# The role of elastica in geometry

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*Abstract:* We discuss the role of elastica in several geometric contexts: the relation to integrable geometric evolution equations, construction of certain Willmore surfaces, and the construction of pseudospherical surfaces.

# Introduction

In this article, we describe some examples which indicate a fundamental role for *elastica* in differential geometry. We give a formal mathematical definition of elastica quite soon, but one should immediately think of equilibrium configurations of inextensible wires, with potential energy due to the deviation of the wire from a straight line (bending energy). This problem is quite old and was one of the early applications of the calculus of variations. Nevertheless, there continue to be interesting and surprising developments relating this problem to differential geometric considerations.

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We will explore the role of elastica in three specific geometric constructions or contexts, which correspond to the three main sections of this paper:

1. Elastica, Killing fields, and integrable geometric evolution equations.
2. Elastica and Willmore surfaces.
3. Elastica and pseudospherical surfaces.

# 1 Elastica, Killing fields, and integrable geometric evolution equations

We start by considering the classical geometry of curves in $\mathbb{R}^3$. Let $\gamma(s)$ be a space curve, parameterized by arc-length $s$. We have the familiar Frenet–Serret equations $T_s = \kappa(s)N$, $N_s = -\kappa(s)T + \tau(s)B$, $B_s = -\tau(s)N$; here, $\{T, N, B\}$ are the tangent, normal, and binormal, $\kappa$ is the curvature, and $\tau$ is the torsion of the curve.

We will consider various *geometric* functionals on space curves, the two most important being

$$L(\gamma) = \int_\gamma ds = \text{length},$$

and

$$E(\gamma) = \int_\gamma \kappa^2(s)\, ds \;=\; \text{bending energy.}$$

Standard arguments in the calculus of variations lead to the following formulae for the Euler–Lagrange operators (or formal $L^2$ gradients) of these functionals: $\nabla L = -\kappa N$, and

$$\nabla E = (\kappa'' + \frac{\kappa^3}{2} - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B,$$

where the prime denotes differentiation with respect to arc-length.

We define an *elastic curve* or *elastica* to be a length-constrained critical point of the bending energy $E$; equivalently, an elastic curve $\gamma$ is a critical point for $E + \lambda L$. It follows immediately from the formulae given above that the curvature and torsion of such a $\gamma$ satisfy

$$\kappa'' + \frac{\kappa^3}{2} - \kappa\tau^2 - \lambda\kappa = 0, \qquad 2\kappa'\tau + \kappa\tau' = 0.$$

From these equations, it easily follows that $\kappa(s)$ and $\tau(s)$ are expressible in terms of elliptic functions.

However, if we are interested in explicit computing $\gamma(s)$, we still need to integrate the Frenet equations. There are various methods for doing this; here, we describe an approach which emphasizes the geometry of the problem:

1. Find *Killing fields* ( = infinitesimal isometries) along $\gamma(s)$ expressible in terms of the local geometric invariants of the curve;

2. Use the Killing fields to facilitate integration.

In [5], Langer and Singer observed that for an elastica $\gamma(s)$, one can write down rather simple expressions for vector fields whose restrictions to $\gamma$ are Killing:

$$X_1 = \kappa B, \qquad X_2 = (\frac{\kappa^2}{2} - \lambda)T + \kappa' N + \kappa\tau B\ .$$

It is not to hard to verify for this for general elastica, but for simplicity, we simply verify that $X_2$ is Killing for *planar* elastica (thus $\tau = 0$ and $\kappa'' + \frac{\kappa^3}{2} - \lambda\kappa = 0$). We have

$$\begin{aligned}
X_2 &= (\frac{\kappa^2}{2} - \lambda)T + \kappa' N; \\
(X_2)' &= \kappa\kappa' T + (\frac{\kappa^2}{2} - \lambda)T' + \kappa'' N + \kappa' N' \\
&= \kappa\kappa' T + (\frac{\kappa^2}{2} - \lambda)\kappa N + \kappa'' N - \kappa\kappa' T \\
&= 0 \text{ (by the Euler–Lagrange equations for planar elastica);}
\end{aligned}$$

thus $X_2$ is *constant* along $\gamma$ and is the infinitesimal generator of translations in the plane.

