector fields on $J^\infty(n,m)$ splits into the direct sum

$$D(J^\infty(n,m)) = D^v(J^\infty(n,m)) \oplus s_0(\mathcal{C})$$

$D^v(J^\infty(n,m))$ being the algebra of vertical fields. This presentation induces a splitting of the algebra $s(\mathcal{C})$ and we obtain

$$s(\mathcal{C}) = \operatorname*{sym}_{n,m} \oplus s_0(\mathcal{C}). \tag{11}$$

From the latter it easily follows

PROPOSITION 2. *Elements of* $\operatorname{sym}_{n,m}$ *are identified with vertical vector fields* X *such that*

$$[X, D_\alpha] = 0 \tag{12}$$

for all $\alpha = 1, \ldots, n$.

Using this result, let us obtain an explicit description of symmetries on $J^\infty(n,m)$. Let $X = \sum_{\beta,\sigma} B^\beta_\sigma \frac{\partial}{\partial u^\beta_\sigma}$ be a vertical field. Then

$$\begin{aligned}
[X, D_\alpha] &= \left[\sum_{\beta,\sigma} B^\beta_\sigma \frac{\partial}{\partial u^\beta_\sigma}, \frac{\partial}{\partial x_\alpha} + \sum_{\beta,\sigma} u^\beta_{\sigma+1_\alpha} \frac{\partial}{\partial u^\beta_\sigma}\right] \\
&= \sum_{\beta,\sigma} B^\beta_{\sigma+1_\alpha} \frac{\partial}{\partial u^\beta_\sigma} - \sum_{\beta,\sigma} D_\alpha(B^\beta_\sigma) \frac{\partial}{\partial u^\beta_\sigma},
\end{aligned}$$

from where one has $B^\beta_{\sigma+1_\alpha} = D_\alpha(B^\beta_\sigma)$, or $B^\beta_\sigma = D_\sigma(B^\beta_{(0,\ldots,0)})$.

Let $B^\beta \stackrel{\text{def}}{=} B^\beta_{(0,\ldots,0)}$.

THEOREM 2. [3] *Any symmetry* $X \in \operatorname{sym}_{n,m}$ *is of the form*

$$\Theta_B = \sum_{\beta,\sigma} D_\sigma(B^j) \frac{\partial}{\partial u^\beta_\sigma}, \tag{13}$$

where $B = (B^1, \ldots, B^m)$, B^β *being arbitrary functions on* $J^\infty(n,m)$.

DEFINITION 6. Vector fields Θ_B are called *evolution fields,* B being called *the generating function.*

REMARK 1. Suppose, a field Θ_B is integrable[5], i.e. possesses a trajectory in $J^\infty(n,m)$. Then evolution of u^β_σ along these trajectories would be governed by equations

$$\frac{\partial u_\sigma}{\partial \tau} = D_\sigma(B)$$

[5]Contrary to the finite dimensional case, this is not usually true for infinite dimensional manifolds $J^\infty(n,m)$ or $\mathcal{E}^\infty$.

and, in particular,

$$\frac{\partial u}{\partial \tau} = B(x, \ldots, u_\sigma, \ldots),$$

which explains the name "evolution" for fields Θ_B and is in complete agreement with the naïve Definition 1!

Recall that $\mathrm{sym}_{n,m}$ is a Lie algebra, i.e. is closed under the commutator of vector fields. Hence, for any Θ_{B_1} and Θ_{B_2} their commutator $[\Theta_{B_2}, \Theta_{B_2}]$ is again an evolution field Θ_B for some new B. This function B is called *the Jacobi bracket* of B_1 and B_2 and is denoted by $\{B_1, B_2\}$. From (13) it follows that

$$B^j = \sum_{\beta,\sigma} \left(D_\sigma(B_1^\beta) \frac{\partial B_2^j}{\partial u_\sigma^\beta} - D_\sigma(B_2^\beta) \frac{\partial B_1^j}{\partial u_\sigma^\beta} \right). \tag{14}$$

Clearly, the function algebra on $J^\infty(n,m)$ is a Lie algebra with respect to the Jacobi bracket.

