On the geometric structure of integrable flows in Grassman manifolds

O.E. Hentosh, Y.A. Prykarpatsky, D.L. Blackmore

2 Subotivska Str. # 13, 290052 L'viv, Ukraina
e-mail: `tolik@mailcity.com, prykan@cybergal.com`

Abstract: Basing on the general approach to finding Lax type integtable flows on Grassmann manifolds, one describes a hierarchy of central extended Poisson structures on adjoint space to unitary loop Lie algebra. They give rise to an exact infinite set of integrable Hamiltonian systems on loop Grassmann manifolds, generalizing Heisenberg and Landau–Lifshitz flows.

1 Introduction

1. In [1,5] some of Lax type integrable flows constrained to live on a loop Grassmann manifold $C_{\mathbf{S}^1}(M)$, where $M := G(H)$ — the Grassmann manifold [7] viewed as the coset space of the unitary group $U(H)$, consisting of the projection operators $P : H \to H$ on the linear Hermitian space H over $\mathbb{C}$, and $C_{\mathbf{S}^1}(M) := \prod_{x \in \mathbf{S}^1} G(H)$ — the direct product parametrized by the circle $\mathbf{S}^1$ points were considered. Basing on the fact that on the manifold $C_{\mathbf{S}^1}(M)$ one can naturally define $U(H)$-action

$$P :\xrightarrow{ad\, a} aPa^+, \quad P^+ = P, \tag{1.1}$$

where $a \in U(H)$, $aa^+ = 1$, in [1] the following important theorem was stated.

THEOREM 1. *The loop Grassmann manifold $C_{\mathbf{S}^1}(M)$ is symplectic with the symplectic structure $\Omega^{(2)} \in \Lambda^2(C_{\mathbf{S}^1}(M))$ being given by the following expression*

$$\Omega^{(2)} = \int_0^{2\pi} dx\, Sp(PdP \wedge dPP), \tag{1.2}$$

and obtained via the standard Dirac type reduction procedure from the next canonical Lie-Poisson structure on the adjoint loop space $\mathcal{G}^* = C_{\mathbf{S}^1}(u^*(H))$: for any γ and $\mu \in \mathcal{D}(\mathcal{G}^*)$

$$\{\gamma, \mu\}_0 := (l,\ [\nabla\gamma(l), \nabla\mu(l)]), \tag{1.3}$$

where $(.,.)$ — the usual Killing form on $C_{\mathbf{S}^1}(u(H))$, $il(P) = P$ — the corresponding momentum mapping from the loop manifold $C_{\mathbf{S}^1}(M)$ into $C_{\mathbf{S}^1}(u^*(H))$, satisfying the constraint $l^2(P) + il(P) = 0$, $P \in C_{\mathbf{S}^1}(M)$.

Proof: The proof of this Theorem is fulfflled by straightforward computing reduced via Dirac procedure [8,9] Poisson bracket (1.3) upon the loop Grassmannian $C_{\mathbf{S}^1}(M)$ and making use of the constraint imposed in the Theorem. □

2. The above construction of a symplectic structure on the loop Grassmannian was generalized in [1] to the case when the loop Lie algebra $\mathcal{G} = C_{\mathbf{S}^1}(u(H))$ is centrally

extended by means of the standard Maurer–Cartan cocycle [8]. Then the corresponding Lie–Poisson bracket on the space $\hat{\mathcal{G}} := \mathcal{G} \oplus \mathbb{C}$ is given as follows:

$$\{\gamma, \mu\}_\lambda := (l,\ [\nabla\gamma(l), \nabla\mu(l)]) + \lambda(\nabla\gamma(l), d\nabla\mu(l)/dx), \tag{1.4}$$

where $\gamma, \mu \in \mathcal{D}(\hat{\mathcal{G}})$, "$\nabla$" — the corresponding gradient operation, $\lambda \in \mathbb{C}$ and $l \in \mathcal{G}^*$, giving rise to the bracket (1.3) at $\lambda = 0$. The problem arises what a symplectic structure generalizing (1.2) does correspond to the Lie–Poisson brackets pencil (1.4) under the restriction $l^2(P) + il(P) = 0,\ P \in C_{\mathbf{S}^1}(M)$, imposed on the momentum mapping

$$C_{\mathbf{S}^1}(M) : \stackrel{l}{\longrightarrow} \hat{\mathcal{G}}^*, \tag{1.5}$$

supposed to be equivariant with respect to a unitary loop group $G := C_{\mathbf{S}^1}(U(H))$ -action on the loop Grassmannian $C_{\mathbf{S}^1}(M)$.

3. The next problem of interest is the development of the analogous study for the intrinsic loop Grassmannian structure and corresponding dual momentum mappings via the construction suggested in [1]. There were treated in detail geometric properties of the loop Grassmannian realized as a submanifold imbedded into a loop matrix manifold $\tilde{C}_{\mathbf{S}^1}(M) := C_{\mathbf{S}^1}(M_{m,N}) \times C_{\mathbf{S}^1}(M_{m,N})$, endowed with a canonical symplectic structure

$$\tilde{\Omega}^{(2)} := \int_0^{2\pi} dx\ Sp(dF \wedge dQ^T), \tag{1.6}$$

where "T" is the transposition sign, $(F, Q) \in C_{\mathbf{S}^1}(M_{m,N}) \times C_{\mathbf{S}^1}(M_{m,N})$ – the space of all loop matrices from Hom$(\mathbb{C}^N; \mathbb{C}^m)$ with maximal rank equal to $m \in \overline{1,N}$, $N \in \mathbb{Z}_+$. As is simply to realize, an arbitrary point $P \in C_{\mathbf{S}^1}(M)$ can be represented as the composition $(F, Q) : \rightarrow\ P := Q^T F$ with the following constraints:

$$F^T Q = (Q^T F)^*, \quad F^T Q F^T Q = F^T Q, \tag{1.7}$$

where the sign "$*$" denotes here the usual complex conjugation. In case when $Q = F^* \in C_{\mathbf{S}^1}(M_{m,N})$ one gets that $P = F^+ F \in C_{\mathbf{S}^1}(M)$ if the constraint $FF^+ = 1$ is imposed additionally on the loop manifold $\tilde{C}_{\mathbf{S}^1}(M)$. Since the symplectic loop manifold $\tilde{C}_{\mathbf{S}^1}(M)$ is invariant with respect to the standard dual unitary loop group actions

$$a :\ (F, Q) \rightarrow (F_a, Q_a), \tag{1.8}$$

where $F_a := Fa^+$, $Q_a^T := aQ^T$ for $a \in C_{\mathbf{S}^1}(U(H))$, and $F_a := aF$, $Q_a^T := Q^T a^+$ for $a \in C_{\mathbf{S}^1}(U(m))$, in [1] the corresponding momentum mappings were calculated explicitely:

$$\tilde{C}_{\mathbf{S}^1}(M) \ni (F, Q) :\stackrel{l}{\longrightarrow} Q^T F \in C_{\mathbf{S}^1}(u^*(H)), \tag{1.9}$$

for the right unitary loop group actions and

$$\tilde{C}_{\mathbf{S}^1}(M) \ni (F, Q) :\stackrel{l}{\longrightarrow} FQ^T \in C_{\mathbf{S}^1}(u^*(m)), \tag{1.10}$$

for the left unitary loop group action. Thereby the following theorem was stated in [1].

THEOREM 2. *The symplectic structure* (1.2) *on the loop Grassmannian* $C_{\mathbf{S}^1}(M)$ *is obtained via a usual Dirac-type reduction from the standard Lie–Poisson structure on the adjoint space* $C_{\mathbf{S}^1}(u^*(H))$ *to the unitary loop Lie algebra* $C_{\mathbf{S}^1}(u(H))$ *under the following first-type constraints*

$$F^* = Q, \quad QF^T = 1, \quad P = F^+F. \tag{1.11}$$

Having imposed on the loop Lie algebra $\mathcal{G} := C_{\mathbf{S}^1}(u(H))$ *the standard Maurer–Cartan central extension* $\mathcal{G} :\to \mathcal{G} \oplus \mathbb{C}$, *on the basis of the momentum mappings* (1.9) *and* (1.10) *one can build a new symplectic structure on the Grassmannian* $C_{\mathbf{S}^1}(M)$ *properly involving the* d/dx *— differentiation, and thereby giving rise to new kinds of Lax type integrable flows on the loop Grassmann manifolds.*

The problem of constructing these new centrally extended symplectic structures on the loop Grassmannian $C_{\mathbf{S}^1}(M)$ described above, as well as a one of generating the corresponding Lax type integrable flows on it, will be the main themes of our further proceeding study.

2 Centrally extended symplectic structures and integrable flows on the loop Grassmann manifolds

1. Let us consider the Lie–Poisson bracket (1.4) on the adjoint loop space $\mathcal{G}^* = C_{\mathbf{S}^1}(u^*(H))$. It is an easy task to prove that Poisson brackets $\{.,.\}_0$ and $\{.,.\}_1 := d/d\lambda\{.,.\}_\lambda|_{\lambda=0}$ are independent and compatible on $\mathcal{G}^*$. But this could not be the case after the reduction to the loop Grassmannian $C_{\mathbf{S}^1}(M)$ via the usual Dirac procedure. Thus we need to reduce the whole Lie–Poisson bracket on the loop Grassmannian $C_{\mathbf{S}^1}(M)$. To do this effectively let us consider a Casimir functional $h(\lambda) \in I_\lambda(\hat{\mathcal{G}}^*)$ satisfying for all $\lambda \in \mathbb{C}$ the following Novikov–Marchenko equation:

$$d\nabla h(\lambda)/dx = [1/\lambda l, \nabla h(\lambda)]. \tag{2.1}$$

Supposing that $|\lambda| \to 0$, we can try to find an asymptotic expansion of the Casimir functional $h(\lambda) \in I(\hat{\mathcal{G}}^*)$:

$$h(\lambda) \simeq h_0 + \sum_{j\geq 1}^{\infty} h_j \lambda^j, \tag{2.2}$$

where by definition $h_0 \in I_0(\hat{\mathcal{G}}^*)$, that is for all $l \in \mathcal{G}^*$ $[l, \nabla h_0(l)] = 0$. Substituting the expansion (2.2) into (2.1) we obtain the following recurrent hierarchy of gradientlike relations:

$$d\nabla h_j/dx = [l, \nabla h_{j+1}]. \tag{2.3}$$

for all $j \in \mathbb{Z}_+$. It is evident that finding solutions to the hierarchy (2.3) is a very complex problem when the element $l \in \mathcal{G}^*$ is arbitrary. But for our case we need to take into account the projector constraint $l^2 + il = 0$, $l \in \mathcal{G}^*$, making the reduction on the loop Grassmannian $C_{\mathbf{S}^1}(M)$, what is demanded in this study.

Suppose now $il \in C_{\mathbf{S}^1}(M)$, that is $l^2 + il = 0$. This obviously means that for any vector field $X \in \mathcal{T}(C_{\mathbf{S}^1}(M))$ the following identity

$$lX + Xl + iX = 0 \tag{2.4}$$

holds at any point $il \in C_{\mathbf{S}^1}(M)$. