Equi-centroaffinely homogeneous surfaces with vanishing Pick invariant in $\mathbb{R}^4$

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Abstract: A nondegenerate equi-centroaffine surface in $\mathbb{R}^4$ is called homogeneous if for any two points p and q on the surface there exists an equi-centroaffine transformation in $\mathbb{R}^4$ which takes the surface to itself and takes p to q. In this paper we determine the definite equi-centroaffinely homogeneous surfaces with vanishing Pick invariant in $\mathbb{R}^4$ up to centroaffine transformations.

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1 Equi-centroaffine surfaces in $\mathbb{R}^4$

The equi-centroaffine differential geometry of surfaces in $\mathbb{R}^4$ is the study of the properties of surfaces which are invariant under the equi-centroaffine group Γ in $\mathbb{R}^4$. Γ is the subgroup of the equiaffine transformation group in $\mathbb{R}^4$ which keeps the origin $0 \in \mathbb{R}^4$ invariant.

Let $x : \mathbf{M} \to \mathbb{R}^4$ be a oriented, immersed surface such that $x(p) \notin \mathrm{d}x(T_pM)$ for all $p \in \mathbf{M}$ and $x(\mathbf{M})$ does not lie in a hyperplane of $\mathbb{R}^4$. For any oriented, local basis $\sigma = \{E_1, E_2\}$ of TM with the dual basis $\{\theta_1, \theta_2\}$ we define

$$G^\sigma = [E_1(x), E_2(x), x, \mathrm{d}^2x] = \sum_{i,j=1}^{2} [E_1(x), E_2(x), x, E_iE_j(x)]\theta_i \otimes \theta_j, \tag{1.1}$$

where $[\ \ ]$ is the standard determinant in $\mathbb{R}^4$. G^σ is a symmetric 2-form. We assume that G^σ is nondegenerate. If τ is another local basis of TM, we have

$$|\det G^\tau|^{-\frac{1}{4}}\, G^\tau = |\det G^\sigma|^{-\frac{1}{4}}\, G^\sigma,$$

where $|\ \ |$ denotes the absolute value of the real number (or the norm of the complex number). Therefore

$$G = |\det G^\sigma|^{-\frac{1}{4}}\, G^\sigma \tag{1.2}$$

is independent of the choice of the basis σ and thus a globally defined symmetric 2-form.

From (1.1) we know that G is an invariant under equi-centroaffine transformations. G is called the equi-centroaffine metric of $x : \mathbf{M} \to \mathbb{R}^4$. Let Δ denote the Laplacian of G. $\{x, \Delta x\}$ is called the equi-centroaffine normalization of $x : \mathbf{M} \to \mathbb{R}^4$ ([4]).

Remark: The definitions of equi-centroaffine metric G of $x : \mathbf{M} \to \mathbb{R}^4$ here and the equiaffine metric of $x : \mathbf{M} \to \mathbb{R}^4$ in [8] are different. Some equi-centroaffine surfaces in $\mathbb{R}^4$ are degenerate surfaces under equiaffine metric, cf. [4].

We call x a nondegenerate equi-centroaffine surface if G^σ is nondegenerate. x is called definite or indefinite if G^σ is definite or indefinite, respectively. In this paper we consider only definite surfaces in $\mathbb{R}^4$.

For the equi-centroaffine surface x, let $\nabla = \{\Gamma_{ij}^k\}$ and $\widehat{\nabla} = \{\widehat{\Gamma}_{ij}^k\}$ be the induced connection and the Levi–Civita connection of the equi-centroaffine metric G. We define the Fubini–Pick form C by

$$C_{ij}^k := \Gamma_{ij}^k - \widehat{\Gamma}_{ij}^k, \quad i, j, k = 1, 2. \tag{1.3}$$