Now we restrict everything above said onto $\mathcal{E}^\infty$. Suppose, as in (3), $\mathcal{E}$ is given by relations $F^1 = 0, \ldots, F^r = 0$. Then a field $X \in D(J^\infty(n,m))$ is tangent to $\mathcal{E}^\infty$ if and only if one has

$$X(F^j) = \sum_{\beta,\sigma} a_\sigma^{j\beta} D_\sigma(F^\beta)$$

for some functions $a_\sigma^{j\beta}$. In particular, if $X = \Theta_B$, the last equation rewrites as

$$\Theta_B(F^j) = \sum_{\beta,\sigma} a_\sigma^{j\beta} D_\sigma(F^\beta),$$

or, if we use the notation

$$\ell_F(B) \stackrel{\text{def}}{=} \Theta_B(F), \tag{15}$$

as

$$\ell_{F^j}(B) = \sum_{\beta,\sigma} a_\sigma^{j\beta} D_\sigma(F^\beta). \tag{16}$$

DEFINITION 7. The operator

$$\ell_F^j = \sum_\beta \frac{\partial F}{\partial u_\sigma^j} D_\sigma, \quad j = 1, \ldots, m, \tag{17}$$

is called *the operator of universal linearization.*

Thus, Θ_B is a symmetry of $\mathcal{E}$ if and only if B satisfies equation (16).

Restrict (16) onto $\mathcal{E}^\infty$. Then the right-hand side vanishes by definition, and we obtain the following important result.

THEOREM 3. [3] *Let $\mathcal{E} \subset J^k(n,m)$ be a differential equation given by relations $F^1 = 0, \ldots, F^r = 0$ and Θ_B be an evolution vector field. Denote by $\bar{B}$ restriction of B onto $\mathcal{E}^\infty$. Then Θ_B is a symmetry of $\mathcal{E}$ if and only if*

$$\ell_{\mathcal{E}}(\bar{B}) = 0, \tag{18}$$

where $\ell_{\mathcal{E}}$ is the restriction of $\ell_F = (\ell_{F^1}, \ldots, \ell_{F^r})$ onto $\mathcal{E}^\infty$. In other words,

$$\operatorname{sym}(\mathcal{E}) = \ker(\ell_{\mathcal{E}}).$$

Note that solutions of (18) are closed under the Jacobi bracket restricted onto $\mathcal{E}^\infty$.

Equation (18) is the basis for direct computations of symmetries. We illustrate its use in the next section.

2 Example: the Burgers equation

Consider the Burgers equation $\mathcal{B}$

$$u_t = u_{xx} + uu_x \tag{19}$$

and choose *internal coordinates* on $\mathcal{B}^\infty$ by setting $u_k = u_{(k,0)}$. Below we use the method described in [3].

Defining equations. Rewrite restrictions onto $\mathcal{B}^\infty$ of all basic concepts in these coordinate system. For the total derivatives we obviously obtain

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i}, \tag{20}$$

$$D_t = \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} D_x^i(u_3 + u_0 u_1)\frac{\partial}{\partial u_i}. \tag{21}$$

The operator of universal linearization for $\mathcal{B}$ then looks as

$$\ell_{\mathcal{B}} = D_t - u_1 - u_0 D_x - D_x^2 \tag{22}$$

and, as it follows from Theorem 3, evolution field

$$\Theta_\varphi = \sum_{i=1}^{\infty} D_x^i(\varphi)\frac{\partial}{\partial u_i} \tag{23}$$

is a symmetry for $\mathcal{B}$ if and only if $\varphi = \varphi(x, t, u_0, \ldots, u_k)$ satisfies the equation