We can compute the norm of $X_2$ to find

$$|X_2|^2 = (\frac{\kappa^2}{2} - \lambda)^2 + (\kappa')^2 = \lambda^2$$

(which is valid if we assume $\kappa$ and $\kappa'$ go to zero at infinity (the *asymptotically linear* curve case). Without loss of generality, we can assume $\partial_x$ = generator of translation along $x$-axis $= -\frac{1}{\lambda}((\frac{\kappa^2}{2} - \lambda)T + \kappa' N)$. For any planar curve, we have $T = \frac{dx}{ds}\partial_x + \frac{dy}{ds}\partial_y$; thus $\langle T, \partial_x \rangle = \frac{dx}{ds} = \frac{-1}{\lambda}(\frac{\kappa^2}{2} - \lambda)$ which implies immediately that $x(s)$ is expressible in terms of elliptic integrals. Similar calculations apply to $y(s)$. Analogous calculations, although more complicated, can be performed on elastica in $\mathbb{R}^3$. We mention that appropriate choice of the parameters in the various elliptic integrals allows one to determine the *closed* elastica in $\mathbb{R}^3$, and this work was done in [5].

The discussion above was related to a geometric equilibrium problem; we now discuss a related dynamical problem. Let $\gamma(s,t)$ be an evolving space curve ($s$ is again arclength, $t$ the time parameter) with evolution $\gamma_t = \kappa B$. This is called the *localized induction equation*, or LIE. LIE is a highly idealized model of the evolution of an isolated, thin vortex tube in an inviscid, imcompressible fluid [1]. One can immediately ask if there are any simple solutions to this equation; for example, it is clear that a circle evolves by simple translation along its axis of symmetry. Hasimoto was the first to point out the connection between LIE and elastica [2]: if $\gamma(s, 0)$ is an elastic curve, then the associated solution $\gamma(s,t)$ to LIE is (for any time $t$) simply a rigid body motion applied to $\gamma(s, 0)$. Note the dual role of $\kappa B$ — it defines the evolution of general curves, and serves as a Killing field for elastica.

This fact suggests investigating the geometric meaning of LIE. Following Marsden and Weinstein [7], we write

$$\gamma_t = \kappa B = (-T) \times (-\kappa N) = (-T) \times \nabla L.$$

Marsden and Weinstein interpret this equation as follows: let $\Gamma$ denote the space of curves in $\mathbb{R}^3$, with tangent space $T_\gamma \Gamma$ identified with the normal vector fields along $\gamma$. Then $\nabla L$ is the gradient of the length functional, and $(-T)\times$ is a skew-symmetric operator on the tangent space $T_\gamma \Gamma$; LIE has the form of a hamiltonian evolution equation, with *$L$ as its Hamiltonian*, and indeed Marsden and Weinstein check that the implied symplectic structure exists.

This hamiltonian interpretation of LIE becomes more interesting in light of a second discovery of Hasimoto [3]: define a transform from curves to "complex wave functions" via $\gamma(s) \to \psi(s) = \kappa(s)e^{i\int^s \tau(u)du}$. If $\gamma(s,t)$ satisfies LIE, then $\psi(s,t)$ satisfies the *cubic nonlinear Schrödinger equation* (NLS): $\frac{1}{i}\psi_t = \psi_{ss} + \frac{1}{2}|\psi|^2\psi$. NLS is a well-known example of an integrable PDE, and so we have *LIE is a geometric realization of NLS*.

This beautiful connection between geometry and integrable systems was apparently discovered by Hasimoto in an *ad hoc* manner. We mention there is now a systematic procedure for "geometricizing" integrable evolution equations [11].