We know that C_{ij}^k is symmetric for i and j. The Pick invariant is defined by

$$J = \frac{1}{2} G^{ij} C_{ik}^l C_{lj}^k. \tag{1.4}$$

Let $x : \mathbf{M} \to \mathbb{R}^4$ be an equi-centroaffine surface with definite equi-centroaffine metric G. We choose a local basis $\sigma = \{E_1, E_2\}$ of TM such that

$$G = 2\epsilon e^w (\mathrm{d}u^2 + \mathrm{d}v^2), \tag{1.5}$$

where $\epsilon = \pm 1$. Then $\{E_1(x), E_2(x), x, \Delta x\}$ forms a local moving frame for $\mathbb{R}^4$ along $\mathbf{M}$, where Δ is the Laplacian of G. Since $\{G^\sigma\}$ defines a conformal structure on $\mathbf{M}$, in each point we can introduce a complex coordinate z such that $z = u + iv$. We denote

$$\begin{array}{ll} x_z = \frac{1}{2}(x_u - i x_v), & x_{\bar{z}} = \frac{1}{2}(x_u + i x_v), \\ \mathrm{d}z = \mathrm{d}u + i\mathrm{d}v, & \mathrm{d}\bar{z} = \mathrm{d}u - i\mathrm{d}v. \end{array}$$

Then (1.5) can be written as

$$G = 2\epsilon e^w (\mathrm{d}u^2 + \mathrm{d}v^2) = 2\epsilon e^w |\mathrm{d}z|^2 = \epsilon e^w (\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z). \tag{1.6}$$

Therefore $[x_z, x_{\bar{z}}, x, x_{zz}] = [x_z, x_{\bar{z}}, x, x_{\bar{z}\bar{z}}] = 0$ and

$$\begin{aligned} G^\sigma &= [E_1(x), E_2(x), x, \mathrm{d}^2 x] \\ &= [x_u, x_v, x, \mathrm{d}^2 x] \\ &= -2i[x_z, x_{\bar{z}}, x, x_{z\bar{z}}](\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z). \end{aligned} \tag{1.7}$$

By (1.2), (1.6) and (1.7) we obtain

$$-2i[x_z, x_{\bar{z}}, x, x_{z\bar{z}}] = \epsilon e^{2w}. \tag{1.8}$$

We assume that $x_{zz} = \theta x_z + e^{-w} \varphi x_{\bar{z}} + \psi x$. (1.8) yields $\theta = w_z$. Hence

$$x_{zz} = w_z x_z + e^{-w} \varphi x_{\bar{z}} + \psi x. \tag{1.9}$$

PROPOSITION 1.1: *Let Φ, Ψ be the forms on* $\mathbf{M}$ *defined by*

$$\Phi = \varphi \mathrm{d}z^3 = e^w \frac{[x_z, x_{zz}, x, x_{z\bar{z}}]}{[x_z, x_{\bar{z}}, x, x_{z\bar{z}}]} \mathrm{d}z^3, \tag{1.10}$$

$$\Psi = \psi \mathrm{d}z^2 = \frac{[x_z, x_{\bar{z}}, x_{zz}, x_{z\bar{z}}]}{[x_z, x_{\bar{z}}, x, x_{z\bar{z}}]} \mathrm{d}z^2. \tag{1.11}$$

Then Φ, Ψ are globally defined on $\mathbf{M}$ *and equi-centroaffine invariants.*

PROPOSITION 1.2: (i) *If the definite equi-centroaffine surfaces $x, \tilde{x} : \mathbf{M} \to \mathbb{R}^4$ are equi-centroaffinely equivalent, then*

$$G = \tilde{G}, \qquad \Phi = \tilde{\Phi}, \qquad \Psi = \tilde{\Psi}. \tag{1.12}$$

(ii) *Conversely, if $x, \tilde{x} : \mathbf{M} \to \mathbb{R}^4$ are definite equi-centroaffine surfaces such that* (1.12) *holds, then x and $\tilde{x}$ are equi-centroaffinely equivalent.*