$$D_t\varphi = u_1\varphi + u_0 D_x\varphi + D_x^2\varphi, \tag{24}$$

where D_t, D_x are given by (20), (21). Computing $D_x^2\varphi$ we obtain

$$D_x^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + 2\sum_{i=1}^{k} u_{i+1}\frac{\partial^2\varphi}{\partial x\partial u_i} + \sum_{i,j=0}^{k} u_{i+1}u_{j+1}\frac{\partial^2\varphi}{\partial u_i\partial u_j} + \sum_{i=0}^{k} u_{i+2}\frac{\partial\varphi}{\partial u_i},$$

while

$$D_x^i(u_0u_1+u_3)=\sum_{\alpha=0}^{i}\binom{i}{\alpha}u_\alpha u_{i-\alpha+1}+u_{i+3}.$$

Hence, (24) transforms to

$$\frac{\partial\varphi}{\partial t}+\sum_{i=1}^{k}\sum_{\alpha=1}^{i}\binom{i}{\alpha}u_\alpha u_{i-\alpha+1}\frac{\partial\varphi}{\partial u_i}=$$

$$u_1\varphi+u_0\frac{\partial\varphi}{\partial x}+\frac{\partial^2\varphi}{\partial x^2}+2\sum_{i=1}^{k}u_{i+1}\frac{\partial^2\varphi}{\partial x\partial u_i}+\sum_{i,j=0}^{k}u_{i+1}u_{j+1}\frac{\partial^2\varphi}{\partial u_i\partial u_j}. \qquad (25)$$

Higher order terms. Note now that the left-hand side of (25) is independent of u_{k+1} while the right-hand one is quadratic in this variable and is of the form

$$u_{k+1}^2\frac{\partial^2\varphi}{\partial u_k^2}+2u_{k+1}\left(\frac{\partial^2\varphi}{\partial x\partial u_k}+\sum_{i=0}^{k-1}u_{i+1}\frac{\partial^2\varphi}{\partial u_i\partial u_k}\right).$$

It means that

$$\varphi=Au_k+\psi, \qquad (26)$$

where $A=A(t)$ and $\psi=\psi(t,x,u_0,\dots,u_{k-1})$. Substituting (26) into (25) one obtains

$$\dot{A}u_k+\frac{\partial\psi}{\partial t}+\sum_{i=1}^{k-1}\sum_{\alpha=1}^{i}\binom{i}{\alpha}u_\alpha u_{i-\alpha+1}\frac{\partial\psi}{\partial u_i}+\sum_{i=1}^{k}\binom{k}{i}u_iu_{k-i+1}A=$$

$$u_1(Au_k+\psi)+u_0\frac{\partial\psi}{\partial x}+\frac{\partial^2\psi}{\partial x^2}+2\sum_{i=1}^{k-1}u_{i+1}\frac{\partial^2\psi}{\partial x\partial u_i}+\sum_{i,j=0}^{k-1}u_{i+1}u_{j+1}\frac{\partial^2\psi}{\partial u_i\partial u_j},$$

where $\dot{A}\overset{\text{def}}{=}\frac{dA}{dt}$. Here again everything is at most quadratic in u_k, and equating coefficients at u_k^2 and u_k we get

$$\frac{\partial^2\psi}{\partial u_{k-1}^2}=0, \qquad 2\left(\sum_{i=0}^{k-2}u_{i+1}\frac{\partial^2\psi}{\partial u_i\partial u_{k-1}}+\frac{\partial^2\psi}{\partial x\partial u_{k-1}}\right)=ku_1A+\dot{A}.$$

Hence,

$$\psi=\tfrac{1}{2}(ku_0A+\dot{A}x+\dot{a})u_{k-1}+O[k-2],$$

where $a=a(t)$ and $O[l]$ denotes a function independent of $u_i, i>l$. Thus

$$\varphi=Au_k+\frac{1}{2}(ku_0A+\dot{A}x+\dot{a})u_{k-1}+O[k-2] \qquad (27)$$

which gives the "upper estimation" for solutions of (24).

Estimating Jacobi brackets. Let $\varphi = \varphi(t, x, u_0, \dots, u_k)$, $\psi(t, x, u_0, \dots, u_l)$ be two symmetries of $\mathcal{B}$. Then their Jacobi bracket restricted onto $\mathcal{B}^\infty$ looks as

$$\{\varphi, \psi\} = \sum_{i=0}^{l} D_x^i(\varphi)\frac{\partial\psi}{\partial u_i} - \sum_{i=0}^{k} D_x^j(\psi)\frac{\partial\varphi}{\partial u_j}. \tag{28}$$

Suppose φ is of the form (27) and similarly

$$\psi = Bu_l + \tfrac{1}{2}(lu_0B + \dot{B}x + \dot{b})u_{l-1} + O[l-2]$$

and compute (28) for these functions temporary denoting $ku_0A + \dot{A} + a$ and $lu_0B + \dot{B} + b$ by $\bar{A}$ and $\bar{B}$ respectively. Then we have:

$$\begin{aligned}
\{\varphi, \psi\} &= D_x^l(Au_k + \tfrac{1}{2}\bar{A}u_{k-1})B + \tfrac{1}{2}D_x^{l-1}(Au_k + \tfrac{1}{2}\bar{A}u_{k-1})\bar{B} + \\
&\quad -D_x^k(Bu_l + \tfrac{1}{2}\bar{B}u_{l-1})A - \tfrac{1}{2}D_x^{k-1}(Bu_k + \tfrac{1}{2}\bar{B}u_{l-1})\bar{A} + O[k+l-1] \\
&= \tfrac{1}{2}(lD_x(\bar{A})u_{k+l-2} + \bar{A}u_{k+l-1})\bar{B} + \tfrac{1}{2}(Au_{k+l-1} + \tfrac{1}{2}\bar{A}u_{k+l-2})B + \\
&\quad -\tfrac{1}{2}(kD_x(\bar{B})u_{k+l-2} + \bar{B}u_{k+l-1})\bar{A} - \tfrac{1}{2}(Bu_{k+l-1} + \tfrac{1}{2}\bar{B}u_{k+l-2})A \\
&\quad +O[k+l-3],
\end{aligned}$$

or in short,

$$\{\varphi, \psi\} = \tfrac{1}{2}(l\dot{A}B - k\dot{B}A)u_{k+l-2} + O[k+l-3]. \tag{29}$$

Low-order symmetries. Take $k = 2$ and solve equation (24) directly. Then one obtains five independent solutions which are

$$\begin{aligned}
\varphi_1^0 &= u_1, \\
\varphi_1^1 &= tu_1 + 1, \\
\varphi_2^0 &= u_2 + u_0u_1, \\
\varphi_2^1 &= tu_2 + (tu_0 + \tfrac{1}{2}x)u_1 + \tfrac{1}{2}u_0, \\
\varphi_2^2 &= t^2u_2 + (t^2u_0 + tx)u_1 + tu_0 + x.
\end{aligned} \tag{30}$$

Action of low-order symmetries. Let us compute the action

$$T_i^j \stackrel{\text{def}}{=} \{\varphi_i^j, \cdot\} = \Theta_{\varphi_i^j} - \ell_{\varphi_i^j}$$

of symmetries φ_i^j on other symmetries of the equation $\mathcal{B}$.

