Weingarten congruences and non-auto-Bäcklund transformations for hyperbolic surfaces

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Abstract: : This note contains a complete formula of Bäcklund transformation for a system of nonlinear PDE-s describing hyperbolic surfaces in E^3

This short note serves as a concrete and advanced exemplification of the ideas presented in the previous contribution [1]. The geometric content of the example is by no means new. It belongs to the theory of the so called Weingarten (W-) congruences. It was Luigi Bianchi who introduced this important notion into the differential geometry (see § 4 of [2]). This subject — notably within the projective differential geometry — belongs to the most developed parts of the classical differential geometry [3–5].

In the sequel the standard terminology of the classical differential geometry of surfaces (see Sec. 4 of [6] and literature quoted therein) and the theory of congruences [1] is employed. The result of the note have been obtained within Cartan's method of moving frame in the Finikov formulation. For the sake of simplicity our discussion does not involve these technicalities.

W-congruence is defined by the property that a map

$$\mathcal{F}_1 \ni F_1 \xrightarrow{\beta} F_2 \in \mathcal{F}_2,$$

(see Fig. 4 of [1]) maps asymptotic lines of $\mathcal{F}_1$ on asymptotic lines of $\mathcal{F}_2$ (compare the first Ribaucour theorem discussed in [1]). Here we tacitly assume that both focal surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$ are hyperbolic (their Gaussian curvatures K_1 and K_2 are negative everywhere). This guarantees the existence of asymptotic lines (coordinates) on both focal surfaces.

In [1] it is pointed out that the theory of rectilinear congruences is well suited to treat Bäcklund transformations. Indeed, the first to notice that W-congruence admit a soliton interpretation was B. Cenkl [7]. In his paper Cenkl presents formulae (denoted by (β)) which are not completely satisfactory: they seem to be incomplete. We believe however that Cenkl's formulae are quite important in the soliton context. They could serve as a pretty general Bäcklund transformation containing a lot of specific reductions.

This note is aimed at 1) a correction of Cenkl's formulae and 2) a presentation of a novel and simpler form of Bäcklund transformation in the discussed case.

The surface $\mathcal{F}_1$ when equipped with asymptotic coordinates u and v is implicitly described by its fundamental forms

$$\begin{array}{l} I = r^2(s^2du^2 + 2st\cos\psi \; dudv + t^2dv^2) \\ II = 2rst\sin\psi \; dudv \end{array} . \tag{1}$$

Here the four functions $s(u,v)$, $t(u,v)$, $\psi(u,v)$ and $r(u,v)$ satisfy the Gauss-Mainardi-Codazzi (GMC) system

$$
\begin{array}{rcl}
0 &=& \psi,_{uv} + (\frac{r,_v s \sin\psi}{2rt}),_v + (\frac{r,_u t \sin\psi}{2rs}),_u - ts\sin\psi, \\
s,_v &=& \frac{1}{2r}(t\cos\psi r,_u - sr,_v), \\
t,_u &=& \frac{1}{2r}(s\cos\psi r,_v - tr,_u).
\end{array}
\tag{2}
$$

Notice that $K_1 = -\frac{1}{r^2}$. At the outset we do not impose any restriction on r (thus on Gaussian curvature as well). In other words we work with the system of *three* PDE-s on *four* unknowns. It is quite remarkable that under this condition a quite general system (2) is integrable in the sense that it admits Bäcklund transformation which are described as follows.

The fundamental forms of $\mathcal{F}_2$ are given by

$$
\begin{array}{rcl}
I &=& \tilde r^2(\tilde s^2 du^2 + 2\tilde s\tilde t\cos\tilde\psi\; dudv + \tilde t^2 dv^2), \\
II &=& 2\tilde r\tilde s\tilde t\sin\tilde\psi\; dudv,
\end{array}
\tag{3}
$$

while four functions $\tilde s(u,v)$, $\tilde t(u,v)$, $\tilde\psi(u,v)$ and $\tilde r(u,v)$ satisfy GMC system

$$
\begin{array}{rcl}
0 &=& \tilde\psi,_{uv} + (\frac{\tilde r,_v \tilde s \sin\tilde\psi}{2\tilde r\tilde t}),_v + (\frac{\tilde r,_u \tilde t \sin\tilde\psi}{2\tilde r\tilde s}),_u - \tilde t\tilde s\sin\tilde\psi, \\
\tilde s,_v &=& \frac{1}{2\tilde r}(\tilde t\cos\tilde\psi \tilde r,_u - \tilde s\tilde r,_v), \\
\tilde t,_u &=& \frac{1}{2\tilde r}(\tilde s\cos\tilde\psi \tilde r,_v - \tilde t\tilde r,_u).
\end{array}
\tag{4}
$$

Both quadruples (s,t,ψ,r) and $(\tilde s,\tilde t,\tilde\psi,\tilde r)$ can be always linked by means of the following *first-order* system (Bäcklund transformation).

