Toda systems and W-algebras

A. Ovchinninkov

3-6 Batyreva str., Podolsk, 142103 Moscow, Russia
e-mail: popov@ap.phys.msu.su

1 Introduction

There exists a remarkable connection between the conformal field theory, the theory of KdV-type equations and Toda-type systems.

Two dimensional conformally invariant solvable field theories are based on various extensions of the chiral Virasoro algebras. One of the best known extensions is polynomial extension of the Virasoro algebra by higher spin fields. The Virasoro algebra itself is the infinite-dimensional Lie algebra generated by the generators L_n, $n \in \mathbb{Z}$, and relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \,,$$

where c is a central element. The generatrix of L_n's

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

is the energy-momentum tensor of the conformally invariant field theory.

However the Virasoro algebra is only a part of the conformal algebra (that is the algebra of symmetries of conformal field theory, CFT) in the general case. Many examples of more general algebras are known, for example, the Neveu–Schwarz algebra, Wess–Zumino–Witten algebra, parafermionic algebra of Zamolodchikov and so on.

Classification of all possible kinds of conformal algebras is the first step to classification of all possible types of CFT. This problem is a very important one and its total investigation has not been completed yet. A very interesting class of such algebras was considered firstly by A. B. Zamolodchikov. It is the so-called W-algebras, which can be obtained by means of polynomial extensions of the Virasoro algebra.

A large class of polynomial extensions of the Virasoro algebra can be constructed by quantizing the second Gel'fand–Dikii Poisson bracket structure of Lax operators, used in the theory of integrable systems.

Another approach is that Toda theories also provide a realization of W-algebras. Toda theories may be regarded as Wess–Zumino models (based on maximally non-compact simple real Lie groups) reduced by certain conformally invariant constraints. This lecture is devoted to these problems.

2 Virasoro algebra in conformal field theory

In statistical physics, in the theory of critical phenomena at the second-order phase transition points, the global scaling symmetry has been known and extensively used for many years. In the scaling theory a given physical system is classified by a set of basic operators $\{\phi_i(x)\}$, having anomalous dimensions $\{\Delta_i\}$. Under scaling transformations of the space

$$x \to \lambda x$$

(λ being a constant parameter) the basic operators transform like:

$$\phi_i(x) \to \lambda^{\Delta_i} \phi_i(\lambda x).$$

It means that if we made this transformation then the correlation function for the basic operators

$$\langle \phi_1(x_1)\phi_2(x_2)\cdots \rangle$$

will not change.

The scaling transformation above is global, $\lambda = \text{const}$, and the corresponding scaling symmetry of critical phenomena has been well establish. The idea that critical point theories should also be invariant by local scalings like above, in which λ depends on space point x, was put forward by Polyakov. This is a conformal symmetry of critical phenomena.

In two-dimensional conformal theories the group of local rescalings can be extended to an infinite-dimensional group of analytic transformations:

$$z \to \tilde{z} = f(z), \qquad \bar{z} \to \bar{\tilde{z}} = \overline{f(z)},$$

where

$$z = x_1 + ix_2, \qquad \bar{z} = x_1 - ix_2.$$

Correspondingly, the basic operators transform as:

$$\phi_i(z, \bar{z}) \to (\lambda(z))^{\Delta_i} \phi_i(f(z), \overline{f(z)}), \quad \lambda(z) = \left| \frac{df(z)}{dz} \right|.$$

The conformal theory for 2D systems has been developed by Belavin, Polyakov, and Zamolodchikov [2]. The energy-momentum tensor of the theory is generated by the conformal transformations of the correlation functions, and because of conformal invariance its trace equals to zero. Its independent components

$$T = T_{11} - T_{22} + 2iT_{12}, \quad \bar{T} = T_{11} - T_{22} - 2iT_{12}$$

satisfy Cauchy–Riemann equations $\partial_{\bar{z}} T = \partial_z \bar{T} = 0$, so that $T = T(z)$, $\bar{T} = \bar{T}(\bar{z})$. Conformal group can be regarded as direct product $\Gamma(z) \times \Gamma(\bar{z})$ of groups of analytical transforms of variables z and $\bar{z}$, respectively, and fields $T(z)$ and $\bar{T}(\bar{z})$ are connected with the generators of subgroups $\Gamma(z)$ and $\Gamma(\bar{z})$.

