Equations defined on the lattice and their solutions

Decio Levi

Dipartimento di Fisica, Università di Roma Tre, Via della Vasca Navale 84, 00146 Rome, Italy
e-mail: levi@amaldi.fis.uniroma3.it

Abstract: In this work we introduce difference equations in their various setting and we show how one can solve them, at first in the linear case by discrete Fourier analysis or Z transform and later in the nonlinear case by a discretization of the spectral transform introduced to integrate the Korteweg–de Vries equation.
The nonlinear integrable model we consider in detail is the Toda Lattice, obtained as an approximation of the anharmonic oscillator on the lattice introduced in the fifties by Fermi, Pasta and Ulam. We briefly discuss the spectral problem for the Toda lattice and than we go over to discuss its algebraic properties, Bäcklund transformations, Darboux matrices, connection with integrability of partial differential equations and symmetry group.

1 Introduction

The main object of this report is to present an introduction to the field of linear and nonlinear difference or differential difference equations [1, 2, 3, 4].

By a difference equation we mean a functional relation, linear or nonlinear, between functions calculated at different points of a lattice.

Why are these systems important? They appear in many applications. First of all they can be written down as a discretization of a differential equation when one is trying to solve it with a computer. In such a case one reduces the differential equation to a recurrence relation:

$$\frac{du}{dx} = f(x, u) \qquad \Rightarrow \qquad v(n+1) = g(n, v(n)).$$

On the other hand we can consider dynamical systems defined on a lattice, i.e. systems where the real independent fields depend on a set of independents variables which vary partly on the integers and partly on the reals. For example we can have

$$\frac{d^2 u(n,t)}{dt^2} = F(t, u(n,t), u(n-1,t), \ldots, u(n-a,t), u(n+1,t), \ldots, u(n+b,t)).$$

This kind of equations can appear in many different settings; among all, those which are more interesting from a scientific point of view, are associated to the evolution of many body problems, to the study of crystals, to biological systems, etc.

As an example of possible applications we consider the problem of the transmission of energy in one-dimensional molecular systems, problem which is of particular relevance for understanding the functioning of physical systems of biological interest [5]. This is a particularly hot topic as it has been argued that some relevant biological processes require the transport of energy with low dispersion along essentially one-dimensional chains or substructures of macromolecular systems that can be regarded as quasi one-dimensional chains (such as the spines in an α helix [6]). A mechanism for the non-dispersive transport of vibrational energy along hydrogen bonded chains was proposed by Davydov and its continuous limit for small lattice spacing gave rise to a nonlinear Schöringer equation with soliton solutions [7]. Whether such soliton-like solutions are valid also at biological temperatures is an open problem. In the case of diatomic nonlinear lattices we can describe such systems by the following dynamical systems

$$\begin{aligned} M_1\ddot{x}_n - k_1(y_n - x_n) + k_2(x_n - y_{n-1}) - \epsilon\beta_1(y_n - x_n)^2 & \\ +\epsilon\beta_2(x_n - y_{n-1})^2 = 0&, \\ M_2\ddot{y}_n + k_1(y_n - x_n) - k_2(x_{n+1} - y_n) + \epsilon\beta_1(y_n - x_n)^2 & \\ -\epsilon\beta_2(x_{n+1} - y_n)^2 = 0&, \end{aligned} \tag{1.1}$$

where M_1 and M_2 are the different values of the two atomic masses, ϵ is a small parameter while k_1, k_2, β_1 and β_2 are four constants of order 1. When $k_2 \ll k_1$ these equations represent a molecular chain with intramolecular interaction stronger than the intermolecular one. This is the case, for example, of an hydrogen-bonded polypeptide chain.

In [5] this problem has been considered from two different points of view:

1. a molecular dynamics numerical simulation,

2. reductive perturbative analysis of the dynamics of the diatomic chain.

In the second case the development led to a nonlinear Schrödinger equation and to a soliton-like solution that carries energy without dispersion. Moreover one can show that localized initial excitations of generic form can evolve into soliton-like solutions that travel along the chain. The numerical calculations show that such solutions are robust with respect to thermal disorder at biological relevant temperatures, i.e. energy can be transported with no appreciable dispersion even at room temperature.

Such results could be obtained either by numerical calculations or by doing a continuum limit. The numerical results (see Fig. 1–3) show clearly the relevance of the nonlinear terms in this non-dispersive energy transport. The analytic calculations, hint to the result but involving a continuum limit they are not sufficient.

In Section 2 we present some results on linear difference equations and their methods of solutions. In Section 3 we go over to consider integrable nonlinear differential difference equations and their methods of solution. Section 4 is devoted to the introduction of group theoretical methods for solving difference and differential difference equations and then in Section 5 we present some conclusive remarks.

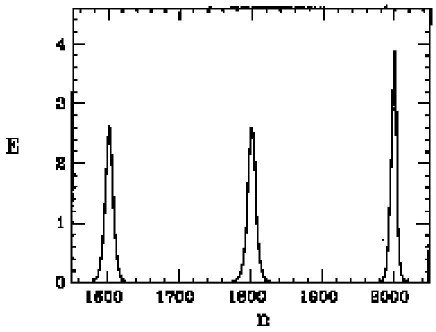

Fig. 1: Energy profile of the solitonic excitation at times $t = 0$ (right), $t = 2.65 \times 10^4$ (middle), $t = 5.3 \times 10^4$ (left). Temperature $T = 0K$.

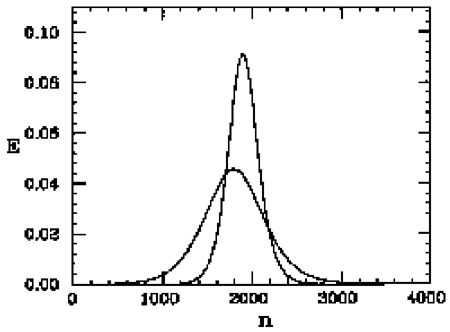

Fig. 2: Energy profile in the linear model, at times $t = 1.35 \times 10^4$ (narrow curve) and $t = 2.65 \times 10^4$ (broad curve). The initial energy profile is the same as in Fig. 1. Temperature $T = 0K$.

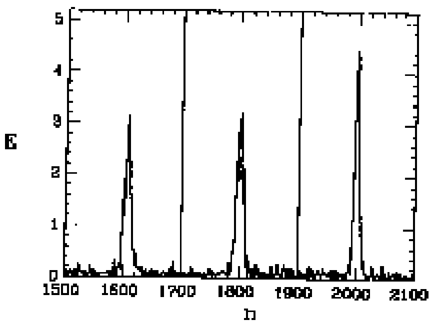

Fig. 3: Energy profile of the same solitonic excitation as of Fig. 1, at times $t = 0$ (right), $t = 2.65 \times 10^4$ (middle), $t = 5.3 \times 10^4$ (left). Temperature $T = 300K$. The vertical bars separate different configurations.

2 Linear difference equations

In this Section we will introduce some difference equations and show how to integrate them [1, 2]. Here we will consider, for the sake of simplicity, just the case of linear difference equations in one discrete dependent variable. In this context difference equations are nothing else but functional relations that allows us to compute the value of a function *recursively* from a given set of values. This is very frequent in mathematics when we define a function through its expansion in series of a given set of orthonormal functions. For example we may want to solve the Airy equation

$$\frac{d^2y}{dx^2}(x) = xy(x). \tag{2.1}$$

Let us define the Airy function as a power series solution to (2.1)

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{2.2}$$

Substituting (2.2) into (2.1) we get for the constants a_k the following linear ordinary difference equation of third order with variable coefficients:

$$(k+3)(k+2)a_{k+3} = a_k \tag{2.3}$$

with the boundary condition

$$a_2 = 0. \tag{2.4}$$

A solution of (2.1) can be obtained by solving the recurrence relation (2.3) with the initial condition (2.4). It is written in terms of two arbitrary constants and thus it corresponds to the general solution of the Airy equation

$$y(x) = a_0 \left[1 + \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} + \ldots\right] + a_1 \left[x + \frac{x^4}{4\cdot 3} + \frac{x^7}{7\cdot 6\cdot 4\cdot 3} + \ldots\right].$$

By the ratio test

$$\lim_{k\to\infty} \frac{a_{k+3}}{a_k} = \lim_{k\to\infty} \frac{1}{(k+3)(k+2)} = 0,$$

we evince that the resulting power series converges for all values of x.

An alternative way of solving the difference equation is given by the Z-transform or discrete Fourier transform [1]. Using the Z-transform we get the recurrence formula already summed up. Let us consider a sequence of numbers parameterized by an index k, i.e. $\{y_k\}$. We can associate to it a function given by the following invertible transformation:

$$Y(z) = \sum_{k=0}^{\infty} \frac{y_k}{z^k}, \quad y_k = \frac{1}{2\pi i} \oint dz z^{k-1} Y(z),$$

where the integration is on the unit circle. This direct and inverse transformation can be read as a discrete Fourier transform once we substitute z by $z = e^{i\lambda}$.

The Z-transform possesses all the properties of the Fourier transform (mutatis mutandis) like the convolution theorem, multiplication theorem etc. plus some extra ones associated to the peculiarity of the discrete equations as, for example, the shifting theorem. For example if

$$Y(z) = Z(\{y_k\}) \qquad \text{for} \qquad |\, z \,| > r$$

then

$$Z[\{(k+n-1)^{(n)} y_k\}] = (-1)^n z^n \frac{d^n Y(z)}{dz^n}, \quad n > 0,$$

and a similar formula holds for $n < 0$. Here we have introduced the factorial power function or Pohammer symbol, defined as

$$(k+n-1)^{(n)} = (k+n-1)(k+n-2)\cdots(k+1)k. \tag{2.5}$$

The shifting theorem tells us that

$$Z(\{y_{k+n}\}) = z^n Y(z) - \sum_{m=0}^{n-1} y_m z^{n-m}.$$

Let us end this short section on the solution of linear difference equations by giving a few examples of the use of the Z-transform for solving initial values problems and Volterra summation equations.