At the outset we select any solution $X(u,v)$ to the following *linear* equation

$$
X,_{uv} + \frac{r,_v}{2r}X,_u + \frac{r,_u}{2r}X,_v - ts\cos\psi X = 0.
\tag{5}
$$

Next we construct the following system (Bäcklund transformation) for unknowns δ, Θ and ρ

$$
\begin{array}{rcl}
\frac{\delta,_u}{2} &=& \frac{\psi,_u}{2} + \frac{r,_v s\sin\psi}{2rt} + s\sin\frac{\delta+\psi}{2}(\Theta - \frac{r}{2\rho}), \\
\frac{\delta,_v}{2} &=& -\frac{\psi,_v}{2} - \frac{r,_u t\sin\psi}{2rs} - t\sin\frac{\delta-\psi}{2}(\Theta + \frac{r}{2\rho}), \\
\Theta,_u &=& -\frac{r}{2\rho}(\log X),_u + s\cos\frac{\delta+\psi}{2}\{-\frac{r}{2\rho}\Theta + (1+\Theta^2)\}, \\
\Theta,_v &=& \frac{r}{2\rho}(\log X),_v - t\cos\frac{\delta-\psi}{2}\{\frac{r}{2\rho}\Theta + (1+\Theta^2)\}, \\
(\log\rho),_u &=& -(\log X),_u - s\cos\frac{\delta+\psi}{2}(\Theta - \frac{r}{2\rho}), \\
(\log\rho),_v &=& -(\log X),_v + t\cos\frac{\delta-\psi}{2}(\Theta + \frac{r}{2\rho}).
\end{array}
\tag{6}
$$

Finally, the quadruple $(\tilde s,\tilde t,\tilde\psi,\tilde r)$ can be computed by resolving the following system which involves yet another function $\varphi(u,v)$ $(\cot 2\varphi = \Theta)$

$$
\tilde r = \frac{1}{r}\left(\frac{\rho}{\sin\varphi\cos\varphi}\right)^2
\tag{7}
$$

$$
\begin{aligned}
\tilde{s}\sin\frac{\tilde{\delta}+\tilde{\psi}}{2} &= sr^{\frac{\sin\varphi\cos\varphi}{\rho}}\sin\tfrac{\delta+\psi}{2},\\
\tilde{t}\sin\frac{\tilde{\delta}-\tilde{\psi}}{2} &= -tr^{\frac{\sin\varphi\cos\varphi}{\rho}}\sin\tfrac{\delta-\psi}{2},\\
\tilde{s}\cos\frac{\tilde{\delta}+\tilde{\psi}}{2} &= 2\varphi,_u+s\cos\tfrac{\delta+\psi}{2},\\
\tilde{t}\cos\frac{\tilde{\delta}-\tilde{\psi}}{2} &= -2\varphi,_v+t\cos\tfrac{\delta-\psi}{2}.
\end{aligned}
\tag{8}
$$

A few comments are in order. The equation (7) which relates Gaussian curvatures K_1 and K_2 at the corresponding points of W-congruence is called Halphen–Ribaucour formula. It can be rewritten as follows

$$K_1K_2=\frac{1}{|L_1L_2|^4}. \tag{9}$$

According to [1] L_1 and L_2 are limit points of the ray.

Let us come back to the GMC system (2). One can view it as a system of *three* PDE-s on *three* function $s(u,v)$, $t(u,v)$ and $\psi(u,v)$ with $r(u,v)$ (or Gaussian curvature) playing the role of a *parameter* of the system. Within this interpretations (in general $r\neq\tilde{r}$) the system (6–8) is not auto–Bäcklund transformation anymore. This observation explains completely the title of the note.

One can obtain again an auto-Bäcklund transformation under the restrictions $r=\tilde{r}$. If so one neccesarily has

$$r=f(u)+g(v). \tag{10}$$

Hyperbolic surfaces for which

$$K=-\frac{1}{\{f(u)+g(v)\}^2} \tag{11}$$

are known in the classical differential geometry as Bianchi surfaces [2]. Their soliton interpretations has been given in [9]. Surely the Bianchi case is a specific and still integrable reduction of the GMC system (2). For instance Bäcklund transformation for Bianchi surfaces given in [9] can be derived from our formalism.

A final remark concerns a method we used to derive formulae (6–8). Within the Cartan's moving reper method [5] one can show that in order two surfaces to be focal surfaces of a W–congruence it is necessary and sufficient that fundamental forms of both focal surfaces satisfy GMC equations and moreover additional constraint (Bäcklund transformation) is to be satisfied. In the end we present a novel and simpler (linear!) form of the formulae (6).

$$
\partial_u\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0 & -s\cos\frac{\psi}{2} & -(\frac{\psi,_u}{2}+\frac{r,_v s\sin\psi}{2rt})\\ s\cos\frac{\psi}{2} & 0 & -s\sin\frac{\psi}{2}\\ (\frac{\psi,_u}{2}+\frac{r,_v s\sin\psi}{2rt}) & s\sin\frac{\psi}{2} & 0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}+\frac{r}{2}\begin{bmatrix}s\cos\frac{\psi}{2}X\\ -X,_u\\ -s\sin\frac{\psi}{2}X\end{bmatrix}
$$

$$
\partial_v\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0 & t\cos\frac{\psi}{2} & (\frac{\psi,_v}{2}+\frac{r,_u t\sin\psi}{2rs})\\ -t\cos\frac{\psi}{2} & 0 & -t\sin\frac{\psi}{2}\\ -(\frac{\psi,_v}{2}+\frac{r,_u t\sin\psi}{2rs}) & t\sin\frac{\psi}{2} & 0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}+\frac{r}{2}\begin{bmatrix}t\cos\frac{\psi}{2}X\\ X,_v\\ t\sin\frac{\psi}{2}X\end{bmatrix}
$$

where

$$\begin{aligned} y_1 &= X\rho\cos\tfrac{\delta}{2}, \\ y_2 &= X\rho\cot 2\varphi, \\ y_3 &= X\rho\sin\tfrac{\delta}{2}. \end{aligned}$$

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