Transformation properties of any field A_j are defined by singular terms of operator product expansion $T(\zeta)A_j(z,\bar{z})$. In quantum field theory there exists a set of "primary" fields $\Phi_l(z,\bar{z})$ which are characterized by expansion

$$T(\zeta)\Phi_l(z,\bar{z}) = \frac{\Delta_l}{(\zeta - z)^2}\Phi_l(z,\bar{z}) + \frac{1}{\zeta - z}\frac{\partial}{\partial z}\Phi_l(z,\bar{z}) + O(1),$$

and similarly for $\bar{T}(\bar{\zeta})\Phi_l(z,\bar{z})$. Here Δ_l (and $\bar{\Delta}_l$) is numerical parameter called the conformal dimension of the field. Combinations $s_l = \Delta_l - \bar{\Delta}_l$ and $d_l = \Delta_l + \bar{\Delta}_l$ are spin and scaling dimension of field Φ_l, respectively.

Operator product expansion for the fields $T(z)$ is

$$T(z)T(z') = \frac{c}{2(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{T'(z')}{z-z'} + O(1),$$

where prime denotes derivative and c is the parameter of the theory. Define the operators $L_n(z)$ acting on local fields $A_j(z,\bar{z})$:

$$L_n(z)A_j(z,\bar{z}) = \oint_z d\zeta(\zeta - z)^{n+1}T(\zeta)A_j(z,\bar{z}).$$

(Integration over small contour around z.) These operators form the Virasoro algebra (it follows from operator product expansion for $T(z)T(z')$):

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}.$$

3 Extra symmetries in conformal field theory

In the case when there exists a field Q_s with dimensions $(\Delta,\bar{\Delta}) = (s,0)$ we can speak about infinite extra symmetry of such field theory. Indeed, such a field satisfies equation $\partial_{\bar{z}}Q_s = 0$, i.e., $Q_s = Q_s(z)$, and we have an infinite number of conserved currents

$$J_\omega^{(s)}(z) = \omega(z)Q_s(z), \quad \partial_{\bar{z}}J_\omega^{(s)}(z) = 0,$$

where $\omega(z)$ is an analytic function. These currents can be considered as generators of extra symmetry. The example of conformal field theory possessing the extra symmetry with respect to current algebra (Kac–Moody algebra) is the Wess–Zumino model [5].

A. B. Zamolodchikov investigated [6] infinite extra symmetries generated by currents with spins $s = 1/2, 1, \ldots, 3$. For the cases $s = 5/2$ and $s = 3$ he found the symmetry generators to form the associative algebras with quadratic determining relations.

In the case $s = 3$ the symmetry algebra is formed by Virasoro generators (note that Virasoro algebra is always subalgebra of full symmetry algebra) and the set of new generators $W_n^{(3)}$ which are components of a spin 3 field

$$W^{(3)}(z) = \sum_{n\in\mathbb{Z}} \frac{W_n^{(3)}}{z^{n+3}}.$$

Commutation relations in this case have the following form:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \tfrac{c}{12}(n^3-n)\delta_{n+m,0}, \\
[L_n, W^{(3)}_m] &= (2n-m)W^{(3)}_{n+m}, \\
[W^{(3)}_n, W^{(3)}_m] &= \tfrac{16}{22+5c}(n-m)\Lambda_{n+m} \\
&\quad +(n-m)\left(\tfrac{(n+m+2)(n+m+3)}{15} - \tfrac{(n+2)(m+3)}{6}\right)L_{n+m} \\
&\quad +\tfrac{c}{360}(n^2-4)(n^2-1)n\delta_{n+m,0},
\end{aligned}$$

where

$$\Lambda_n \stackrel{\text{def}}{=} \sum_k : L_k L_{n-k} : + \frac{1}{5}x_n L_n, \qquad x_{2l} = (l+1)(1-l); \qquad x_{2l+1} = (2+l)(1-l),$$

and $:\ :$ denotes normal ordering (operators L with greater values of subscripts stand on right).

