Isothermic surfaces of k-th order

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Abstract: This paper gives the answer to the question:

What is between 1 *and* ∞ *?*

1 denotes isothermic surfaces in E^3 (the ones admitting infinitesimal isometric deformation preserving principal curvatures) and ∞ denotes Bonnet surfaces (the ones admitting one-parameter isometric deformations preserving principal curvatures). Isothermic surfaces of k-th order are introduced so as to fill up the gap between 1 and ∞. The main theorem shows that the answer to the above question is very simple: *there is nothing.* In other words, we prove in the second statement: $2 = \infty$. We also present some properties of deformations of Bonnet surfaces.

This article is based on a part of the paper [2]. It presents a solution to the problem posed by Professor Antoni Sym during one of his lectures for Student Nonlinear Physics Research Group.

1 Isothermic surfaces of k-th order

In what follows, ε denotes a parameter of a surface deformation, $-\varepsilon_0 < \varepsilon < \varepsilon_0$, $\varepsilon_0 \geq 0$, and $\varepsilon = 0$ corresponds to the initial surface.

Following the traditional notation [1] we denote the coefficients of first and second fundamental forms[1] of a surface $S = r(u, v)$ by

$$E = r_{,u} \cdot r_{,u}\,, \quad F = r_{,u} \cdot r_{,v}\,, \quad G = r_{,v} \cdot r_{,v}\,, \tag{1a}$$

and

$$D = r_{,uu} \cdot n, \quad D' = r_{,uv} \cdot n, \quad D'' = r_{,vv} \cdot n, \tag{1b}$$

respectively, where comma denotes differentiation and n denotes the normal vector

$$n = \frac{r_{,u} \times r_{,v}}{||r_{,u} \times r_{,v}\,||}.$$

Moreover, we assume that deformations under consideration are always analytic in ε, i.e. all the functions

$$E(u, v; \varepsilon), \quad F(u, v; \varepsilon), \quad G(u, v; \varepsilon), \tag{2a}$$

$$D(u, v; \varepsilon), \quad D'(u, v; \varepsilon), \quad D''(u, v; \varepsilon), \tag{2b}$$

[1] The coefficients of first and second fundamental forms that satisfy Gauss and Codazzi equations describe the surface uniquely up to a rigid motion [1].

((u,v) — local parameters), are analytic at $\varepsilon = 0$. To be more specific, we confine ourselves to the case of isometric deformations and thus we put

$$\begin{aligned} E(u,v;\varepsilon) &= E(u,v;0) = E(u,v), \\ F(u,v;\varepsilon) &= F(u,v;0) = F(u,v), \\ G(u,v;\varepsilon) &= G(u,v;0) = G(u,v), \end{aligned} \tag{3a}$$

and

$$\begin{aligned} D(u,v;\varepsilon) &= D(u,v) + \sum_{j=1}^{\infty} \varepsilon^j \Gamma_j(u,v), \\ D'(u,v;\varepsilon) &= D'(u,v) + \sum_{j=1}^{\infty} \varepsilon^j \Gamma'_j(u,v), \\ D''(u,v;\varepsilon) &= D''(u,v) + \sum_{j=1}^{\infty} \varepsilon^j \Gamma''_j(u,v). \end{aligned} \tag{3b}$$

Guided by the result of Bianchi mentioned above we propose the following definition:

Definition: A surface in E^3 which admits non-trivial isometric deformations preserving mean curvature up to the k-th order of deformation parameter ε (i.e. $H(u,v;\varepsilon) = H(u,v) + \sum_{j=k+1}^{\infty} \varepsilon^j H_j(u,v)$) is called *an isothermic surface of k-th order*. Obviously, isothermic surfaces of 1-st order are just isothermic surfaces and isothermic surfaces of ∞-order are Bonnet surfaces.

This definition leads us to conditions on the surface and the deformation. We consider the surface in curvature coordinates (u,v). It follows that first and second fundamental forms read

$$I = Edu^2 + Gdv^2, \tag{4a}$$

$$II = k_1 Edu^2 + k_2 Gdv^2, \tag{4b}$$

it means $F = D' = 0$, $D = k_1 E$ and $D'' = k_2 G$, where k_1, k_2 are principal curvatures.

The second fundamental form is subject to the following constraints:

1. k conditions resulting from

$$H(u,v;\varepsilon) = H(u,v) + o(\varepsilon^{k+1}), \tag{5a}$$

where $o(\varepsilon^m)$ denotes small of m-th order. We compute

$$H(u,v;\varepsilon) \equiv H_\varepsilon = \frac{k_1 + k_2}{2} = \frac{D_\varepsilon G + D''_\varepsilon E}{2EG}, \tag{5b}$$

and equate lowest terms to zero.

2. The Gauss equation

$$K_\varepsilon = -\frac{1}{2\sqrt{EG}}\left(\partial_u \frac{G,_u}{\sqrt{EG}} + \partial_v \frac{E,_v}{\sqrt{EG}}\right). \tag{6a}$$

It is sufficient to use invariance of the Gaussian curvature under isometries ($K_\varepsilon = K = k_1 k_2$). Hence

$$D_\varepsilon D''_\varepsilon - (D'_\varepsilon)^2 = DD''. \tag{6b}$$

3. Codazzi equations

$$\frac{\partial D_\varepsilon}{\partial v} - \frac{\partial D'_\varepsilon}{\partial u} - \Gamma^1_{12} D_\varepsilon + (\Gamma^1_{11} - \Gamma^2_{12}) D'_\varepsilon + \Gamma^2_{11} D''_\varepsilon = 0, \tag{7a}$$

$$\frac{\partial D''_\varepsilon}{\partial u} - \frac{\partial D'_\varepsilon}{\partial v} + \Gamma^1_{22} D_\varepsilon + (\Gamma^2_{22} - \Gamma^1_{12}) D'_\varepsilon - \Gamma^2_{12} D''_\varepsilon = 0, \tag{7b}$$

where Γ^k_{ij} are second set of Christoffel symbols [1].

Firstly, we consider the case $k = 1$. (5a) and (6b) reduce to

$$G\Gamma''_1 + E\Gamma_1 = 0, \qquad k_2 G\Gamma''_1 + k_1 E\Gamma_1 = 0. \tag{8}$$

This gives $\Gamma_1 = \Gamma''_1 = 0$, we assumed $k_1 \neq k_2$ (we omit umbilic points). (7) reduces to

$$\frac{\partial \Gamma'_1}{\partial u} = \left(\frac{1}{2}\ln\frac{E}{G}\right),_u \Gamma'_1, \qquad \frac{\partial \Gamma'_1}{\partial v} = \left(\frac{1}{2}\ln\frac{G}{E}\right),_v \Gamma'_1. \tag{9}$$

The integrability condition for (9) results in

$$\partial_u \partial_v \left(\ln\frac{E}{G}\right) = 0. \tag{10}$$

Thus we can change the coordinates such that the coefficients of first and second fundamental forms would read

$$E = G = e^{2\theta}, \quad D = k_1 e^{2\theta}, \quad D'' = k_2 e^{2\theta}. \tag{11}$$

Therefore the surface is isothermic.

Now we consider the case $k > 1$. We assume that $1 \leq j \leq k$. From (5) we obtain

$$\Gamma_j + \Gamma''_j = 0, \tag{12}$$

and from (6)

$$\Gamma_j = A \sum_{l=1}^{j-1} (\Gamma_l \Gamma_{j-l} + \Gamma'_l \Gamma'_{j-l}), \tag{13a}$$

where

$$A = \frac{1}{e^{2\theta}(k_2 - k_1)}. \tag{13b}$$

With the help of (11) and (12) we can reduce Codazzi equations to Cauchy–Riemann equations

$$\frac{\partial \Gamma'_j}{\partial u} = \frac{\partial \Gamma_j}{\partial v}, \qquad \frac{\partial \Gamma'_j}{\partial v} = -\frac{\partial \Gamma_j}{\partial u}, \tag{14}$$

hence Γ'_j and Γ_j must be harmonic functions ($\Gamma'_j + i\Gamma_j$ is a holomorfic function in $z = u + iv$).

Summarizing — the conditions for isothermic surfaces of k-th order read

(i) the surface is an isothermic surface,

(ii) there is a nontrivial deformation satisfying (12–14).

These considerations lead to a theorem.

2 Main theorem

Surprisingly enough, one can prove the following theorem.

THEOREM. *1. The class of umbilic-free isothermic surfaces of 2-nd order is described in terms of the fundamental forms referred to curvature coordinates* (u, v) *as follows*

$$I = e^{2\theta}(du^2 + dv^2), \tag{15a}$$

$$II = e^{2\theta}(k_1 du^2 + k_2 dv^2), \tag{15b}$$

$$\Delta A \equiv (\partial_u^2 + \partial_v^2)\frac{1}{e^{2\theta}(k_2 - k_1)} = 0. \tag{15c}$$

2. The class of umbilic-free isothermic surfaces of k-th order $(k \geq 2)$ *— in particular the class of umbilic-free Bonnet surfaces — is identical with umbilic-free isothermic surfaces of 2-nd order.*

3. There exists a unique deformation (described above) modulo an analitical invertible change of the parameter ε.

4. Once (15) *hold, there exist a power series* (3b) *almost uniformly convergent within the domain of local coordinates* (u, v).

Proof: The first statement follows immediately from (12–14) for $k = 2$.

The second statement (the main part of the theorem) is proved by induction. The crucial ingredient of the proof is the following unexpected though rather elementary recursion formula

$$\Gamma_k = \sum_{j=1}^{k-1} c_j (A\Gamma'_{k-j} + \tilde{A}\Gamma_{k-j}), \tag{16a}$$

$$\Gamma'_k = c_k + \sum_{j=1}^{k-1} c_j(\tilde{A}\Gamma'_{k-j} - A\Gamma_{k-j}), \tag{16b}$$

where c_j are constant, $\tilde{A}$ is harmonic and $\tilde{A} + iA$ is a holomorfic function in $z = u + iv$.

When $k = 1$ then $\Gamma_1 = 0$, $\Gamma'_1 = c_1$ so (16) holds.

Assume the formula (16) holds for k, where $1 \leq k \leq n$; we will prove it for $n+1$. From (13)

$$\Gamma_{n+1} = A\sum_{j=1}^{n}(\Gamma_j\Gamma_{n+1-j} + \Gamma'_j\Gamma'_{n+1-j}). \tag{17}$$

We use (16) for Γ_{n+1-j} and Γ'_{n+1-j} to obtain

$$\begin{aligned} c_1A\Gamma'_n + A\sum_{j=1}^{n-1}\Bigg(\Gamma_j\sum_{i=1}^{n-j}\Big(Ac_i\Gamma'_{n+1-j-i} + \tilde{A}c_i\Gamma_{n+1-j-i}\Big) \\ +\Gamma'_j\left(c_j + \sum_{i=1}^{n-j}\Big(\tilde{A}c_i\Gamma'_{n+1-j-i} - Ac_i\Gamma_{n+1-j-i}\Big)\right)\Bigg) \\ = \quad A\sum_{i=1}^{n} c_i\Gamma'_{n+1-i} + \tilde{A}\sum_{i=1}^{n} c_i\Gamma_{n+1-i} \\ +A^2\sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\left(\Gamma_j c_i\Gamma'_{n+1-j-i} - \Gamma'_j c_i\Gamma_{n+1-j-i}\right). \end{aligned}$$

The last term is equal to zero, so we arrive at (16a)

$$\Gamma_{n+1} = \sum_{i=1}^{n} c_i(A\Gamma'_{n+1-i} - \tilde{A}\Gamma_{n+1-i}).$$

We check at once that Γ_{n+1} is harmonic, which is clear from (16a). Next, we try to obtain Γ'_{n+1} from (14) (we know that such a function is unique modulo constant). It is easily seen that (16b) satisfies (14) and c_{n+1} denotes uniquely function Γ'_{n+1}. This ends the proof of the second statement.