As the first example, let us consider the following initial value problem:

$$(k+1)y_{k+1} - (50-k)y_k = 0, \quad y_0 = 1. \tag{2.6}$$

By taking the Z-transform and using the shifting theorem and the differentiation theorem we can rewrite equation (2.6) as:

$$-z(z+1)\frac{dY(z)}{dz} = 50Y(z).$$

Taking into account the boundary condition we obtain

$$Y(z) = \left(\frac{z+1}{z}\right)^{50},$$

whose inverse transform provide the required solution

$$y_k = \begin{pmatrix} 50 \\ k \end{pmatrix}.$$

Let us solve, as the second and final example, a Volterra discrete equation

$$y_k = 1 + 16\sum_{m=0}^{k-1}(k-m-1)y_m, \quad k \geq 0.$$

By a trivial transformation and by taking into account the convolution theorem and the shifting theorem we get

$$zY(z) - z = \frac{z}{1-z} + 16\frac{z}{(z-1)^2}Y(z) \quad \rightarrow \quad Y(z) = \frac{1}{2}\left[\frac{z}{z-5} + \frac{z}{z+3}\right]$$

from which we get

$$y_k = \frac{1}{2}[5^k + (-3)^k].$$

3 Nonlinear integrable differential difference equations and their properties

3.1 The Fermi-Pasta-Ulam system

After the Second World War, during one of his frequent summer visits to the National Laboratories of Los Alamos (New Mexico, U.S.A.), Fermi became interested in the potentialities of the electronic computing machines. In 1952 with Ulam and Pasta he decided to try them out in problems which did not possessed any closed form analytic solutions, i.e. in the study of the long time behaviour of nonlinear physical systems. The motivation was to observe the rate of mixing and the thermalization of such systems. The calculations where carried out in 1953. The results showed that the prevalent belief in the universality of mixing in nonlinear systems may not always be justified. Fermi died in November 1954 and Pasta and Ulam wrote up those results in a preprint of the National Laboratories of Los Alamos which came out in May 1955 [8].

In their work they considered the simplest nontrivial nonlinear model, a one-dimensional string with fixed ends where the forces are acting on each element of the string. To be able to carry out the numerical calculation they reduced the string to a finite number of points, which in their calculation was set to be at most 64. In such a way the partial differential equation which describes the string is reduced to a system of coupled ordinary differential equations for the single displacements x_i, where $i = 1, 2, \dots, 64$. By choosing the simplest possible nonlinear potential, $V(r) = \frac{1}{2}r^2 + \frac{\alpha}{3}r^3$, they wrote down the following set of equations:

$$\ddot{x}_i = V'(x_{i+1} - x_i) - V'(x_i - x_{i-1}), \tag{3.1}$$

i.e.

$$\ddot{x}_i = (x_{i+1} + x_{i-1} - 2x_i) + \alpha[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2]. \tag{3.2}$$

The solution of the corresponding linear problem is a periodic vibration of the string. If the initial position of the string is, say, a single sine wave, than the string will oscillate in this mode indefinitely.

To be able to attack the nonlinear problem numerically they had to discretize also the time variable so as to reduce the problem to a set of recursive relations. The string was initially at rest with either a single sine wave shape or a sawtooth wave shape. In Fig. 4 (resp. Fig. 5) we can see a plotting of the numerical results of Fermi–Pasta–Ulam calculations in the case when the initial condition was given by a single sine wave (resp. sawtooth wave). The results of their numerical calculations were summarized in the following way in their report: "Let us say here that the results of our computations show features which were, from the beginning, surprising to us. Instead of a gradual, continuous flow of energy from the first, 'initial,' mode to the higher modes, all of the problems show an entirely different behaviour." As one can see in Fig. 4 there is initially an increase of energy of the higher modes but at a later time the first mode comes to a few percent of its initial value.

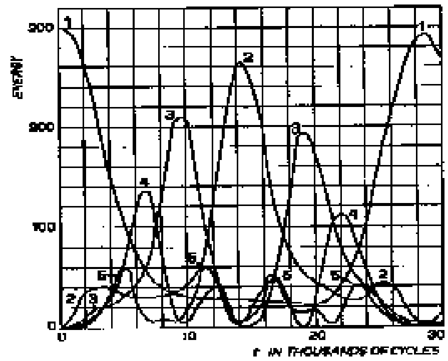

Fig. 4: The quantity plotted is the energy (kinetic plus potential in each of the first five modes). The units for energy are arbitrary. $N = 32$; $\alpha = 1/4$; $\delta t^2 = 1/8$. The initial form of the string was a single sine wave. The higher modes never exceeded in energy 20 of our units. About 30,000 computation cycles were calculated.

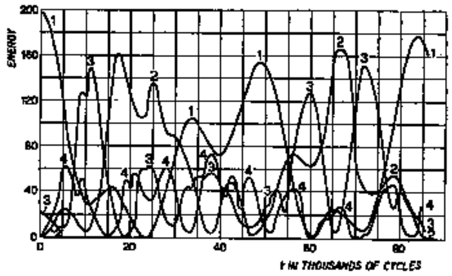

Fig. 5: Same condition as in Fig. 4, but the initial configuration of the string was a sawtooth triangular-shaped wave. Already at $t = 0$, therefore, energy was present in some modes other than 1. However, modes 5 and higher never exceeded 40 of our units.

In the subsequent years such system was studied with great detail, both numerically and analytically. Zabusky and Kruskal [9], among other things, considered the case of smooth waves, such that the changes of $x_{i+1} - x_i$ are small. In such a case one can approximate the lattice by a continuum system. If α is small, defining $x_i(t) = x(y, t)$ with $y = ih$, at lowest order in α and in the lattice spacing h, we can reduce equation (3.2) to

$$\frac{d^2x}{dt^2} = h^2 \frac{d^2x}{dy^2}.$$

Choosing a wave speed $c_0 = \pm h$ and introducing the new variables $\eta = y - c_0 t$, $\tau \propto t$ and $u \propto x$ we get that equation (3.2) can be rewritten as

$$\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial \eta} + \delta^2 \frac{\partial^3 u}{\partial \eta^3} = 0, \tag{3.3}$$

i.e. the Korteweg–de Vries equation [10]. It has been shown that this equation has an infinite number of conserved quantities in involution, it possesses a denumerable set of exact solutions and can be written as an Hamiltonian system so that one calls it an *integrable* partial differential equation. This result paved the way to a renewal of interests in studying integrable nonlinear partial differential equations and in rediscovering their properties. Gardner, Greene, Kruskal and Miura [11] where able to show that one could solve the Cauchy problem for the Korteweg–de Vries equation (3.3) in the case of bounded solutions at infinity by solving an associated spectral problem at the initial time, solving the linear ordinary differential equations which govern the time evolution of the spectrum and then recovering the solution at any time by solving a linear integral equation.

3.2 Nonlinear integrable models on the lattice

Fermi, Pasta and Ulam showed that an harmonic oscillator with small quadratic nonlinearity has periodic solutions and stable pulses. These results suggested that there must exist a nonlinear lattice which exhibit exact almost periodic waves as in the case of partial differential equations the Korteweg–de Vries equation (3.3) does.

So, at the beginning of the sixties Toda started looking for those potentials $V(r)$ such that equation (3.1) be integrable. In 1967 in a sequence of papers in the Journal of the Physical Society of Japan [12] he presented his final result:

$$V(r) = \frac{e^{-br}}{b} + r,$$

which gives rise to what was afterwards denoted as the Toda Lattice equation

$$\frac{d^2 x_n}{dt^2} = e^{-b[x_n - x_{n-1}]} - e^{-b[x_{n+1} - x_n]}. \tag{3.4}$$

This equation has soliton solutions

$$x_n = \frac{1}{b} \ln \left(\frac{1 + e^{2(kn - k \pm \beta t)}}{1 + e^{2(kn \pm \beta t)}} \right),$$

periodic solutions, associated spectral problem, etc. as any other integrable nonlinear partial differential equation.

Solution of the spectral problem for the Toda Lattice

Following the result of Toda, in 1974 Flaschka and Manakov [13, 14] solved the spectral problem associated to the equation (3.4). To do so they first wrote down the Lax pair

$$\begin{aligned} \mathcal{L}\psi_n(t) &= \lambda \psi_n(t), \\ (\mathcal{L}\psi)_n(t) &= a_{n-1}(t)\psi_{n-1}(t) + a_n(t)\psi_{n+1}(t) + b_n \psi_n(t), \\ \frac{\partial \psi_n(t)}{\partial t} &= a_{n-1}(t)\psi_{n-1}(t) - a_n(t)\psi_{n+1}(t). \end{aligned} \tag{3.5}$$

The Lax pair (3.5) give rise to the nonlinear differential difference equations

$$\begin{aligned} \frac{\partial a_n(t)}{\partial t} &= a_n(t) \left(b_n(t) - b_{n+1}(t) \right), \\ \frac{\partial b_n(t)}{\partial t} &= a_{n-1}(t) - a_n(t), \end{aligned}$$

which reduce to the Toda lattice equation (3.4) once we make the identifications: $a_n(t) = \exp\{-b[x_{n+1}(t) - x_n(t)]\}$ and $b_n(t) = x_{n,t}(t)$. This transformation is invertible if $x_n \to 0$ as $n \to -\infty$.

The initial value problem of the Toda lattice is solved by mapping the initial data $(a_n(0), b_n(0))$ into certain scattering data $\Sigma(0)$ (reflection coefficients, point spectrum and normalization constants). The temporal evolution of the scattering data, $\Sigma(t)$, being linear and decoupled can be calculated explicitly. Starting from $\Sigma(t)$, the inverse problem is solved via a discrete analogue of the Gelfand–Levitan–Marchenko integral equation.

The natural boundary conditions for $a_n(t)$ and $b_n(t)$, corresponding to $x_n(t) \to 0$ as $n \to -\infty$, are $a_n(t) \to 1$ and $b_n(t) \to 0$ as $n \to -\infty$.