Let us note that this algebra is not a Lie-type algebra because of the quadratic terms in the right-hand side of $[W, W]$ commutators. However these commutation relations satisfy very rigid limitations following from the associativity of current algebra. For example:

$$[W_n, [W_m, W_k]] + \text{permutations} = 0.$$

In the general case W-algebra is generated by several fields

$$W^{(j)}(z) = \sum_{n\in\mathbb{Z}} W^{(j)}_n z^{-j-n}$$

with higher (integer or half-integer) spins $3 \le j \le N$. Commutation relations have the following form

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \tfrac{c}{12}(n^3-n)\delta_{n+m,0}, \\
[L_n, W^{(j)}_m] &= ((j-1)n-m)W^{(j)}_{n+m},
\end{aligned}$$

and they have polynomial form for commutators $[W, W]$.

The question arises: how many W-algebras exist? The other problem includes constructions of representations of these algebras and the related models of CFT. (Note that Virasoro algebra itself is the simplest W-algebra generated by the current of spin $s = 2$.)

4 Classical W-algebras associated with KdV-type equations

The following fact may be useful to answer these questions. It turns out that W-algebra can be considered as quantum version of a certain structure in the theory of KdV-type equations. The KdV equation has the form

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x}.$$

This equation is Hamiltonian. This means that it can be written in the Poisson bracket form

$$\frac{\partial u}{\partial t} = \{H, u\},$$

where

$$H = \int u^2 \, dx$$

and the Poisson bracket is defined by

$$\{u(x), u(y)\} = \left(u' + 2u \frac{d}{dx} + \alpha \frac{d^3}{dx^3} \right) \delta(x - y).$$

If one takes a Fourier decomposition of $u(x)$ (provided that $u(x)$ is periodic) then this Poisson bracket is converted to the Virasoro form

$$\{L_n, L_m\} = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},$$

where constant c can be expressed in terms of α. So, this Poisson bracket can be thought of as the classical limit of the Virasoro algebra.

The KdV equation admits a generalization to a system of equations for several functions $u_1(t, x), \ldots, u_N(t, x)$. In these cases the equations are also Hamiltonian and Poisson brackets are of the form

$$\{u_i(x), u_j(y)\} = \widehat{\mathcal{D}}_{ij}\delta(x - y),$$

where $\widehat{\mathcal{D}}_{ij}$ is a certain differential operator whose coefficients are polynomials in $u_i(x)$ and their derivatives. For $n = 3$ one has two functions and their Fourier components L_n and W_n have the following Poisson brackets

$$\begin{aligned}
\{L_n, L_m\} \quad &= (n - m)L_{n+m} + \tfrac{c}{12}(n^3 - n)\delta_{n+m,0}, \\
\{L_n, W^{(3)}_m\} \quad &= (2n - m)W^{(3)}_{n+m}, \\
\{W^{(3)}_n, W^{(3)}_m\} \quad &= \tfrac{16}{5c}(n - m)\textstyle\sum_k L_k L_{n-k} \\
&\quad +(n - m)\left(\tfrac{(n+m+2)(n+m+3)}{15} - \tfrac{(n+2)(m+3)}{6}\right) L_{n+m} \\
&\quad +\tfrac{c}{360}(n^2 - 4)(n^2 - 1)n\delta_{n+m,0}.
\end{aligned}$$

These relations provide a semiclassical limit of the commutation relations for the spin 3 W-algebra as it was noticed by Feigin and Hovanova.

These examples lead us to the natural conjecture that there exists a similar relation in the general case. If we would have classification and explicit construction of the Hamiltonian structure for general KdV-type equations, we could try to classify and construct

W-algebras by means of a kind of quantization of these classical objects. In fact the explicit construction of general KdV-type equations exists. It was achieved by Drinfeld and Sokolov by means of the Hamiltonian reduction of a natural Hamiltonian structure connected with affine Lie algebras.