As a consequence of the LIE-NLS correspondence, LIE enjoys the existence of an infinite list of conserved quantities and commuting evolution equations. We list the first few:

$$
\begin{array}{llr}
X_1 = & \kappa B, & I_1 = \int_\gamma \, ds, \\
X_2 = & \frac{\kappa^2}{2} T + \kappa' N + \kappa\tau B, & I_2 = \int_\gamma -\tau \, ds, \\
X_3 = & \kappa^2\tau T + (2\kappa'\tau + \kappa\tau')N + (\kappa\tau^2 - \kappa'' - \frac{1}{2}\kappa^3)B, & I_3 = \int_\gamma \frac{1}{2}\kappa^2 \, ds, \\
X_4 = & (-\kappa\kappa'' + \frac{1}{2}(\kappa')^2 + \frac{3}{2}\kappa^2\tau^2 - \frac{3}{8}\kappa^4)T & I_4 = \int_\gamma \frac{1}{2}\kappa^2\tau \, ds, \\
& +(-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \frac{3}{2}\kappa^2\kappa')N & \\
& +(\kappa\tau^3 - 3(\kappa'\tau)' - \frac{3}{2}\kappa^3\tau - \kappa\tau'')B, &
\end{array}
$$

Details of the structure and consequences of the LIE-NLS correspondence appear in [4]. In that paper we presented a generalization of the relation observed above for elastica (the evolution vector field $\kappa B$ is a Killing field): Let $\gamma$ be a critical point for $I_n + \lambda I_1$, $n > 2$. Then $X_{n-1}$ and $X_{n-2}$ along $\gamma$ are restrictions of Killing fields in $\mathbb{R}^3$. Thus elastica serve as the first examples of distinguished initial conditions for the hierarchy of integrable evolution equations commuting with LIE.

# 2 Elastica and Willmore surfaces

Let $N_3$ be a three-manifold with metric $g$, and $M_2$ a two-dimensional submanifold. We define the *Willmore functional* $W(M_2) = \int_{M_2}(H^2 + \bar{K})dA$, where $H$ is the mean curvature and $\bar{K}$ is the sectional curvature of $N$ along $M_2$. It is a fact that $W$ is invariant with respect to *conformal* changes of the metric $g$ (thus, if we replace $g(p)$ by $\tilde{g}(p) = \lambda(p)g(p)$, where $\lambda(p)$ is some smooth function, then the value of $W$ does not change). A *Willmore surface* is a critical point for $W$.

Following [6], we wish to find Willmore surfaces in $\mathbb{R}^3$ which are surfaces of revolution. Let us begin by relating the value of $W$ for such a surface, to the geometry of a planar generating curve.

It is convenient to work in cylindrical coordinates in $\mathbb{R}^3$: $\theta, r, z$. We consider a generating curve as lying in the $r-z$ half-plane. The standard metric for $\mathbb{R}^3$ can be written $g_1 = dz^2 + dr^2 + r^2 d\theta^2$. Notice that this metric is conformally equivalent to the metric $g_2 = \frac{dz^2+dr^2}{r^2} + d\theta^2$; and the restriction of this metric to the $r-z$ half-plane is just the standard metric for the Poincaré half-plane! Thus, $(R^3 - \{z\text{-axis}\}, g_1)$ is conformally equivalent to $(\mathbb{R}^3 - \{z\text{-axis}\}, g_2)$ which is isometric to $P \times S^1$, where $P$ denotes the Poincaré half-plane and $S^1$ the unit circle. If $T$ denotes a surface of revolution (around the $z$-axis) then we have $W(T, g_1) = W(T, g_2)$ and $T$ can be considered as a "right-cylinder" in $P \times S^1$. We now compute $W$ with respect to $g_2$:

$$
\begin{aligned}
W(T, g_2) &= \int_T (H_2^2 + \bar{K}_2)\, dA_2 \\
&= \int_T (H_2^2)\, dA_2 \quad \text{(since } \bar{K}_2 \text{ is zero)} \\
&= \int_T \frac{1}{4}(\kappa_g + 0)^2\, dA_2 \ (\kappa_g \text{ is the geodesic curvature of the}
\end{aligned}
$$

generating curve in $P$; the meridians have zero curvature)

$$= \frac{\pi}{2}\int_\gamma (\kappa_g)^2\, ds.$$

This last expression is simply the bending energy $E$ in the Poincaré half-plane. The critical points for $E$ in $P$ have been classified [6], and we conclude by either direct calculation or the principal of symmetric criticality that the surfaces obtained by revolving the Poincaré elastica around the $z$-axis give critical points for the Willmore functional.