Under the condition

$$\prod_{n=-\infty}^{+\infty} a_n(t) < \infty, \qquad \left| \sum_{n=-\infty}^{+\infty} b_n(t) \right| < \infty \tag{3.6}$$

we can easily formulate the scattering problem for the Toda lattice equation [3, 15]. Let us mention here that the expressions appearing in equation (3.6) are obviously integrals of motions; thus (3.6) need to be satisfied just for the initial data and then they will be satisfied at any time.

The continuum spectrum of $\mathcal{L}$ is degenerate, filling the interval $-2 \leq \lambda \leq 2$; in general, $\mathcal{L}$ has also a discrete spectrum. By defining $\lambda = \frac{1}{2}(z + \frac{1}{z})$ we get that the discrete spectrum is given by the set $z_j \in (-1, 1)$ for $j = 1, \ldots, N$ and the continuum spectrum by $z = e^{ik}$ for all $k \in [0, 2\pi]$. The eigenfunctions for the discrete spectrum are square summable while those of the continuum spectrum are bounded for all n. We can define the Jost solutions $\psi_n(z)$, $\phi_n(z)$ and the bound states solutions $\zeta_n(z_j)$, which are special solutions of the spectral problem which satisfy the following boundary conditions

$$\begin{aligned} \phi_n(z) &\simeq z^n \qquad \text{as} \quad n \to +\infty \\ \psi_n(z) &\simeq z^{-n} \qquad \text{as} \quad n \to -\infty \\ \sum_{n=-\infty}^{+\infty} \zeta_n^2(z_j) &= 1 \qquad \text{for} \quad j = 1, \ldots, N \end{aligned}$$

For any n, the functions $\phi_n(z)$ and $\psi_n(z)$ are analytic inside the unit circle of the complex z-plane.

By the following alternative boundary conditions we can define a new set of Jost functions $\tilde{\phi}_n(z)$ and $\tilde{\psi}_n(z)$:

$$\begin{aligned} \tilde{\phi}_n(z) &\simeq z^{-n} \qquad \text{as} \quad n \to +\infty, \\ \tilde{\psi}_n(z) &\simeq z^{n} \qquad \text{as} \quad n \to -\infty. \end{aligned}$$

On the unit circle we have $\tilde{\phi}_n(z) = \bar{\phi}_n(z)$, $\tilde{\psi}_n(z) = \bar{\psi}_n(z)$ where by $\bar{\psi}$ we mean the complex conjugate of the function ψ. By continuing these relations to the whole complex plane, we get

$$\phi_n\left(\frac{1}{z}\right) = \tilde{\phi}_n(z) = \bar{\phi}_n\left(\frac{1}{\bar{z}}\right), \quad \tilde{\psi}_n(z) = \bar{\psi}_n\left(\frac{1}{\bar{z}}\right) = \psi_n\left(\frac{1}{z}\right).$$

Since the spectral problem is given by a second-order difference equations which admits at most two independent solutions, the two sets of solutions defined above are not independent and thus we can define the following relations among them:

$$\begin{aligned}\psi_n(z) &= \beta(z)\phi_n(z) + \alpha(z)\phi_n\left(\frac{1}{z}\right),\\ \phi_n(z) &= -\beta\left(\frac{1}{z}\right)\psi_n(z) + \alpha(z)\psi_n\left(\frac{1}{z}\right).\end{aligned}$$

which imply the following asymptotic conditions

$$\begin{aligned}&\psi_n(k) \simeq \beta(k)e^{ink} + \alpha(k)e^{-ink} \quad \text{as} \quad n \to +\infty,\\ &\zeta_n(z_j) \simeq c_j(z_j)^n \quad \text{as} \quad n \to +\infty.\end{aligned}$$

By calculating the Wronskian between two independent solutions of the spectral problem we get the following analyticity properties:

1. $\mid \alpha(k) \mid^2 = 1 + \mid \beta(k) \mid^2$,
2. $\alpha(z)$ is analytic for $\mid z \mid < 1$, continuous up to the unit circle, except for poles in the points $z = \pm 1$. In this domain there are N points z_j, $j = 1, \ldots, N$ such that $\alpha(z_j) = 0$.

Taking all these results into account we can define the scattering data Σ

$$\Sigma = \{\beta(k), \alpha(k)\,\text{for}\, k \in [0, 2\pi);\, z_j, c_j\,\text{for}\, j = 1, \ldots, N\},$$

which evolve in time according to the following equations:

$$\begin{aligned}\dot{\beta}(k,t) &= 2i\sin[k\beta(k,t)],\\ \dot{z}_j(t) &= 0,\\ \dot{c}_j(t) &= \frac{1}{2}\left[z_j(t) - \frac{1}{z_j(t)}\right]c_j(t).\end{aligned}$$

By taking into account the analytic properties of the solutions of the Spectral Problem we can define uniquely a function, analytic everywhere, which has simple poles at the points z_j where the function $\alpha(z)$ vanishes, which has a jump on the unit circle and with well-defined boundary conditions at infinity. In such a way we easily arrive to define the following Gelfand–Levitan–Marchenko linear "integral" equation for the functions $\chi_n(z)$ and $\Gamma_n^{(j)}$ in terms of the scattering data:

$$\begin{aligned}\bar{\chi}_n(z) &= 1 + \sum_{k=1}^{N} \frac{z_k^{2n}\Gamma_n^{(k)}c_k}{\alpha'(z_k)(z - z_k)} + \frac{1}{2}z^{2n}\frac{\beta(z)}{\alpha(z)}\chi_n(z)\\ &\quad -\frac{1}{2\pi i}P\int \frac{y^{2n}\beta(y)\chi_n(y)dy}{\alpha(y)(y - z)},\\ \Gamma_n^{(j)} &= 1 + \sum_{m=1}^{N} \frac{z_m^{2n}c_m\Gamma_n^{(m)}}{\alpha'(z_m)(\frac{1}{z_j} - z_m)} - \frac{1}{2\pi i}\int_{|z|=1} \frac{z^{2n}\beta(z)\chi_n(z)dz}{\alpha(z)(z - \frac{1}{z_j})}.\end{aligned}$$

The solution of the Toda lattice is then obtained as

$$a_n(t) = \frac{\theta_{n-1}}{\theta_n},$$

where

$$\theta_n = 1 - \sum_{m=1}^{N} \frac{z_m^{2n} c_m \Gamma_n^{(m)}}{\alpha'(z_m)} - \frac{1}{2\pi i} \int z^{2n-1} \frac{\beta(z)}{\alpha(z)} \chi_n(z) dz.$$

Bäcklund transformations, Darboux matrices and nonlinear equations on the lattice

Let us start from the Lax pair associated to the Toda Lattice and let us rewrite it in matrix form. To do so let us introduce the vector

$$\Phi_n(t) = \begin{pmatrix} \psi_n(t) \\ \psi_{n-1}(t) \end{pmatrix}. \tag{3.7}$$

Taking equation (3.7) into account the Lax pair associated to the Toda Lattice equation reads

$$\begin{aligned} \Phi_{n+1}(t) &= \mathcal{U}_n(\lambda, a_n(t), b_n(t))\phi_n(t), \\ \dot{\Phi}_n(t) &= \mathcal{V}_n(\lambda, a_n(t), b_n(t))\phi_n(t), \end{aligned} \tag{3.8}$$

where

$$\mathcal{U}_n = \begin{pmatrix} \lambda - b_n(t), & -a_{n-1}(t) \\ a_n(t), & 0 \end{pmatrix}, \quad \mathcal{V}_n = \begin{pmatrix} -\lambda + b_n(t), & 2a_{n-1}(t) \\ -a_{n-1}(t), & \lambda - b_{n-1}(t) \end{pmatrix}.$$

From now on we will always consider nonlinear equations where the Lax pair is written in matrix form. As we shall see the matrix form is more useful to describe the general properties of such systems. In fact, we have in all generality that given a Lax pair of the form (3.8), the nonlinear evolution equations on the lattice read [16]

$$\dot{\mathcal{U}}_n + \mathcal{U}_n \mathcal{V}_n = \mathcal{V}_{n+1} \mathcal{U}_n. \tag{3.9}$$

Let us recall [17] that any completely integrable nonlinear partial differential equation can be described by a Lax pair which can be written as

$$\begin{aligned} \Phi_{,\xi}(\lambda, \xi, \eta) &= \mathcal{W}(\lambda, q(\xi, \eta))\Phi(\lambda, \xi, \eta), \\ \Phi_{,\eta}(\lambda, \xi, \eta) &= \mathcal{Z}(\lambda, q(\xi, \eta))\Phi(\lambda, \xi, \eta). \end{aligned} \tag{3.10}$$

In terms of $\mathcal{W}$ and $\mathcal{Z}$ the nonlinear partial differential equation reads

$$\mathcal{W}_{,\eta} + \mathcal{W}\mathcal{Z} = \mathcal{Z}_{,\xi} + \mathcal{Z}\mathcal{W}. \tag{3.11}$$

How do we represent a Bäcklund Transformation (BT) in this context?