Let $\overset{0}{A}_1$ be the Lie algebra formed by smooth mappings $u : S^1 \to A_1$ from the unit circle into Lie algebra A_1 of 2×2-matrices with zero trace. The generators of this algebra are $X_a(x)$, $a = 1, 2, 3$, $x \in S^1$ and commutation relations are

$$[X_a(x), X_b(y)]_0 = \varepsilon_{abc} X_c(x) \delta(x - y).$$

The simple Lie algebra A_1 possesses the bilinear ad-invariant Killing form

$$\langle A, B \rangle = \operatorname{tr} AB, \quad A, B \in A_1,$$

which allows one to construct the central extension $\widehat{A}_1$ of algebra $\overset{0}{A}_1$ by means of 2-cocycle of Maurer–Cartan

$$\omega(u, v) = \oint_{S_1} dx \, \langle u(x), \partial_x v(x) \rangle.$$

The algebra $\widehat{A}_1$ is the simplest Kac–Moody algebra. Its elements are pairs $\widehat{u}(x) = (u(x), \lambda)$, $\lambda \in \mathbb{C}$ with commutators $[\widehat{u}, \widehat{v}] = ([u, v]_0, \omega(u, v))$. Elements $(0, \lambda)$ form the center of algebra $\widehat{A}_1$.

Generators of Kac–Moody algebra $\widehat{A}_1$ are $\widehat{X}_a(x)$ and I with commutation relations

$$\begin{aligned} \left[\widehat{X}_a(x), \widehat{X}_b(y)\right] &= \varepsilon_{abc} \widehat{X}_c(x) \delta(x - y) + \delta_{ab} \delta'(x - y) I, \\ \left[\widehat{X}_a(x), I\right] &= 0. \end{aligned}$$

The basis $X_a \in A_1$ is such that $\langle X_a, X_b \rangle = \delta_{ab}$. Basis in $\widehat{A}_1$ allows one to decompose any element $\widehat{u}$; components of $\widehat{u}$ be $(u_a(x), \lambda)$. A linear functional $\widehat{u}^*$ on $\widehat{A}_1$ (i.e., an element of dual space $\widehat{A}_1^*$) can be written in form

$$\widehat{u}^*(\widehat{u}) = \sum_{a=1}^{3} \oint_{S_1} dx \, u_a^*(x) u_a(x) - i\alpha\lambda,$$

so its coordinates in dual basis are $(u_a^*(x), -i\alpha)$.

Let's consider the co-adjoint representation of algebra $\widehat{A}_1$. It acts in dual space $\widehat{A}_1^*$ by the rule

$$\operatorname{ad}_{\widehat{u}}^* \widehat{u}^* = \big(\varepsilon_{abc} u_c^*(x) u_b(x) - i\alpha \partial_x u_a(x), 0\big).$$

The dual space $\widehat{A}_1^*$ can be identified with $\widehat{A}_1$ because the Killing form is nondegenerate. Then the last relation can be written as

$$\operatorname{ad}_{\widehat{u}}^* \widehat{u}^* = -i\alpha \partial_x \widehat{u} + [\widehat{u}, \widehat{u}^*].$$

Co-adjoint action ad^* of algebra $\widehat{A}_1$ can be lifted to co-adjoint action Ad^* of the group $\widehat{\mathrm{SL}}(2,\mathbb{C})$ of smooth mappings $g : S^1 \to \mathrm{SL}(2,\mathbb{C})$:

$$\mathrm{Ad}_g^*\,\widehat{u}^* = -i\alpha\partial_x g(x)g^{-1}(x) + g(x)u^*(x)g^{-1}(x).$$

Now we construct the differential operator $\widehat{L} = -i\alpha\partial_x - u^*(x)$. Then

$$\widehat{L}' = \mathrm{Ad}_g^*\,\widehat{L} = g(x)\widehat{L}g^{-1}(x).$$

This transformation is called gauge transformation, and operators $\widehat{L}$ and $\widehat{L}'$ are gauge equivalent. Denote by E the manifold of classes of gauge equivalent operators $\widehat{L}$, i.e. the manifolds of orbits of co-adjoint action of group $\widehat{\mathrm{SL}}(2,\mathbb{C})$. Any operator $\widehat{L}$ can be reduced by means of gauge transformation to the form

$$\widehat{L} = -\alpha\partial_x + \frac{1}{\sqrt{2}}\begin{pmatrix} \phi(x) & 0 \\ 0 & -\phi(x) \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\phi : S^1 \to \mathbb{C}$. Applying to the operator $\widehat{L}$ the gauge transform with matrix

$$g(x) = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}}\phi(x) \\ 0 & 1 \end{pmatrix},$$

we obtain

$$\widehat{L} = -\alpha\partial_x + \begin{pmatrix} 0 & W(x) \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where function $W(x)$ is defined by the relation

$$\left(\frac{\alpha}{i}\partial_x + \frac{\phi(x)}{\sqrt{2}}\right)\left(\frac{\alpha}{i}\partial_x - \frac{\phi(x)}{\sqrt{2}}\right) = \left(\frac{\alpha}{i}\partial_x\right)^2 + W(x).$$