For φ_1^0 one has

$$T_1^0 = \Theta_{u_1} - \ell_{u_1} = \sum_{i\ge 0} u_{i+1}\frac{\partial}{\partial u_i} - D_x = -\frac{\partial}{\partial x}.$$

Hence, if $\varphi = Au_k + O[k-1]$ is a function of the form (27), then one has

$$T_1^0\varphi = -\tfrac{1}{2}\dot{A}u_{k-1} + O[k-2].$$

Consequently, if φ is a symmetry, then, since $\mathrm{sym}(\mathcal{B})$ is closed under the Jacobi bracket,

$$(T_1^0)^{k-1}\varphi = \left(-\frac{1}{2}\right)^{k-1} \frac{d^{k-1}A}{dt^{k-1}} u_1 + O[0]$$

is a symmetry as well. But from (30) one sees that first-order symmetries are linear in t. Thus, we obtain

PROPOSITION 3. *If $\varphi = Au_k + O[k-1]$ is a symmetry of the Burgers equation, then A is a k-th degree polynomial in t.*

Final description. Note that direct computations show that the equation $\mathcal{B}$ possesses a third-order symmetry of the form

$$\varphi_3^0 = u_3 + \frac{3}{2}u_0u_2 + \frac{3}{2}u_0^2 + \frac{3}{4}u_0^2u_1.$$

Using the actions T_2^2 and T_3^0, one can see that

$$((T_2^2)^i \circ (T_3^0 \circ T_2^2)^{k-1})u_1 = \left(-\frac{3}{2}\right)^{k-1} \frac{k!(k-1)!}{(k-i)!} u_k + O[k-1] \tag{31}$$

is a symmetry, since u_1 is the one.

THEOREM 4. *The symmetry algebra* $\mathrm{sym}(\mathcal{B})$ *for the Burgers equation $\mathcal{B} = \{u_t = uu_x + u_{xx}\}$ is generated, as a vector space, by elements of the form*

$$\varphi_k^i = t^i u_k + O[k-1], \quad k \geq 1, \quad i = 0, \dots, k,$$

which are polynomial in all variables. For the Jacobi bracket one has

$$\{\varphi_k^i, \varphi_l^j\} = \frac{1}{2}(li - kj)\varphi_{k+l-2}^{i+j-1} + O[k+l-3]. \tag{32}$$

The algebra $\mathrm{sym}(\mathcal{B})$ *is simple and has φ_1^0, φ_2^2 and φ_3^0 as its generators.*

Proof: It only remains to prove that all φ_k^i are polynomials and that $\mathrm{sym}(\mathcal{B})$ is simple. The first fact follows from (31) and from the obvious observation that coefficients of both T_2^2 and T_3^0 are polynomials.

Let us prove that $\mathrm{sym}(\mathcal{B})$ is a simple Lie algebra. To do this, introduce an order in the set $\{\varphi_k^i\}$ defining

$$\Phi_{\frac{k(k+1)}{2}+i} \stackrel{\mathrm{def}}{=} \varphi_k^i.$$

Then any symmetry may be represented as $\sum_{\alpha=1}^{s} \lambda_\alpha \Phi_\alpha$, $\lambda \in R$. Let $I \subset \mathrm{sym}(\mathcal{B})$ be an ideal and $\Phi = \Phi_s + \sum_{\alpha=1}^{s-1} \lambda_\alpha \Phi_\alpha$ be its element. Assume $\Phi_s = \varphi_k^i$ for some $k \geq 1$ and $i \leq k$.

Note now that

$$T_1^1 = \sum_{\alpha \geq 0} D_x^\alpha (tu_1 + 1) \frac{\partial}{\partial u_\alpha} - tD_x = \frac{\partial}{\partial u_0} - t\frac{\partial}{\partial x}$$

and

$$T_2^0 = \sum_{\alpha \geq 0} D_x^\alpha (u_2 + u_0 u_1) \frac{\partial}{\partial u_\alpha} - D_x^2 - u_0 D_x - u_1 = -\frac{\partial}{\partial t}.$$

Therefore,

$$((T_1^1)^{k-1} \circ (T_2^0)^i)\Phi = c\varphi_1^0,$$

where the coefficient c does not vanish. Hence, I contains φ_1^0. But due to (31) the latter, together with φ_2^2 and φ_3^0, generates the whole algebra. □

Further details on the structure of $\mathrm{sym}(\mathcal{B})$ one can find in [3].