A BT is nothing else but a relation between two different solutions q and $\tilde{q}$ of (3.11), where each solution can be represented as a solution of a Spectral Problem (say $\Phi_{,\xi} = \mathcal{W}\Phi$) corresponding to the Lax pair (3.10). Thus if q is a solution of (3.11) corresponding to the Lax pair (3.10) and to the Spectral Problem $\Phi_{,\xi} = \mathcal{W}(\lambda, q(\xi,\eta))\Phi$, than $\tilde{q}$ will be a solution of the spectral problem $\tilde{\Phi}_{,\xi} = \mathcal{W}(\lambda, \tilde{q}(\xi,\eta))\tilde{\Phi}$. If a BT exists than there must be a relation between the two solutions q and $\tilde{q}$ which implies the existence of a relation between the two Spectral Problems and thus between the two "wave functions" Φ and $\tilde{\Phi}$. We can write down this relation as

$$\tilde{\Phi}(\lambda, \tilde{q}) = \mathcal{D}(\lambda, q, \tilde{q}), \Phi(\lambda, q)$$

and then the BT reads

$$\mathcal{D}_{,\xi} = \tilde{\mathcal{W}}\mathcal{D} - \mathcal{D}\mathcal{W}. \tag{3.12}$$

The second half of the BT, which ensures that both q and q^- satisfy equation (3.11), reads

$$\mathcal{D}_{,\eta} = \tilde{\mathcal{Z}}\mathcal{D} - \mathcal{D}\mathcal{Z}.$$

Comparing (3.12) and (3.9) we see that we can identify a BT for a completely integrable nonlinear partial differential equation with a nonlinear differential equation on the lattice [18], provided:

$$q = q_n, \quad \tilde{q} = q_{n+1}, \quad \xi = t, \quad \mathcal{W} = \mathcal{V}_n, \quad \tilde{\mathcal{W}} = \mathcal{V}_{n+1}, \quad \mathcal{D} = \mathcal{U}_n.$$

In this way we can, starting from an integrable nonlinear partial differential equation which possesses a Lax pair, construct the Lax pair for a differential difference equation which will thus possess all the integrability properties as the starting nonlinear equation. In particular we will be able to construct a set of BT which will give rise to integrable difference difference equations. In fact, by requiring that there exist a new set of solutions $(\tilde{a}_n, \tilde{b}_n)$ which satisfy the same Spectral Problem (3.8), we find that there must exist a Darboux matrix $\mathcal{D}_n(\lambda, a_n, b_n, \tilde{a}_n, \tilde{b}_n)$ such that

$$\tilde{\Phi}_n = \mathcal{D}_n(\lambda, a_n, b_n, \tilde{a}_n, \tilde{b}_n)\Phi_n. \tag{3.13}$$

In correspondence with equation (3.13) we can write down the corresponding BT

$$\mathcal{D}_{n+1}\mathcal{U}_n = \tilde{\mathcal{U}}_n\mathcal{D}_n \tag{3.14}$$

and

$$\mathcal{D}_{n,t} = \tilde{\mathcal{V}}_n\mathcal{D}_n - \mathcal{D}_n\mathcal{V}_n.$$

Under the identification

$$q_n = q_{n,m}, \quad \tilde{q}_n = q_{n,m+1}, \quad \Phi_n = \Phi_{n,m}, \quad \tilde{\Phi}_n = \Phi_{n,m+1},$$

eqs. (3.8, 3.13) give the Lax pair for the difference difference equation (3.14).

We can solve the so obtained BT to construct soliton solutions. In fact we can, in all generality, represent the Darboux matrix $\mathcal{D}_n$ (and similarly $\mathcal{D}$) in terms of the wave function Φ_n of the associated Spectral Problem in the following way

$$\mathcal{D}_n = \chi_n \left(1 - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0} P_n, \right)$$

where χ_n is a matrix corresponding to the asymptotic normalization of the spectral problem, λ_0 and μ_0 are the parameters which characterize the spectrum while P_n is a projection matrix which can be written down in terms of some constant vectors and of the solution of the spectral problem Φ calculated for $\lambda = \lambda_0$ and $\lambda = \mu_0$. Such $\mathcal{D}_n$ are particularly simple solutions of the Hilbert–Riemann problem, which is the fundamental block at the base of the spectral problem [17].

3.3 Integrable nonlinear lattices

We end this Section by presenting a list of some of the best known nonlinear integrable equations on the lattice. This list isn't meant to be exhaustive, but rather to present a set of examples. Moreover we will refer to the literature for the presentation of the complete list of relevant results.

As a first example let us present again the Toda equation (3.4). Being the first nonlinear integrable equation on the lattice discovered, it is also the more studied and, in a certain sense, is the prototype of the integrable nonlinear differential difference equations, in the same way as the Korteweg–de Vries equation is for partial differential equations. Most of the results are presented in the book by Toda [3]. This equation has been extended to the case of two-dimensional flows, giving rise to the so called *two dimensional Toda Lattice* [19]

$$x_{n,ty}(y,t) = e^{-b[x_n(y,t) - x_{n-1}(y,t)]} - e^{-b[x_{n+1}(y,t) - x_n(y,t)]},$$

also an integrable model.

The Toda lattice is just the first member of a hierarchy of nonlinear integrable differential difference equations [20] one of whose most interesting members is the *Volterra* equation

$$x_{n,t} = x_n \left(x_{n+1} - x_{n-1} \right).$$

One can introduce a discrete Zakharov–Shabat spectral problem [4]. Among the nonlinear differential difference equations we can associate to it we find the discrete Nonlinear Schrödinger equation [4]

$$i\psi_{n,t} = (\psi_{n+1} + \psi_{n-1} - 2\psi_n) + \mid \psi_n \mid^2 (\psi_{n+1} + \psi_{n-1}),$$

the discrete sine-Gordon equation [21]

$$\theta_{,t}(n+1) - \theta_{,t}(n) = \frac{1}{2}[\sin\theta(n+1) + \sin\theta(n)]$$

and the discrete modified Korteweg–de Vries equation [4]

$$q_{n,t} + (q_{n-1} + q_{n-2} - q_{n+1} - q_{n+2}) + q_n^2(q_{n-1} - q_{n+1}) + q_{n+1}^2(q_n + q_{n+2}) \\ -q_{n-1}^2(q_n + q_{n-2}) = 0.$$

All these equations can be extended to include coefficients depending explicitly on n [22–26].

A different interesting nonlinear differential difference equation, only recently discovered, is the discrete Krichever–Novikov equation [27–29]

$$\begin{aligned} u_{n,t} &= \frac{P(u_n)u_{n+1}u_{n-1} + Q(u_n)(u_{n+1} + u_{n-1}) + R(u_n)}{u_{n+1} - u_{n-1}}, \\ P(u_n) &= \alpha u_n^2 + 2\beta u_n + \gamma, \\ Q(u_n) &= \beta u_n^2 + \lambda u_n + \delta, \\ R(u_n) &= \gamma u_n^2 + 2\delta u_n + \omega, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \omega$ and λ are 6 arbitrary constants. As far as one can say up to now no algebraic properties are known for this equation apart from the fact that it possesses higher conserved densities and flows.

4 Group theory and differential difference equations

4.1 Introduction

In this Section we show how the theory of Lie groups can be used to study and solve difference equations.

The power of Lie group theory in the treatment of differential equations is well known [30–33]. Group theory is used to obtain new solutions from old ones, to reduce ordinary differential equations to lower order ones, to reduce the number of independent variables in partial differential equations, to classify equations into equivalence classes and for many other purposes.

Applications to DDE's involve the following aspects:

1. Constructing an efficient algorithm for finding the connected component of the symmetry group.

2. Performing symmetry reduction (to equations with fewer variables) using both continuous and discrete subgroups of the symmetry group.

3. The concept of *conditional symmetries*, i.e. group transformations which leave only a subclass of solutions invariant [34–36] can be extended with profit to DDE's and applied to obtain reductions of equations and also Bäcklund transformations [37].

4. It is possible to construct DDE's with prescribed symmetry groups, just as one can do for differential equations [38, 39].

5. It is possible to find transformations between given DDE's which have the same symmetry algebra, just as one can do for differential equations [40].

6. The group classification of DDE's according to their symmetries can be performed [41].

We shall discuss some of those aspects below on the example of the Toda lattice and for others one can look into the referred literature where those results are reported.

A simplified approach for constructing the symmetry group [42–45] consists in considering a function $u(x,n)$, $u \in \mathbb{R}^p$, $x \in \mathbb{R}^q$, $n \in \mathbb{Z}^m$, depending on a set of continuous variables x and a set of discrete ones n. The function is subject to a differential-difference equation of the form

$$E^{\mu}\big(x, n, u(n+k,x), u_{x_i}(n+k,x), u_{x_i x_j}(n+k,x)\big) = 0, \tag{4.1}$$

$(i,j) = 1,2,\ldots,q$, $\mu = 1,\ldots,N$, where $k \in \mathbb{Z}^m$ varies between some finite (vector) limits. Continuous and discrete symmetries are considered separately. The continuous ones are found using an infinitesimal method, a modification of the prolongation techniques used for differential equations [30-33]. Discrete symmetries, specially those acting on the discrete variables n, are then found separately, not using infinitesimal techniques, essentially by inspection. The discrete and continuous symmetries are combined into one symmetry group $G = G_D \rhd G_C$, with the (local) continuous part as an invariant subgroup. Subgroups of G can then be used for instance to perform symmetry reduction, i.e. to reduce the number of independent variables in the equation. "Non-splitting subgroups" that are themselves not semi-direct products of a subgroup of G_D with a subgroup of G_C are of particular interest in this context.

A more complete approach is when all independent variables are viewed as being continuous, however some, or all, of them vary discretely. In this context the simplified approach corresponds to the case when the continuous variables take values only on integer multiples of the spacing, starting from an arbitrary initial point. Thus, instead of derivatives, the equations will involve variations, or discrete derivatives $\Delta_x u$ with respect to the variables which vary discretely. We have

$$\Delta_x u(x) = \frac{u(x+\sigma) - u(x)}{\sigma},$$

where σ is the finite spacing in the x direction (the spacing can be different for each variable, i.e. σ_i for x_i). We also introduce a shift operator T_x with action

$$T_x u(x) = u(x+\sigma),$$

and we then have

$$\Delta_x = \frac{T_x - 1}{\sigma}.$$

In the case when the increments of all variables x_i are discrete we write the equations to be studied as

$$E^{\mu}\big(x, \sigma, T^{\alpha} u(x), T^{\beta_i} \Delta_{x_i} u, T^{\gamma_{ij}} \Delta_{x_i} \Delta_{x_j} u, \ldots\big) = 0, \tag{4.2}$$

$(i, j) = 1, \ldots, q$, $\mu = 1, \ldots, N$. with, for example

$$T^{\alpha} = T_{x_1}^{\alpha_1} T_{x_2}^{\alpha_2} \ldots T_{x_q}^{\alpha_q}, \quad a_i \leq \alpha_i \leq b_i, \quad i = 1, \ldots, q, \tag{4.3}$$

where α_i are integers, varying over some finite ranges and T^{β_i}, $T^{\gamma_{ij}}$, $\cdots$ are defined similarly. Continuous transformations among solutions of equation (4.2) are now treated infinitesimally in terms of "discrete vector fields" that can be written in the form

$$X = \left[\xi^i(x, u)\Delta_{x_i} u + \phi(x, u)\right] \delta_u. \tag{4.4}$$

Below we shall describe prolongation techniques for this case [46].

A similar approach, restricted to linear difference equations, was recently developed in terms of commuting linear difference operators [47, 48].

4.2 Differential operator symmetry algebras for difference equations

The formalism

Let us consider equation (4.1) and find its Lie point symmetries satisfying

$$\widetilde{x} = \Lambda_g(x, u_n), \quad \widetilde{u}_n = \Omega_g(x, n, u_n), \quad \widetilde{n} = n,$$

where g denotes the group parameters, Λ_g and Ω_g are some locally smooth and invertible functions, such that if $u_n(x)$ is a solution $\tilde{u}_n(\tilde{x})$ is also a solution. For simplicity, here, we consider u_n to be a scalar ($q = 1$), n a single discrete variable ($m = 1$), while x can be a vector. The variable n at this stage is not transformed ($\widetilde{n} = n$). We shall actually look for infinitesimal transformations, i.e. put

$$g = I + \varepsilon, \quad \widetilde{x} = x + \varepsilon\xi(x, u_n), \quad \widetilde{u}_n = u_n + \varepsilon\phi(x, n, u_n). \tag{4.5}$$

The requirement that solutions be transformed into solutions is best formulated in terms of vector fields

$$\widehat{X} = \xi^i(x, u_n)\partial_{x_i} + \phi(x, n, u_n)\partial_{u_n}$$

(same ξ and ϕ as in equation (4.5)). The prolongation $\mathrm{pr}^{(N)}\widehat{X}$ will have the form

$$\begin{aligned} \mathrm{pr}^{(N)}\widehat{X} \quad = \quad & \sum_{i=1}^{q} \xi_i(x, u_n)\partial_{x_i} + \sum_{k=-a}^{b} \phi(x, n+k, u_{n+k})\partial_{u_{n+k}} \\ & + \sum_{i=1}^{q} \sum_{k=-a_i}^{b_i} \phi^{x_i}(x, n+k, \ldots)\partial_{u_{n+k, x_i}} \\ & + \sum_{i,j} \sum_{k=-a_{ij}}^{b_{ij}} \phi^{x_i x_j}(x, n+k, \ldots)\partial_{u_{n+k, x_i x_j}} + \cdots . \end{aligned} \tag{4.6}$$

In equation (4.6) N is the order of the equation, ϕ^{x_i}, $\phi^{x_i x_j}$, ... are expressed in terms of total derivatives of the coefficients ϕ, and ξ just as in the purely continuous case [30-33]. The only specific feature, characterizing the fact that equation (4.1) is a differential-difference equation, is the summation over k in equation (4.6), i.e. a summation over the different values of the discrete variable, figuring in equation (4.1).

The invariance requirement for equation (4.1) is translated into the condition that $\mathrm{pr}^{(N)}\widehat{X}$ should annihilate the equation on its solution set

$$\mathrm{pr}^{(N)}\widehat{X}E\mid_{E=0}=0. \tag{4.7}$$

The determining equations for the Lie symmetries, i.e. for the coefficients ξ_i and ϕ are then read off from equation (4.7), as coefficients of linearly independent expressions in the derivatives u_{n,x_i}, $u_{n,x_i x_j}$, ... and differences, $u_{n+1}-u_n$, u_n-u_{n-1}, ...

Example of the Toda Lattice

Let's apply the techniques introduced in the previous parts of this Section to the case of equation (3.4). In this case equation (4.7) reduces to an over-determined system of determining equation obtained by equating to zero the coefficients of $[x_{n,t}]^k$ with $k=0,1,2,3$ and of $x_{n\pm1}$. They imply

$$\begin{aligned}\xi &= at+d,\\ \phi &= b+2an+ct, \quad a,b,c,d \quad \text{real constants},\end{aligned}$$

corresponding to a finite-dimensional Lie algebra generated by the vector fields:

$$\widehat{D}=t\partial_t+2n\partial_{x_n}, \quad \widehat{T}=\partial_t, \quad \widehat{W}=t\partial_{x_n}, \quad \widehat{U}=\partial_{x_n}.$$

The group transformation which will leave equation (3.4) invariant is hence

$$\tilde{x}_n(\tilde{t})=x_n(\tilde{t}e^{-\frac{\lambda_4}{2}}-\lambda_3)+\lambda_2(\tilde{t}e^{-\frac{\lambda_4}{2}}-\lambda_3)+\lambda_4 n+\lambda_1, \tag{4.8}$$

where λ_j with $j=1,2,3,4$ are arbitrary real group coefficients.

To the transformation (4.8) we can add a discrete one [49]

$$\tilde{n}=n+N \tag{4.9}$$

and consequently we can write down the symmetry group of equation (3.4) as

$$G=G_D \triangleright G_C,$$

where G_D are the discrete transformation (4.9) and the invariant subgroup G_C corresponds to the Lie algebra (4.2). A complete classification of the one-dimensional subgroups of G is obtained by complementing the Lie algebra (4.2) with the vector field $\widehat{Z}=\partial_n$ and requiring, at the end of the calculations, that the corresponding group parameter be integer. In such a way the symmetry algebra becomes

$$[\widehat{Z},\widehat{D}]=2\widehat{U}, \quad [\widehat{D},\widehat{T}]=-\widehat{T}, \quad [\widehat{D},\widehat{W}]=\widehat{W}, \quad [\widehat{T},\widehat{W}]=\widehat{U}, \tag{4.10}$$

and the one-dimensional subalgebras are:

$$\begin{array}{lll} \{\widehat{Z}+a\widehat{D}+b\widehat{U}\}, & \{\widehat{Z}+a\widehat{T}+k\widehat{W}\}, & \{\widehat{Z}+\epsilon\widehat{W}\}, \\ \{\widehat{Z}+a\widehat{U}\}, & \{\widehat{T}+a\widehat{W}\}, & \{\widehat{D}+a\widehat{U}\}, \\ \{\widehat{U}\}, & \{\widehat{Z}\}, & \{\widehat{W}\}, \end{array}$$

where $(a,b)\in R$, $a\neq 0$, $k=0,\pm 1$, $\epsilon=\pm 1$.

Nontrivial solutions, corresponding to reductions with respect to continuous subgroups, $G_0 \subset G_C$, are obtained by considering invariance of the Toda lattice under $\{\widehat{T}+a\widehat{W}\}$ and $\{\widehat{D}+a\widehat{U}\}$ and read

$$\begin{aligned} x_n(t) &= b-\frac{1}{2}at^2-\sum_{j=1}^{n}\log(c-aj), \\ x_n(t) &= b+2(n+a)\log(t)-\sum_{j=0}^{n}\log[c+(2a-1)j+j^2], \end{aligned}$$

where b and c are arbitrary integration parameters.

Reduction by a purely discrete subgroup, $G_0 \subset G_D$, implies the invariance of equation (3.4) under discrete translations of n and makes it possible to impose the periodicity condition

$$x_{n+N}(t)=x_n(t).$$

In such a case we can reduce the differential difference equation (3.4) to an ordinary differential equation. For example, for $N=2$, defining

$$x_{n+1}(t)=\frac{1}{2}(\phi(t)+\alpha t+\beta), \quad x_n(t)=-\frac{1}{2}(\phi(t)-\alpha t-\beta)$$

we get

$$\phi_{,tt}=-4\sinh(\phi),$$

while for $N=3$, by appropriate definition we get the ordinary Bullough–Dodd–Mikhailov–Shabat differential equation

$$q_{,tt}=e^{-2q}-e^{q}.$$

Let us now consider a subgroup $G_0 \subset G$ that is contained neither in G_C nor in G_D, i.e. a nonsplitting subgroup of G. In such a case, considering a reduction invariant under $\widehat{Z}+a\widehat{D}+b\widehat{U}$ we get

$$F''(y)=e^{-b}[e^{F(ye^{a})-F(y)+a}-e^{F(y)-F(ye^{-a})-a}], \tag{4.11}$$

where the symmetry variables $F(y)$ and y are given by:

$$y=te^{-an}, \quad x_n(t)=an^2+bn+F(y).$$

Under $\widehat{Z} + a\widehat{T} + k\widehat{W}$ we get:

$$F''(y) = e^{F(y+a)-F(y)} - e^{F(y)-F(y-a)} - \frac{k}{a} \tag{4.12}$$

with

$$y = t - an, \quad x_n(t) = \frac{kt^2}{2a} + F(y).$$

Equation (4.11) can be called a differential *dilation-delay* type equation; it involves one independent variable y, but the function F and its derivatives are evaluated at the point y and at the dilated points ye^a and ye^{-a}. Equation (4.12) instead is a differential delay equation which has as interesting solutions for $k = 0$ the soliton and periodic solutions of the Toda lattice.

The other two nonsplitting subgroups give rise to linear delay equations which can be solved explicitly and give trivial solutions.

The same calculations we have carried out for the Toda lattice (3.4) can be carried out for other nonlinear equations like the inhomogeneous Toda lattice [50]:

$$y_{n,\tilde{t}\tilde{t}} = ne^{y_{n-1}-y_n} - (n+1)e^{y_n - y_{n+1}} + 1. \tag{4.13}$$

In such a case we get:

$$\widehat{D} = \tilde{t}\partial_{\tilde{t}} + (\tilde{t}^2 + 2n)\partial_{y_n}, \quad \widehat{T} = \partial_{\tilde{t}} + \tilde{t}\partial_{y_n}, \quad \widehat{W} = \tilde{t}\partial_{y_n}, \quad \widehat{U} = \partial_{y_n}.$$

These vector fields have the same symmetry algebra as (4.10), i.e. that of the Toda lattice (3.4). This is a necessary condition for the existence of a point transformation between the two equations. In fact, by comparing the two sets of vectors, we get the following transformation between a solution $x_n(t)$ of (3.4) and one $y_n(\tilde{t})$ of (4.13)

$$\tilde{t} = t; \quad y_n(\tilde{t}) = x_n(t) + \frac{1}{2}\tilde{t}^2 + \log(n!).$$

Group classification of discrete dynamical systems

As an application of the methods of Section 4.2.1, let us consider the differential difference equation

$$E_n \equiv \ddot{u}_n(t) - F_n\big(t, u_{n-1}(t), u_n(t), u_{n+1}(t)\big) = 0. \tag{4.14}$$

The equation is of the form (4.1) with $N = p = q = m = 1$. We wish to classify such equations into conjugacy classes, where each class is characterized by the symmetry group of the equations in the class. The classification will be under "allowed transformations" that take equation (4.14) into an equation of the same form, but possibly with a different function $\widetilde{F}_n\big(\tilde{t}, \widetilde{u}_{n-1}, \widetilde{u}_n, \widetilde{u}_{n+1}\big)$. We restrict the allowed transformations to the form

$$u_n(t) = \Omega_n(\widetilde{u}_n(\tilde{t}), t, g), \quad \tilde{t} = \tilde{t}(t, g), \quad \widetilde{n} = n, \tag{4.15}$$

where Ω_n and $\tilde{t}$ are some locally smooth and invertible functions; g are group parameters. Lie point symmetries of equation (4.6) will be realized by vector fields as in equation (4.2), or more specifically

$$\widehat{X} = \tau(t, u_n)\partial_t + \phi_n(t, u_n)\partial_{u_n}. \tag{4.16}$$

The invariance condition is as in equation (4.7).

Equations of the form (4.14) arise in physical applications, for instance in studies of molecular chains [5, 7]. The interaction functions F_n are usually not completely specified in the models considered. A symmetry classification of these interactions can be used to single out particularly tractable, or even integrable models.

We make several restrictions in our analysis. The interaction F_n involves only nearest neighbours (u_{n+k} with $k = 0, \pm 1$). The interaction is nonlinear and coupled, i.e.

$$\frac{\partial^2 F_n}{\partial u_i \partial u_k} \neq 0, \quad \left(\frac{\partial F_n}{\partial u_{n-1}}, \frac{\partial F_n}{\partial u_{n+1}}\right) \neq (0, 0)$$

(the first relation holds for at least one pair i, k).

To calculate the Lie point symmetry algebra of equation (4.14) we calculate the relevant terms of $\mathrm{pr}^{(2)}\widehat{X}$:

$$\begin{aligned} \mathrm{pr}^{(2)}\widehat{X} &= \tau(t, u_n)\partial_t + \sum_{k=n-1}^{n+1} \phi_k(t, u_k)\partial_{u_k} + \phi_n^{tt}\partial_{u_{n,tt}}, \\ \phi_n^{tt} &= D_t^2\phi_n - (D_t^2\tau)u_{n,t} - 2(D_t\tau)u_{n,tt}. \end{aligned}$$

Applying the algorithm (4.7) and substituting for $u_{n,tt}$ from equation (4.14), we find that the vector field (4.16) has the form

$$\widehat{X} = \tau(t)\partial_t + \left[\left(\frac{1}{2}\dot{\tau}(t) + a_n\right)u_n + \beta_n(t)\right]\partial_{u_n}, \ \dot{a}_n = 0. \tag{4.17}$$

The interaction function F_n, the function of t, $\tau(t)$, the function of n and t, $\beta_n(t)$, and the function of n, a_n, are related by one remaining determining equation, namely

$$\begin{aligned} &\frac{1}{2}\dddot{\tau}u_n + \ddot{\beta}_n + \left(a_n - \frac{3}{2}\dot{\tau}\right)F_n - \tau F_{n,t} \\ &\qquad - \sum_{\alpha}\left[\left(\frac{1}{2}\dot{\tau} + a_\alpha\right)u_\alpha + \beta_\alpha\right]F_{n,u_\alpha} = 0. \end{aligned} \tag{4.18}$$

The allowed transformations (4.15) turn out to have the form

$$u_n(t) = \frac{A_n}{\sqrt{\tilde{t}_t}}\tilde{u}_n(\tilde{t}) + B_n(t), \quad \tilde{t} = \tilde{t}(t), \quad A_{n,t} = 0, \quad \tilde{t}_t \neq 0, \quad A_n \neq 0, \quad \tilde{n} = n.$$

where $B_n(t)$, A_n and $\tilde{t}(t)$ can be freely chosen. Allowed transformations will simultaneously transform the equation (4.14) and the elements (4.17) of its symmetry algebra.

Since F_n is a priori not specified, we find it more efficient to use the allowed transformations to simplify the vector fields.

Indeed, let us assume that the interaction F_n is such that a nontrivial symmetry algebra L exists, at least a one-dimensional one.

We choose an element $\widehat{X} \in L$; it will have the form (4.17) with coefficients satisfying equation (4.18). Allowed transformations take $\widehat{X}$ into

$$\widehat{X} = \tau(t)\tilde{t}_t \partial_{\tilde{t}} + \left[\frac{\tau}{2}\tilde{t}_{tt}(\tilde{t}_t)^{-1} + \frac{1}{2}\dot{\tau} + a_n\right]\widetilde{u}_n + (\tilde{t}_t)^{1/2} A_n^{-1}\left[\left(\frac{1}{2}\dot{\tau} + a_n\right)B_n + \beta_n - \tau B_{n,t}\right]\partial_{\widetilde{u}_n}.$$

where $\tau(t)$, a_n and $\beta_n(t)$ are to be viewed as given, $\tilde{t}(t)$, A_n and $B_n(t)$ as arbitrary. Using this freedom we can transform the given vector field $\widehat{X}$ into one of three "standard" forms, each of them quite simple. We then view equation (4.18) as a first order linear partial differential equation for F_n and solve it, using the method of characteristics.

This way we find that three different one-parameter symmetry algebras exist. Together with the corresponding interactions, they are represented as follows

$$A_{1,1} : \widehat{X} = \partial_t + a_n u_n \partial_{u_n} \tag{4.19}$$
$$F_n = e^{a_n t} f_n(\xi_{n-1}, \xi_n, \xi_{n+1}), \qquad \xi_k = u_k e^{-a_k t}, \qquad k = n, n \pm 1$$
$$A_{1,2} : \widehat{X} = a_n u_n \partial_{u_n} \tag{4.20}$$
$$F_n = u_n f_n(t, \xi_{n-1}, \xi_{n+1}), \qquad \xi_k = u_k^{a_n} u_n^{-a_k}, \qquad k = n \pm 1$$
$$A_{1,3} : \widehat{X} = \beta_n(t)\partial_{u_n} \tag{4.21}$$
$$F_n = \frac{\ddot{\beta}_n}{\beta_n} u_n + f_n(t, \xi_{n-1}, \xi_{n+1}), \quad \xi_k = \beta_n(t) u_k - \beta_k(t) u_n, \quad k = n \pm 1$$

The algebras $A_{1,1}$, $A_{1,2}$, $A_{1,3}$ correspond to a vector field $\widehat{X}$, originally with $\tau(t) \neq 0$; with $\tau \equiv 0$, $a_n \neq 0$; and with $\tau = 0$, $a_n = 0$, $\beta_n(t) \neq 0$, respectively.

We see that while F_n originally depended on 4 variables, the existence of a one-dimensional symmetry algebra restricts the interaction to one involving an arbitrary function of 3 symmetry variables.

Equations (4.19), (4.20) and (4.21) are the key to a complete symmetry classification for equation (4.14). Indeed, we can first find all interactions with abelian symmetry algebras, starting from each of the three algebras $A_{1,1}$, $A_{1,2}$ and $A_{1,3}$. Thus, element $\widehat{X}_1$ is already in standard form. Further operators $\widehat{X}_j$ are then constructed, commuting with $\widehat{X}_1$ (and amongst each other for $\dim L > 2$). The interactions F_n, or rather the functions f_n, are further constrained, using the additional determining equations (4.18). It turns out [43] that abelian symmetry algebras L exist for nonlinear equations and their dimensions satisfy $1 \leq \dim L \leq 4$. Once the abelian symmetry algebras $A_{n,i}$ are known, we can use them to construct nilpotent nonabelian symmetry algebras. The abelian ones will figure as maximal abelian ideals in the nilpotent ones. In turn, abelian and nilpotent symmetry algebras will serve as nilradicals (maximal nilpotent ideals) in solvable Lie algebras. Finally, nonsolvable symmetry algebras of equation (4.14) will always have a Levi decomposition, with $\mathrm{sl}(2,\mathbb{R})$ as the Levi factor [51].

For complete analysis, see [43]. Here we shall just present some examples of systems of the type (4.14) having interesting symmetry algebras (and groups).

First of all, the symmetry algebra L is at most 7-dimensional. The highest dimensions, $\dim L = 6$ and 7, occur for interactions F_n depending in a prescribed way on the dependent functions u_{n-1}, u_n and u_{n+1} namely via a symmetry variable ξ with

$$\xi = (\gamma_n - \gamma_{n+1})u_{n-1} + (\gamma_{n+1} - \gamma_{n-1})u_n + (\gamma_{n-1} - \gamma_n)u_{n+1}, \tag{4.22}$$

$\gamma_{n+1} \neq \gamma_n$, $\dot{\gamma}_n = 0$.