In the following, we derive the integrability conditions for the equi-centroaffine surface $x : \mathbf{M} \to \mathbb{R}^4$. From (1.9) we have

$$\begin{cases} x_{zz} = w_z x_z + e^{-w} \varphi x_{\bar{z}} + \psi x, \\ x_{\bar{z}\bar{z}} = w_{\bar{z}} x_{\bar{z}} + e^{-w} \bar{\varphi} x_z + \bar{\psi} x, \\ x_{zz\bar{z}} = (w_{z\bar{z}} + e^{-2w} |\varphi|^2) x_z + (e^{-w} \varphi_{\bar{z}} + \psi) x_{\bar{z}} + (e^{-w} \varphi \bar{\psi} + \psi_{\bar{z}}) x + w_z x_{z\bar{z}}, \\ x_{\bar{z}\bar{z}z} = (e^{-w} \bar{\varphi}_z + \bar{\psi}) x_z + (w_{z\bar{z}} + e^{-2w} |\varphi|^2) x_{\bar{z}} + (e^{-w} \bar{\varphi} \psi + \bar{\psi}_z) x + w_{\bar{z}} x_{z\bar{z}}. \end{cases} \tag{1.13}$$

Hence $x_{zz\bar{z}\bar{z}} = x_{\bar{z}\bar{z}zz}$ yields

$$(w_{z\bar{z}} + e^{-2w} |\varphi|^2)_{\bar{z}} + e^{-2w} \bar{\varphi} \varphi_{\bar{z}} = (e^{-w} \bar{\varphi}_z + \bar{\psi})_z + \bar{\psi}_z + w_{\bar{z}} (w_{z\bar{z}} + e^{-2w} |\varphi|^2),$$

$$(e^{-w} \varphi_{\bar{z}} + \psi)_{\bar{z}} + \psi_{\bar{z}} + w_z (w_{z\bar{z}} + e^{-2w} |\varphi|^2) = e^{-2w} \varphi \bar{\varphi}_z + (w_{z\bar{z}} + e^{-2w} |\varphi|^2)_z,$$

$$\begin{aligned} &(e^{-w} \varphi_{\bar{z}} \bar{\psi}) + (e^{-w} \varphi \bar{\psi} + \psi_{\bar{z}})_{\bar{z}} + w_z (e^{-w} \bar{\varphi} \psi + \bar{\psi}_z) = \\ &= (e^{-w} \bar{\varphi}_z \psi) + (e^{-w} \bar{\varphi} \psi + \bar{\psi}_z)_z + w_{\bar{z}} (e^{-w} \varphi \bar{\psi} + \psi_{\bar{z}}), \end{aligned}$$

that is,

$$(e^{-w} w_{z\bar{z}} + e^{-3w} |\varphi|^2)_z + e^{-3w} \varphi \bar{\varphi}_z - e^{-w} (e^{-w} \varphi_{\bar{z}})_{\bar{z}} - 2e^{-w} \psi_{\bar{z}} = 0, \tag{1.14}$$

$$(e^{-2w} \varphi \bar{\psi} + e^{-w} \psi_{\bar{z}})_{\bar{z}} + e^{-2w} \varphi_{\bar{z}} \bar{\psi} = (e^{-2w} \bar{\varphi} \psi + e^{-w} \bar{\psi}_z)_z + e^{-2w} \bar{\varphi}_z \psi. \tag{1.15}$$

Let κ be the Gauss curvature of x, then from (1.6) we have

$$\kappa = -\epsilon e^{-w} w_{z\bar{z}}. \tag{1.14}$$

2 Equi-centroaffinely homogeneous surfaces with $J \equiv 0$ in $\mathbb{R}^4$

An equi-centroaffine surface in $\mathbb{R}^4$ is called equi-centroaffinely homogeneous, if for any two points p and q on the surface there exists an equi-centroaffine transformation in $\mathbb{R}^4$ which takes the surface to itself and takes p to q. Therefore, an equi-centroaffinely homogeneous surface is an orbit of an equi-centroaffine group which acts on a fixed point in $\mathbb{R}^4$. In this section, we prove the following theorem.