3 Recursion operators and nonlocalities

In fact, we were successful in getting a complete description of symmetry algebra for the Burgers equation because $\mathrm{sym}(\mathcal{B})$ is a very rich algebra. In other cases this method may not work, e.g. for the KdV equation (1)[6]. Below we expose another method based on computation of recursion operators.

Recursion operators for the Burgers and KdV equations. Informally speaking, a recursion operator for an evolution equation $\mathcal{E}$

$$\frac{\partial u}{\partial t} = F\left(t, x, u, \dots, \frac{\partial^k u}{\partial x^k}\right) \tag{33}$$

is an expression of the form

$$\mathcal{R} = \sum_{i=-\infty}^{s} a_i \circ D_x^i \circ b_i,$$

a_i, b_i being functions of $x, t, \dots, u_k, \dots$, which preserves symmetries of $\mathcal{E}$, i.e. $\mathcal{R} : \mathrm{sym}(\mathcal{E}) \to \mathrm{sym}(\mathcal{E})$. Usually, integrable equations possess such operators. For example (see [5]), the operator

$$D_x + \frac{1}{2}u_0 + \frac{1}{2}u_1 D_x^{-1}$$

is a recursion for the Burgers equation, while

$$D_x^2 + \frac{2}{3}u_0 + \frac{1}{3}u_1 D_x^{-1}$$

is the one for the KdV equation (1).

[6] Though it will be helpful anyway to estimate the symmetry algebra.

Having a recursion operator and a starting symmetry φ_0, one easily obtains explicit expressions for the whole series $\varphi_k = \mathcal{R}^k\varphi_0$. Thus, having a method for computing recursion operators, we get a powerful means to generate symmetries. Such a method was found in [1, 2] and we discuss it in action below.

Cartan forms. A Cartan form on $\mathcal{E}^\infty$ is a form which vanishes on the distribution $\mathcal{C}_\mathcal{E}$ or, equivalently, which annihilates all total derivatives $D_1, \ldots, D_n$ on $\mathcal{E}^\infty$. Cartan forms form a vector subspace $\mathcal{C}^1\Lambda(\mathcal{E}^\infty)$ in the space $\Lambda^1(\mathcal{E}^\infty)$ of all differential one-forms on $\mathcal{E}^\infty$. For the case of (33), one can choose the forms

$$\omega_s \stackrel{\text{def}}{=} du_s - u_{s+1}dx - D_x^s(F)dt, \tag{34}$$

$s = 0, 1, \ldots$, as a basis in $\mathcal{C}^1\Lambda(\mathcal{E}^\infty)$. It can easily be checked that $i_{D_x}\omega_s = i_{D_t}\omega_s = 0$.

Below we shall need formulae for Lie actions of total derivatives on the forms ω_s. Let us make necessary computations in the case (33). For D_x one has:

$$\begin{aligned} L_{D_x}(\omega_s) &= D_x(du_s - u_{s+1}dx - D_x^s(F)dt) \\ &= d(D_xu_s) - (D_xu_{s+1})dx - u_{s+1}d(D_xx) - D_x^{s+1}(F)dt - D_x^s(F)d(D_xt) \\ &= du_{s+1} - u_{s+2}dx - u_{s+1}d(1) - D_x^{s+1}(F)dt - D_x^s(F)d(0), \end{aligned}$$

or

$$L_{D_x}(\omega_s) = \omega_{s+1}. \tag{35}$$

To apply D_t to ω_s, note first that

$$[D_x, D_t] = D_x \circ D_t - D_t \circ D_x = 0. \tag{36}$$

Hence, using (35) and (36), we obtain

$$L_{D_t}(\omega_s) = D_tD_x^s(\omega_0) = D_x^s(D_t\omega_0). \tag{37}$$

But

$$\begin{aligned} D_t(\omega_0) &= D_t(du_0 - u_1dx - Fdt) = d(D_tu_0) - (D_tu_1)dx - (D_tF)dt \\ &= dF - (D_xF)dx - (D_tF)dt = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial t}dt + \sum_{i=0}^{k}\frac{\partial F}{\partial u_i}du_i \\ &\quad - \left(\frac{\partial F}{\partial x} + \sum_{i=0}^{k} u_{i+1}\frac{\partial F}{\partial u_i}\right)dx - \left(\frac{\partial F}{\partial t} + \sum_{i=0}^{k} D_x^i(F)\frac{\partial F}{\partial u_i}\right)dt, \end{aligned}$$

or

$$D_t\omega_0 = \sum_{i=0}^{k}\frac{\partial F}{\partial u_i}\omega_i.$$

Hence

$$D_t\omega_s = D_x^s\left(\sum_{i=0}^{k}\frac{\partial F}{\partial x_i}\omega_i\right) = \sum_{i=0}^{k}\sum_{\alpha=0}^{s}\binom{s}{\alpha}D_x^{s-\alpha}\left(\frac{\partial F}{\partial u_i}\right)\omega_{i+\alpha}. \tag{38}$$

Action of Cartan forms on symmetries. Let $\omega \in \mathcal{C}^1\Lambda(\mathcal{E}^\infty)$ be a Cartan form and consequently

$$\omega = a_0\omega_0 + a_1\omega_1 + \ldots + a_s\omega_s, \tag{39}$$

a_i being functions on $\mathcal{E}^\infty$. Consider another function φ and the vector field

$$\Theta_\varphi = \sum_{i=0}^{\infty} D_x^i(\varphi)\frac{\partial}{\partial u_i}.$$

Hence, contracting Θ_φ with ω, we obtain

$$\begin{aligned} i_{\Theta_\varphi}(\omega) &= a_0 i_{\Theta_\varphi}\omega_0 + a_1 i_{\Theta_\varphi}\omega_1 \ldots + a_s i_{\Theta_\varphi}\omega_s \\ &= a_0\varphi + a_1 D_x(\varphi) + \ldots + a_s D_x^s(\varphi). \end{aligned}$$

Therefore, any ω of the form (39) determines a differential operator

$$\mathcal{R}_\omega = a_0 + a_1 D_x + \ldots + a_s D_x^s. \tag{40}$$

THEOREM 5. [2] *Restrict operator* (40) *onto symmetries of the equation* $\mathcal{E}$, *i.e. for any* $\Theta_\varphi \in \mathrm{sym}(\mathcal{E})$ *set* $\mathcal{R}_\omega(\Theta_\varphi) \overset{\mathrm{def}}{=} \Theta_{\mathcal{R}_\omega\varphi}$. *Then* $\mathcal{R}_\omega$ *preserves the algebra* $\mathrm{sym}(\mathcal{E})$, *i.e.* $\mathcal{R}_\omega \,\mathrm{sym}(\mathcal{E}) \subset \mathrm{sym}(\mathcal{E})$, *if and only if the form* ω *satisfies the equation*

$$\ell_{\mathcal{E}}^{[1]}\omega = 0, \tag{41}$$

where $\ell_{\mathcal{E}}^{[1]}$ *is the extension of the universal linearization operator (17) to Cartan forms.*

Thus equation (41) may be used to compute recursion operators. We show how it works using the example of the Burgers equation.

Basic relations for $\mathcal{B}$. In this case we obtain

$$\begin{aligned} \ell_{\mathcal{B}}^{[1]}(\omega) &= (D_t - u_1 - u_0 D_x - D_x^2)\left(\sum_{i=0}^{s} a_i\omega_i\right) \\ &= \sum_{i=0}^{s}(\ell_{\mathcal{B}}(a_i)\omega_i + a_i D_t(\omega_i) - u_0 a_i\omega_{i+1} - 2D_x(a_i)\omega_{i+1} - a_i\omega_{i+2}). \end{aligned}$$