The $\dim L = 7$ symmetry algebra occurs for $F_n = \xi^{-3}$ and we have

$$\begin{aligned} &\widehat{X}_1 = \partial_t, \quad &&\widehat{X}_2 = t\partial_t + \frac{1}{2}u_n\partial_{u_n}, \quad &&\widehat{X}_3 = t^2\partial_t + tu_n\partial_{u_n} \\ &\widehat{X}_4 = \partial_{u_n}, \quad &&\widehat{X}_5 = t\partial_{u_n}, &&\widehat{X}_6 = \gamma_n\partial_{u_n}, \\ & &&\widehat{X}_7 = t\gamma_n\partial_{u_n} \end{aligned} \tag{4.23}$$

The algebra has a nontrivial Levi decomposition $L \sim \{\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\} \rhd \{\widehat{X}_4, \widehat{X}_5, \widehat{X}_6, \widehat{X}_7\}$ with $\{\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\} \sim sl(2,\mathbb{R})$ and $\{\widehat{X}_4, \widehat{X}_5, \widehat{X}_6, \widehat{X}_7\}$ abelian.

One of the four-dimensional symmetry algebras is of particular interest. The interaction involves an arbitrary function of ξ with ξ as in equation (4.22).

We have

$$\begin{aligned} &F_n = \exp\left(-2\frac{u_{n+1} - u_n}{\gamma_{n+1} - \gamma_n}\right) f_n(\xi), \\ &\widehat{X}_1 = \partial_{u_n}, \quad \widehat{X}_2 = \partial_t, \quad \widehat{X}_3 = t\partial_{u_n}, \quad \widehat{Y} = t\partial_t + \gamma_n\partial_{u_n}. \end{aligned} \tag{4.24}$$

The algebra is solvable with a nilpotent nilradical $\{\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\}$.

The Toda lattice

$$\ddot{u}_n = e^{u_{n-1} - u_n} - e^{u_n - u_{n+1}}$$

corresponds to a special case of the interaction (4.24), and is obtained by setting

$$\gamma_n = 2n, \quad f_n(\xi) = -1 + e^{1/2\xi}.$$

Thus, the integrable Toda lattice [3] is not singled out by its Lie point symmetry group. It occurs in a family, involving an arbitrary function of ξ and n, and an arbitrary function of n ($f_n(\xi)$ and γ_n, respectively).

Using the techniques presented in [27] one can investigate the integrability properties of all these equations. This, however, is out of the scope of this presentation and will be done elsewhere.

4.3 Difference operator symmetry algebras for difference equations

The formalism

We shall here consider equations of the form (4.2) and symmetry algebras realized by "discrete vector fields", as in equation (4.4). For simplicity (mainly of notation) we consider only scalar equations for a single dependent variable u. The independent variables $x = (x_1, \ldots, x_q)$ are continuous, their increments discrete.

When studying Lie point symmetries of differential equations, we can either use the formalism of ordinary vector fields

$$\widehat{X} = \sum_{i=1}^{q} \xi_i(x,u)\partial_{x_i} + \phi(x,u)\partial_u,$$

or the equivalent one of "evolutionary vector fields"

$$\widehat{X}_e = -Q\partial_u, \quad Q = \xi_i(x,u)u_{x_i} - \phi(x,u). \tag{4.25}$$

In this formalism the determining equations are again read of from the equation

$$\mathrm{pr}^N \widehat{X}_e E \mid_{E=0} = 0. \tag{4.26}$$

and the prolongation formula is

$$\begin{aligned} \mathrm{pr}^N \widehat{X}_e &= -Q\partial_u - Q^{x_i}\partial_{u_{x_i}} - Q^{x_i x_j}\partial_{u_{x_i x_j}} - \ldots, \\ Q^{x_i} &= D_{x_i}Q, \\ Q^{x_i x_j} &= D_{x_i}D_{x_j}Q, \quad \ldots \end{aligned}$$

where D_{x_i} are total derivatives. Commutators of evolutionary fields can be calculated by commuting their first prolongations and then projecting the result onto the Lie algebra.

It is the formalism of evolutionary vector fields that is best adapted for the case of difference equations. In this case we have

$$\widehat{X}_e = -Q\partial_u, \quad Q = \xi_i(x, T^a u)T^b \Delta_{x_i} u + \phi(x, T^c u), \tag{4.27}$$

with T^a, ... as in equation (4.3). In this formulation, the difference between Q in equation (4.27) and in the continuous case (4.25) is the presence of the shift operator T_x in ξ_i and ϕ and $\Delta_{x_i} u$ instead of u_{x_i}.

The algorithm for obtaining the determining equations is given in equation (4.26), once the prolongation of the discrete evolutionary vector field $\widehat{X}_e$ can be easily constructed.

The transformation generated by $\widehat{X}_e$ will take functions $u(x)$ into $\widetilde{u}(\widetilde{x})$. The transformations generated by $\mathrm{pr}^N X_e$ must also transform the variations $\Delta_{x_i} u(x)$, $\Delta_{x_i}\Delta_{x_k} u(x)$, ... into the variations of $\widetilde{u}$ with respect to $\widetilde{x}$ (up to order N) and to do this simultaneously at all points of the lattice figuring in the equation.

The appropriate prolongation formula is

$$\mathrm{pr}^N X_e = -\Big\{ \sum_{\alpha} (T^{\alpha} Q) \partial_{T^{\alpha} u} + \sum_{\beta_i} T^{\beta_i} Q^{x_i} \partial_{T^{\beta_i} \Delta_{x_i} u} + \sum_{\gamma_{ij}} T^{\gamma_{ij}} Q^{x_i x_j} \partial_{T^{\gamma_{ij}} \Delta_{x_i} \Delta_{x_j} u} + \dots \Big\}. \tag{4.28}$$

where the summations are over all sites figuring in the equation (4.2).

In equation (4.28) we have

$$Q^{x_i} = \Delta^T_{x_i} Q, \quad Q^{x_i x_j} = \Delta^T_{x_i} \Delta^T_{x_j} Q,$$

where Δ^T_x is the total variation operator, acting on functions of x, u, $\Delta_x u$, ... as

$$\Delta^T_x f\big(x, u(x), \Delta_x u(x), \dots\big) = \frac{1}{\sigma}\Big[f\big(x+\sigma, u(x+\sigma), (\Delta_x u)(x+\sigma), \dots\big) - f\big(x, u(x), (\Delta_x u)(x), \dots\big)\Big],$$

while

$$\Delta_x f\big(x, u(x), \Delta_x u(x), \dots\big) = \frac{1}{\sigma}\Big[f\big(x+\sigma, u(x), (\Delta_x u)(x), \dots\big) - f\big(x, u(x), (\Delta_x u)(x), \dots\big)\Big].$$

Example of the discrete heat equation

Let us consider the equation

$$\big[\Delta_t - \Delta_{xx} + g(x, t, T_x, T_t)\big] u(x,t) = 0, \tag{4.29}$$

where g is some given function of its arguments (g is actually an operator, acting on $u(x,t)$). We apply the linear formalism, putting

$$\widehat{X}_e = -[\tau \Delta_t u + \xi \Delta_x u + f u]\partial_u = -Q\partial_u$$

with τ, ξ and f functions of x, t, T_x, T_t. Applying $\mathrm{pr}^2 \widehat{X}_e$ to equation (4.29) and elimination $\Delta_{xx} u$, $\Delta_{xxx} u$, and $\Delta_{xxt} u$, using equation (4.29) and its consequences, we obtain an expression depending linearly on $\Delta_{xt} u$, $\Delta_t u$, $\Delta_x u$ and u. Requesting that the coefficient of each of these expressions should vanish, we obtain four determining equations:

$$\begin{aligned} 0 &= \Delta_x \tau, \\ 0 &= -(\Delta_t \tau) T_t + 2(\Delta_x \xi) T_x + [g, \tau], \\ 0 &= -(\Delta_t \xi) T_t + (\Delta_{xx} \xi) T_x^2 + 2(\Delta_x f) T_x + [\xi, g], \\ 0 &= -(\Delta_t f) T_t + (\Delta_{xx} f) T_x^2 + 2(\Delta_x \xi) T_x g + \xi(\Delta_x g) T_x + \tau(\Delta_t g) T_t + [f, g]. \end{aligned} \tag{4.30}$$

Once g is specified, equations (4.30) can be solved. The task simplifies significantly if the commutators $[g, \tau]$, $[\xi, g]$ and $[f, g]$ vanish.

Let us consider three examples, all of which have interesting continuous limits.

A. The "free" discrete head equation: $g = 0$ The solution is

$$
\begin{aligned}
\tau &= t^{(2)}\tau_2 + t\tau_1 + \tau_0,\\
\xi &= \frac{1}{2}x(\tau_1 + 2t\tau_2)T_tT_x^{-1} + t\xi_1 + \xi_0,\\
f &= \frac{1}{4}x^{(2)}\tau_2T_t^2T_x^{-2} + \frac{1}{2}t\tau_2T_t + \frac{1}{2}x\xi_1T_tT_x^{-1} + \gamma,
\end{aligned}
$$

Here and below we are making use of factorial powers $x^{(n)}$, introduced in (2.5), which satisfy the following equations:

$$
\begin{aligned}
\Delta_x x^{(n)} &= nx^{(n-1)}, \qquad n \in \mathbb{Z}^{\geq},\\
x^{(n)} &= x(x-\sigma_x)\dots\big(x-(n-1)\sigma_x\big),
\end{aligned}
$$

$$
\begin{aligned}
\Delta_x x^{(-n)} &= -nx^{(-n-1)},\\
x^{(-n)} &= \frac{1}{x(x+\sigma_x)\cdots(x+(n-1)\sigma_x)}.
\end{aligned}
$$

The integration "constants" are functions of T_x and T_t and thus they are actually shift operators. The symmetry algebra is hence infinite-dimensional. This is due to the fact that the shift operators are also symmetries: if $u(x,t)$ is a solution, then so is $T_x^nT_t^ku(x,t)$, $(n,k \in \mathbb{Z})$.

By appropriately choosing the coefficients τ_i, ξ_i and γ we can pick out a six-dimensional subalgebra of the symmetry algebra, isomorphic to the symmetry algebra of the continuous heat equation.

B. The discrete harmonic oscillator potential $g = k^2X^{(2)}T_x^{-2}$, $k \in \mathbb{R}^+$ As in the continuous case, the symmetry algebra is isomorphic to that of the free heat equation. To solve the determining equations, we must introduce a "discrete exponential" function $E_m(t)$ satisfying

$$
\begin{aligned}
\Delta_t E_m(t) &= mE_m(t)T_t^{-1},\\
E_m(t) &= (1 + mT_t^{-1}\sigma_t)^{t/\sigma_t}.
\end{aligned}
$$

The closed symmetry algebra is generated by the following discrete evolutionary fields:

$$
\begin{aligned}
P_0 &= \tau_0\Delta_t u\partial_u, \quad L = \gamma u\partial_u,\\
T_1 &= E_{4k}(t)\tau_1[\Delta_t u + 2kxT_x^{-1}\Delta_x u + 2k^2x^{(2)}T_x^{-2}u + ku]\partial_u,\\
T_2 &= E_{-4k}(t)\tau_2[\Delta_t u - 2kxT_x^{-1}\Delta_x u + 2k^2x^{(2)}T_x^{-2}u - ku]\partial_u,\\
A_1 &= E_{2k}(t)\alpha_1(\Delta_x u + kxT_x^{-1}u)\partial_u,\\
A_2 &= E_{-2k}(t)\alpha_2(\Delta_x u - kxT_x^{-1}u)\partial_u.
\end{aligned}
$$

For more details on the calculations and other examples, see [46].

5 Conclusions

To end this review on results on equations on the lattice let's go back to the physical example we considered in the Introduction and let us point out some of the open problems that we usually face when solving equations on the lattice. The molecular chain (1.1) has been solved in the given literature in two ways:

1. By carrying out a complete discretization of the system.

2. By carrying out a continuum limit.

At least three questions come naturally to the mind:

1. How can one say if the numerical solution of the discretized equation is a good approximation of the real solution?

2. How can one control it if one does not have sufficient conserved quantities?

3. How can one recover the richness of the discrete model when one carries out the continuum limit?

One of the possible answers to the first problem relays in the construction of an approximation scheme which allowed us to approximate a dynamical system by a simpler one, to which it should reduce in the asymptotic limit. Such a result has not been up to now developed, as any asymptotic analysis of an equation on the lattice give rise to a differential equation as it turns a discrete variable to a continuous one. However, on this, work is in progress [52].

A second direction of research in this field, which could provide some results of interest, consists in classifying all *integrable* nonlinear differential difference equations of possible physical interest. A step in this direction has been reported in Section 4.

To conclude I would like to comment on the third point concerning the richness of the world of the differential difference equations with respect to that of partial differential equations by presenting a result recently obtained by MacKay and Aubry [53]. In a theorem MacKay and Aubry showed that dynamical chains are much richer than their continuum counterparts; almost any Hamiltonian chain of weakly coupled oscillators has a 'breather' solution while the existence of breathers for a nonlinear wave equation is rare. This implies that the discrete world can be richer of interesting solutions and thus worthwhile studying by itself.

Acknowledgements. The author would like to thank the C.R.M. of the Université de Montréal and the Institute of Theoretical Physics of Warsaw University for its hospitality.

The results presented in this review paper are part of the results obtained by the author together with R. Benguria, M. Bruschi, O. Ragnisco, A. Sym, L. Vinet and P. Winternitz; I thank them all for many helpful discussions.

REFERENCES

[1] W. G. Kelley and A. C. Peterson: *Difference equations: an introduction with applications*, Academic Press, New York 1991.

[2] R. P. Agarwal: *Difference equations and inequalities: theory, methods and applications*, Dekker, New York 1992.

[3] M. Toda: *Theory of Nonlinear Lattice* Springer, New York 1988.

[4] M. J. Ablowitz and H. Segur: *Solitons and the Inverse Scattering Transform*, SIAM, Providence 1981.

[5] A. Campa, A. Giansanti, A. Tenenbaum, D. Levi, and O. Ragnisco: *Phys. Rev.* **48** (1993), 10168.

[6] A. S. Davydov: *Phys. Scr.* **20** (1979), 387; *Biology and Quantum Mechanics* (Pergamon, New York 1982).

[7] A. C. Scott: *Phys. Rep.* **217** (1992), 1.

[8] E. Fermi, J. Pasta and S. Ulam: *Los Alamos RpT.* **LA-1940**, (1955); *Collected Papers of Enrico Fermi* (University of Chicago Press, Chicago 1965) Vol. II, p. 978.

[9] N. J. Zabusky and M. D. Kruskal: *Phys. Rev. Lett.* **15** (1963), 240.

[10] D. J. Korteweg and G. de Vries: *Philos. Mag.* **39** (1895), 422.

[11] C. S. Gardner, J. M. Green, M. D. Kruskal and R. M. Miura: *Phys. Rev. Lett.* **19** (1967), 1095.

[12] M. Toda: *J. Phys. Soc. Japan* **22** (1967), 431; **23** (1967), 501.

[13] H. Flaschka: *Progr. Theor. Phys.* **51** (1974), 1924.

[14] S. V. Manakov: *Zhur. Eks. Teor. Fiz.* **67** (1974), 543.

[15] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov: *Theory of Solitons, the inverse scattering transform*,(Consultant Bureau, New York 1984), Ch.I, Sect.7.

[16] D. Levi, O. Ragnisco and M. Bruschi: *Il Nuovo Cimento* **58A** (1980), 56.

[17] V. E. Zakharov and A. B. Shabat: *Funkts. Analiz* **13** (1979), 13.

[18] D. Levi and R. Benguria: *Proc. Natl. Acad. Sci. U.S.A.* **77** (1980), 5025.

[19] A. V. Mikhailov: *Pisma Zh. Eksp. Teor. Fiz.* **77** (1978), 2.

[20] S.C. Chiu and J.F. Ladik: *J. Math. Phys.* **18** (1977), 690.

[21] D. Levi and L. Pilloni: *Phys. Lett.***92A** (1982), 5.

[22] D. Levi and O. Ragnisco: *Lett. Nuovo Cimento* **22** (1978), 691.

[23] D. Levi and O. Ragnisco: *J. Phys. A: Math. Gen.* **12** (1979), L157.

[24] D. Levi and O. Ragnisco: *J. Phys. A: Math. Gen.* **12** (1979), L163.

[25] M. Bruschi, D. Levi and O. Ragnisco: *Il Nuovo Cimento* **48A** (1978), 213.

[26] M. Bruschi, D. Levi and O. Ragnisco: *Il Nuovo Cimento* **53A** (1979), 21.

[27] D. Levi and R. I. Yamilov: *J. Math. Phys.* **38** (1997), 6648.

[28] R. I. Yamilov: *Uspekhi Mat. Nauk* **38** (1983), 155.

[29] I. M. Krichever and S. P. Novikov: *Russian Math. Surveys* **35** (1980), 53.

[30] P. J. Olver: *Applications of Lie Groups to Differential Equations*, Springer, New York, 1991.

[31] P. Winternitz: *Lie Groups and Solutions of Nonlinear Partial Differential Equations* in *Integrable Systems, Quantum Groups and Quantum Field Theories*, Kluwer, Dordrecht, 1993.

[32] G. W. Bluman and S. Kumei: *Symmetries and Differential Equations* Springer, New York, 1989.

[33] G. Gaeta: *Nonlinear Symmetries and Nonlinear Equations*, Kluwer, Dordrecht, 1994.

[34] G. W. Bluman and J. D. Cole: *J. Math. Mech.* **18** (1969), 1025.

[35] P. J. Olver and Ph. Rosenau: *SIAM J. Appl. Math.* **47** (1987), 263.

[36] D. Levi and P. Winternitz: *J. Phys. A:Math. Gen.* **22** (1989), 2915.

[37] A. M. Grundland and G. Rideau: in *Modern Group Theoretical Methods in Physics*, edited by J. Bertrand et. al., Kluwer Academic Press, Dordrecht 1995, p. 167.

[38] D. David, D. Levi and P. Winternitz: *Phys. Lett.* **A129** (1988), 161.

[39] G. Rideau and P. Winternitz: *J. Math. Phys.* **31** (1990), 1095.
[40] D. Levi and P. Winternitz: *Phys. Lett.* **A129** (1988), 165.
[41] J. P. Gazeau and P. Winternitz: *J. Math. Phys.* **33** (1992), 4087.
[42] D. Levi and P. Winternitz: *Phys. Lett.* **A152** (1991), 335; *J. Math. Phys.* **34** (1993), 3713.
[43] D. Levi and P. Winternitz: *J. Math. Phys.* **37**, 5551–5576.
[44] S. Maeda: *Math. Jap.* **25** (1980), 405, **26** (1981), 85, *IMA J. Appl. Math.* **38** (1987), 129.
[45] G. R. W. Quispel, H. W. Capel, and R. Sahadevan: *Phys. Lett.* **A170** (1992), 379.
[46] D. Levi, L. Vinet and P. Winternitz: *J. Phys. A: Math. Gen.* **30**, (1997), 633–649.
[47] R. Floreanini, J. Negro, L. M. Nieto and L. Vinet: *Lett. in Math. Phys.* **36** (1996), 351.
[48] R. Floreanini and L. Vinet: *J. Math. Phys.* **36** (1995), 7024.
[49] D. Levi and M. A. Rodriguez: J. Phys. A: Math. Gen. **25** (1992), L975.
[50] D. Levi and O. Ragnisco: *J. Phys. A: Math. Gen.* **24** (1991), 1729.
[51] N. Jacobson: *Lie Algebras* Dover, New York, 1962.
[52] D. Levi and P. M. Santini: *Asymptotic perturbation of discrete systems, work in progress.*
[53] R. S. MacKay and S. Aubry: *Nonlinearity* **7** (1994), 623.