THEOREM 2.1: *Let $x : \mathbf{M} \to \mathbb{R}^4$ be a nondegenerate definite equi-centroaffinely homogeneous surface with vanishing Pick invariant. Then x is centroaffinely equivalent to* :

$$\begin{cases} x_1^2 - x_2^2 &= 1 \\ x_3^2 + x_4^2 &= 1. \end{cases}$$

This surface can be written as

$$x = (\cosh u, \sinh u, \cos v, \sin v), \qquad u \in \mathbb{R}, v \in [0, 2\pi].$$

Its generating group is

$$\left\{ \begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix} \middle| \, u \in \mathbb{R}, v \in [0, 2\pi] \right\},$$

which acts on the point ${}^t(1,0,1,0)$, and the equi-centroaffine metric is given by

$$G = \mathrm{d}u^2 + \mathrm{d}v^2.$$

Proof: Let $x : \mathbf{M} \to \mathbb{R}^4$ be a nondegenerate definite equi-centroaffinely homogeneous surface with vanishing Pick invariant. Since x is equi-centroaffinely homogenous, φ (or ψ) $\neq 0$ means that $\varphi(p) \neq 0$ (or $\psi(p) \neq 0$) for all points $p \in \mathbf{M}$. When $\psi = 0$, x lies in a hyperplane of $\mathbb{R}^4$. So we may assume that $\psi \neq 0$. From the homogeneity of x, we know that the Gauss curvature κ of x is constant.

By a direct computation we have

$$C_{11}^1 = C_{12}^1 = C_{12}^2 = C_{22}^2 = 0, \qquad C_{11}^2 = e^{-w}\varphi, \qquad C_{22}^1 = e^{-w}\bar{\varphi}. \tag{2.1}$$

Then the Pick invariant of x is given by

$$J = \epsilon e^{-3w}\varphi\bar{\varphi}. \tag{2.2}$$

Hence $J \equiv 0$ if and only if $\varphi \equiv 0$.

From (1.14) and $\varphi = 0$ we have $\psi_{\bar{z}} = 0$. Then by $e^{-2w}\psi\bar{\psi} =$ constant we get

$$\psi_z - 2w_z\psi = 0, \qquad w_{z\bar{z}}\psi = 0,$$

that is $w_{z\bar{z}} = 0$. But $w_{z\bar{z}} = 0$ means that x is flat. When x is flat, we can choose the coordinates such that $w \equiv 0$. Hence $\psi_{\bar{z}} = 0$ and $\psi\bar{\psi} =$ constant give $\psi =$ constant. Therefore, if $\varphi = 0$ we obtain that x is flat and ψ is constant.

By the flatness of x we assume that $w \equiv 0$. Since $\varphi = 0$ and ψ is constant, (1.14) and (1.15) are identically satisfied. From (1.13) we have

$$x_{zz} = \psi x. \tag{2.3}$$

Since ψ is nonzero constant, we may assume that $\psi \equiv 1$ because $x_{\tau\tau} = \lambda^{-2}\varphi\bar{\lambda}x_{\bar{\tau}} + \lambda^{-2}\psi x,\ \tau = \lambda z$. Then we get

$$x_{zz} = x. \tag{2.4}$$

By

$$\begin{cases} x_{uu} &= x_{zz} + x_{\bar{z}\bar{z}} + 2x_{z\bar{z}}, \\ x_{uv} &= i(x_{zz} - x_{\bar{z}\bar{z}}), \\ x_{vv} &= -(x_{zz} + x_{\bar{z}\bar{z}}) + 2x_{z\bar{z}}, \end{cases}$$

we have

$$\begin{cases} x_{uu} - x_{vv} &= 4x, \\ x_{uv} &= 0. \end{cases} \tag{2.5}$$

Setting $(2u, 2v) \to (u, v)$, then (2.5) becomes $x_{uv} = 0, x_{uu} - x_{vv} = x$. From $x_{uv} = 0$ we have

$$x = f(u) + g(v).$$

By $x_{uu} - x_{vv} = x$ we obtain

$$f(u) = c_1 \cosh u + c_2 \sinh u, \tag{2.6}$$

$$g(v) = c_3 \cos v + c_4 \sin v, \tag{2.7}$$

where $c_i \in \mathbb{R}^4$, $i = 1, 2, 3, 4$, $[c_1, c_2, c_3, c_4] \neq 0$. Then

$$x = (c_1, c_2, c_3, c_4)^t(\cosh u, \sinh u, \cos v, \sin v). \tag{2.8}$$

The surface is centroaffinely equivalent to

$$x = (\cosh u, \sinh u, \cos v, \sin v). \tag{2.9}$$

□

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