This relation defines so-called Miura transformation:

$$\mu : \phi(x) \to W(x) = -\frac{1}{2}\phi^2(x) + \alpha\frac{i}{\sqrt{2}}\phi'(x).$$

The Hamiltonian structure on the manifold E has the form

$$\{\phi(x), \phi(y)\} = \delta'(x-y)$$

(Kirillov–Poisson structure). For the W's we obtain

$$\{W(x), W(y)\} = \phi(x)\phi(y)\delta'(x-y) + \frac{i\alpha}{\sqrt{2}}(\phi(x)-\phi(y))\delta''(x-y) + \frac{\alpha^2}{2}\delta'''(x-y).$$

This formula has to be considered in sense of distributions, and we can rewrite it as

$$\{W(x), W(y)\} = -(W(x)+W(y))\delta'(x-y) + \frac{\alpha^2}{2}\delta'''(x-y).$$

Fourier coefficients of the functions $\phi(x)$ and $W(x)$ are

$$a(k) = \oint_{S_1} dx\, x^k \phi(x), \qquad W(k) = \oint_{S_1} dx\, x^{k+1} W(x).$$

Therefore, the Poisson brackets for linear functionals $a(k)$, $W(k)$ have the form

$$\{a(k), a(m)\} = m\delta_{k+m,0},$$

$$\{W(k), W(m)\} = (k-m)W_{k+m} - \frac{\alpha^2}{2}(k^3 - k)\delta_{k+m,0}.$$

Quantization of these relations leads to the formulae

$$[a(k), a(m)] = \hbar m \delta_{k+m,0},$$

$$\frac{1}{\hbar}[W(k), W(m)] = (k-m)W_{k+m} + \left(1 - \frac{6\alpha^2}{\hbar}\right)\frac{(k^3-k)}{12}\delta_{k+m,0}.$$

We used the quantization rule $\{,\} \to \frac{1}{\hbar}[,]$ and the definition of $W(k)$:

$$W(k) = -\frac{1}{2}\sum_{k=m+l} :a(m)a(l): + \frac{i\alpha}{\sqrt{2}}(k+1)a(k),$$

where $:\ :$ denotes the normal ordering.

So, we obtain the simplest W-algebra — the Virasoro algebra. Other W-algebras associated with simple Lie algebras can be obtained in analogous way.

5 W-algebras in the theory of Toda systems

W-algebras arise naturally in the constrained Wess–Zumino formulation of Toda field theories. The Toda equations appear in various physical problems and they can be thought of as a generalization of the Liouville equation:

$$\partial_+\partial_-\phi + Me^{\phi} = 0, \qquad \text{where} \quad M = \text{const}.$$

We use the notions

$$x^{\pm} = \frac{1}{2}(x^0 \pm x^1), \qquad \partial_{\pm} = \partial_0 \pm \partial_1.$$

The Toda equations are given as

$$\partial_+\partial_-\phi^{\alpha} + \frac{1}{2}|\alpha|^2 M^{\alpha} \exp\left(\frac{1}{2}\sum_{\beta\in\Delta} K_{\alpha\beta}\phi^{\beta}\right) = 0,$$

where $K_{\alpha\beta}$ is the Cartan matrix of a simple Lie algebra, Δ denotes the set of simple roots and the M^{α}'s are positive constants. The corresponding Lagrangian is

$$\mathcal{L} = \frac{\kappa}{2}\left(\sum_{\alpha,\beta\in\Delta}\frac{1}{2|\alpha|^2}K_{\alpha\beta}\partial_+\phi^{\alpha}\partial_-\phi^{\beta} - \sum_{\alpha\in\Delta} M^{\alpha}\exp\left[\frac{1}{2}\sum_{\beta\in\Delta}K_{\alpha\beta}\phi^{\beta}\right]\right),$$

where κ is the coupling constant of the theory.

Toda theory can be regarded as constrained WZ model. We start with the WZ action based on a connected real Lie group G with maximally noncompact simple real Lie algebra g

$$S(g) = -\frac{k}{8\pi}\int d^2x\eta^{\nu\mu}\mathrm{tr}\{(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)\} + \frac{k}{12\pi}\int_{B_3}\mathrm{tr}\{(g^{-1}dg)\},$$

where g is a group-valued field and B_3 is a three-dimensional manifold whose boundary is Minkowski space-time; the metric $\eta^{\mu\nu}$ has the signature $(+-)$. We choose the coupling constants κ and k to be related by the equality $k = -4\pi\kappa$.

This action possesses left and right Kac–Moody symmetries. Their Noether currents associated to some Lie algebra element λ are given as

$$\begin{array}{ll} J(\lambda) = \mathrm{tr}(\lambda \cdot J), & J = \kappa(\partial_+ g)g^{-1}, \\ \tilde{J}(\lambda) = \mathrm{tr}(\lambda \cdot \tilde{J}), & \tilde{J} = -\kappa g^{-1}(\partial_- g) \end{array}$$

The field equations are equivalent to the conservation of the left and right currents:

$$\partial_- J = 0, \qquad \partial_+ \tilde{J} = 0.$$

Let now μ^α and ν^α ($\alpha \in \Delta$) be arbitrary positive numbers and let us denote the set of positive roots by Φ^+. By imposing the constraints

$$\begin{array}{lll} J(E_\alpha) = \kappa\mu^\alpha, & \tilde{J}(E_{-\alpha}) = -\kappa\nu^\alpha, & \alpha \in \Delta, \\ J(E_\varphi) = 0, & \tilde{J}(E_{-\varphi}) = 0, & \varphi \in \Phi^+ \setminus \Delta \end{array}$$

the equations of motion of the WZ theory reduce to the Toda field equations (see [7]). Here and below E's and H's are Cartan–Weyl generators of Lie algebra g.

Kac–Moody algebra plays a central role in WZ theory [5]. It is clear that gauge-invariant quantities formed out of the KM current J (and $\tilde{J}$) will also be important in the Toda theory. It can be shown that there is a unique Virasoro algebra in the semidirect product formed by the KM algebra and its associated Sugawara-Virasoro algebra. The energy-momentum density

$$L = L^s - \mathrm{tr}(J'\hat{\rho}), \qquad \text{where} \quad L^s = \frac{1}{2\kappa}\mathrm{tr}(J^2), \quad \hat{\rho} = \frac{1}{2}\sum_{\alpha\in\Phi^+} H_\alpha$$

giving rise to this Virasoro algebra, becomes gauge invariant on the constraint-surface. It follows that L must generate the conformal symmetry of the constrained WZ, i.e., of Toda theory.

There are other gauge-invariant polynomial quantities formed out of the constrained KM currents and their derivatives. A crucial property of these gauge-invariant differential polynomials is that they form a closed algebra under the KM Poisson bracket operation. That is, the Poisson bracket of two gauge-invariant differential polynomials is again expressible in terms of gauge-invariant differential polynomials and δ-distributions. This means that if the quantities W^i form a basis in the ring of gauge-invariant differential polynomials then we have

$$\{W^i(x), W^j(y)\} = \sum_k P_k^{ij}(W)\,\delta^{(k)}(x-y),$$

where the P_k^{ij} are polynomials of the W^i's and their derivatives. This Poisson bracket generates a nonlinear algebra, reminiscent of the universal enveloping algebra. This non-linear algebra of the gauge-invariant differential polynomials always contains the Virasoro algebra, hence it is a polynomial extension of it. The Poisson brackets of the W^i are obtained by first extending their domain P (which is the space of constrained KM currents) to the whole KM phase space K, computing the Poisson brackets on K and then restricting to P. These procedures has been explained in [1].

So, we have seen that W-algebras play the significant role in different branches of theoretical and mathematical physics. The investigations of these objects use various mathematical disciplines and are closely connected with the theory of Lie algebras and Lie groups, theory of dynamical systems, of partial differential equations and so on. The further studies of these algebras seem to be very interesting.

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