Using (38) one can now reduce (41) to a system of equations on functions $a_0, \ldots, a_s$. We shall not do it here in the general form, but write down the equations for $s = 2$. In this case one obtains

$$\begin{aligned} D_x(a_1) &= 0, \\ D_x(a_0) + D_x^2(a_1) + u_0 D_x(a_1) &= u_1 a_1 + D_t(a_1), \\ D_x^2(a_0) + u_0 D_x(a_0) + u_1 a_0 &= D_t(a_0). \end{aligned} \tag{42}$$

A "disappointing" result. Now, if one tries to solve (42) or the equation $\ell_{\mathcal{B}}^{[1]}\omega = 0$ in its general form (see [1] for the proof), one will get the only solution

$$\omega = a_0\omega_0, \ a_0 \in R.$$

Corresponding operator $\mathcal{R}_\omega$ just multiplies symmetries by the constant a_0 and produces no nontrivial recursions.

And this is not unnatural, since the known recursion operator for the Burgers equation contains the term D_x^{-1} (see above), which could not appear in our setting.

Nonlocal setting. Introduce a new variable $u_{-1} \stackrel{\text{def}}{=} D_x^{-1}(u_0) = \int u_0\,dx$ and set

$$D_x u_{-1} = u_0, \quad D_t u_{-1} = \int D_t(u_0)\,dx = u_1 + \frac{1}{2}u_0^2.$$

Then we get "new" total derivatives

$$\widetilde{D}_x = u_0\frac{\partial}{\partial u_{-1}} + D_x, \quad \widetilde{D}_t = (u_1 + \frac{1}{2}u_0^2)\frac{\partial}{\partial u_{-1}} + D_t$$

and the Cartan form

$$\omega_{-1} = du_{-1} - u_0 dx - (u_1 + \frac{1}{2}u_0^2)dt.$$

Let us solve the equation

$$\widetilde{\ell}_{\mathcal{B}}\omega = 0, \tag{43}$$

where $\widetilde{\ell}_{\mathcal{B}}$ is the operator $\ell_{\mathcal{B}}$ in which "old" total derivatives D_x, D_t are substituted by $\widetilde{D}_x$ and $\widetilde{D}_t$ respectively.

PROPOSITION 4. [2] *Equation* (43) *possesses two independent solutions:* ω_0, *and*

$$\omega = \omega_1 + \frac{1}{2}u_0\omega_0 + \frac{1}{2}u_1\omega_{-1}.$$

The operator $\mathcal{R}_{\omega_0}$ is the identity, while $\mathcal{R}_\omega$ coincides with the classical recursion operator for the Burgers equation.

REMARK 2. Try to compute *symmetries* in this setting: a very interesting result awaits you!

General scheme. Below we expose the way of action in the case of a general evolution equation.

1. Solve the equation
$$\ell_{\mathcal{E}}\varphi = 0$$
for low-order φ.

2. Solve the equation
$$\ell_{\mathcal{E}}^{[1]}\omega = 0$$
for low-order Cartan forms. Usually you will get trivial solutions only.

3. Extend the setting with nonlocal variables. To do this, solve the equation
$$\ell_{\mathcal{E}}^{*}\psi = 0, \tag{44}$$
where $\ell_{\mathcal{E}}^{*}$ is the formally adjoint[7] to $\ell_{\mathcal{E}}$. The solutions of (44) are *generating functions of conservation laws* for the equation $\mathcal{E}$ (see [7] for details).

4. Enlarge the setting by adding nonlocal variables $D_x^{-1}\psi = \int \psi\, dx$ for any solution ψ; (44) guarantees the new setting to be well-defined.

5. Solve the equation
$$\widetilde{\ell}_{\mathcal{E}}\omega = 0$$
in the new setting and get the recursion operator.

For integrable equations this scheme works perfectly.

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[7] If $\Delta = \sum_{i=0}^{s} a_i D_x^i$, then $\Delta^* = \sum_{i=0}^{s} (-1)^i D_x \circ a_i$.