Panorama of soliton applications — Physics of solitons

Reinhard Meinel

Max-Planck-Gesellschaft, AG Gravitationstheorie
Max-Wien-Platz 1, D-07743 Jena, Germany
e-mail: meinel@gravi.physik.uni-jena.de

Abstract: The variety of soliton applications is illustrated with four nonlinear partial differential equations: The two most famous soliton equations (sine–Gordon equation and Korteweg–de Vries equation), the equation with the most important technical applications (nonlinear Schrödinger equation), and the most fundamental one (Ernst equation).

I The sine-Gordon equation in classical mechanics

The well-known mechanical soliton model (rubber band model, cf. [1]) may be used for demonstrating the main properties of solitons: localization, propagation without change of form, stability with respect to mutual interactions. These properties are observed experimentally as well as theoretically — according to the exact one-, two- and n-soliton solutions of the sine-Gordon equation. In contrast, a discussion of the linearized sine-Gordon equation reveals the dispersive properties of linear waves leading to the spreading of wave packets. Solitons exist due to a balance between dispersion and nonlinearity. The Linear System (LS) related to the sine-Gordon equation was presented in [2]. Further applications can be found, e.g., in the theory of dislocations [3], superconductivity [4] and nonlinear optics [5]. In all these cases, the sine-Gordon equation arises as an approximation of the relevant basic physical equations.

II The Korteweg–de Vries equation in hydrodynamics

The water-wave solitons first observed by Scott Russell [6] may be described by the Korteweg–de Vries (KdV) equation which was derived as an approximation of the basic hydrodynamic equations [7]. The LS related to the KdV equation was found by Gardner et al. [8] who "discovered" the Inverse Scattering Method (ISM) for solving initial value problems. The KdV equation has applications in plasma physics as well [9].

III The nonlinear Schrödinger equation in nonlinear optics

The propagation of optical pulses in fibers with an intensity-dependent refraction index is described approximately by the nonlinear Schrödinger equation. Its related LS has been presented by Zakharov and Shabat [10]. The generation of solitons in optical fibers was proposed in [11] and realized in [12]. Without doubt, solitons will play an important role in future telecommunication systems.

IV The Ernst equation in general relativity

The Ernst equation follows from the vacuum Einstein equations — without approximations — in the case of rotational symmetry and stationarity [13], [14]. Its solutions describe the exterior gravitational fields of rotating bodies in general relativity. The related LS was found in [15]–[18]. The analogues of solitons are the fields of rotating black holes. The ISM allows, in principle, the solution of boundary value problems. A first example has been given recently with the global solution of the problem of a uniformly rotating disk of dust in general relativity [19], [20].

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