In contrast to the discussion in Section 1, we do not know of any far-reaching generalization of this construction using elastica. We do mention a complementary result due to Pinkall [10]: Let $\pi : S^3 \to S^2$ be the Hopf map. If $\gamma$ is critical for $\int_\gamma (\kappa_g^2 + 1)\, ds$ on the two-sphere $S^2$, then $\pi^{-1}$ is critical for $W$ in $S^3$.

# 3 Elastica and pseudospherical surfaces

We recall some basic definitions from classical surface theory. A surface $M$ is called *pseudospherical* if its Gauss curvature is constant and negative. Let $\nu$ be the unit normal to the surface; a tangent vector $W$ to the surface is called *asymptotic* if $\langle \nabla_W \nu, W\rangle = 0$, where $\nabla$ denotes ordinary covariant differentiation in $\mathbb{R}^3$. For a pseudospherical surface, there are generically two transverse asymptotic directions at each point of the surface. An integral curve for the resulting vector fields on the surface is called an *asymptotic curve.* A theorem of Beltrami–Enneper states that for an asymptotic curve $\gamma(u)$ on a surface, the Gauss curvature of the surface at a point on the asymptotic curve is equal to minus the square of the torsion of the curve, considered as a curve in $\mathbb{R}^3$: $G(\gamma(u)) = -\tau_\gamma^2(\gamma(u))$. As a corollary, if $M$ is pseudospherical, then the torsion is constant along an asymptotic curve.

We use these last remarks to give an elementary dynamical construction of pseudospherical surfaces using elastica, and which admits an interesting generalization. Consider the curve $\gamma(s)$ with curvature $\kappa(s) = 2a\,\mathrm{sech}(as)$ and torsion $\tau = \tau_0$. $\gamma$ is not quite an elastic curve; rather, it has the same curvature as a *planar* elastica, but it has constant nonzero torsion. Nevertheless, $\gamma$ has properties similar to that of an elastica. First of all, $\gamma$ has a variational description: it is a critical point for $I_3 - 2\tau_0 I_2 + (\tau_0^2 + a^2)I_1$, further distinguished because it has constant torsion. Also, $X_1 = \kappa B$ and $X_2 = \frac{\kappa^2}{2}T + \kappa' N + \kappa\tau B$ are again Killing fields for $\gamma$.

Define $\Omega = X_2 - \tau_0 X_1 = \frac{\kappa^2}{2}T + \kappa' N$; $\Omega$ is Killing and has no binormal component. Now consider the evolution equation $\gamma_t = \Omega$, with $\gamma(s,0) = \gamma(s)$. Since $\Omega$ is Killing, the image of $\gamma(s,0)$ for any time $t$ will be a rigid body transformation of the original curve.

By construction, the normal $\nu$ to the resulting swept out surface $M$ is in the direction of the binormal $B$ of $\gamma(s,t)$. We claim $\gamma(s,t)$ is (for fixed $t$) an asymptotic curve for $M$. Note that $\langle \nabla_T \nu, T\rangle = \langle \nabla_T B, T\rangle = \langle -\kappa N, T\rangle = 0$, so $T$ is an asymptotic direction (and $\gamma(s,t)$ is an asymptotic curve). Since the torsion of $\gamma(s,t)$ is $\tau_0$ for any time $t$, we have (by Beltrami–Enneper) that the Gauss curvature $G$ of $M$ is $-\tau_0^2$, hence $M$ is pseudospherical.

This construction of pseudospherical surfaces can be generalized [9]: if $\gamma(s,0)$ has constant torsion $\tau_0$ and is a critical point for

$$\sum_{i=0}^{n}\sum_{j=0}^{2i} a_i(-1)^j \tau_0^{-2i+j} I_{2i+1-j},$$

then the evolution $\gamma_t = T^*$, where

$$T^* = \frac{-1}{a_0}\sum_{i=0}^{n}\sum_{j=0}^{2n-1} a_i(-1)^j \tau_0^{-2i+j} X_{2n-j},$$

sweeps out a pseudospherical surface.

Finally, we mention a related phenomenon: in their study of pseudospherical surfaces and soliton theory, Melko and Sterling [8] observe that the resulting surfaces have curvature lines which are planar elastica.

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