To show the third statement we will introduce a new notation. We will replace the series (3b) by a complex series:

$$\sum_{j=1}^{\infty}\varepsilon^j\gamma_j, \tag{18a}$$

where

$$\gamma_j = \Gamma'_j + i\Gamma_j. \tag{18b}$$

We will prove that for any sequence $(c_n)_{n\in\mathbf{N}}$, which defines uniquely the power series (3b), *formally* holds

$$\sum_{j=1}^{\infty}\varepsilon^j\gamma_j = \frac{k}{1-k\alpha}, \tag{19a}$$

where
$$k=\sum_{n=1}^{\infty} c_n \varepsilon^n, \tag{19b}$$
and
$$\alpha=\tilde{A}+iA. \tag{19c}$$
In the new setting formulae (16) take form
$$\gamma_k=c_k+\sum_{j=1}^{k-1} c_j \alpha \gamma_{k-j}. \tag{20}$$

By means of this formula we can proove that
$$\gamma_k=\sum_{j=1}^{k} \alpha^{j-1}\left[\sum_{m_1,\dots,m_j} c_{m_1}\cdot\ldots\cdot c_{m_j}\right], \tag{21}$$
where $m_1,\dots,m_j$ such $m_1+\ldots+m_j=k$ and $m_l\geq 1$.

We prove (21) by induction. For $k=2$ it holds. Assume (21) for k, where $1\leq k\leq n-1$; we will prove it for n. Let us substitute γ_j for $1\leq j\leq n-1$ in (20) for $k=n$ to obtain
$$\begin{aligned}
\gamma_n &= c_n+\sum_{i=1}^{n-1} c_i \sum_{j=1}^{n-i} \alpha^j\left[\sum_{m_1,m_2,\dots,m_j} c_{m_1}\cdot c_{m_2}\cdot\ldots\cdot c_{m_j}\right]\\
&= c_n+\sum_{i=1}^{n-1}\sum_{j=1}^{n-i} \alpha^j\left[c_i\sum_{m_1,m_2,\dots,m_j} c_{m_1}\cdot c_{m_2}\cdot\ldots\cdot c_{m_j}\right],
\end{aligned}$$
where $m_1+m_2+\ldots+m_j=n-i$ and $m_l\geq 1$ and thus
$$\gamma_n=c_n+\sum_{j=1}^{n-1}\alpha^j\sum_{i=1}^{n-j}\left[c_i\sum_{m_1,\dots,m_j} c_{m_1}\cdot\ldots\cdot c_{m_j}\right].$$

This leads us to (21).

When we develop the right-hand side of (19a), it will be clear that (21) implies (19a). □

The proof of the fourth statement starts with the observation that convergence of series (3b) depends only on the radius of convergence of the series (19b). If the radius is positive (non-zero), then the series (3b) for any compact domain (u,v) has a nontrivial interval $(-k_0,k_0)$ of convergence of (19a).

We take a compact domain Ω of (u,v). There exists $M\in\mathcal{R}_+$ such that $|\alpha|<M$ for all $(u,v)\in\Omega$, hence (19a) is convergent when $k\in(-k_0,k_0)$ where $0<k_0M<1$. From this condition and (19b) we can obtain maximal ε. This ends the proof of the theorem.

3 Concluding remarks

In the paper [2] one can find an application of the main theorem to soliton theory. Using methods from [3] one can produce a linear problem for Bonnet surfaces. Our result shows how many open problems still can be found in classical differential geometry, even in E^3.

REFERENCES

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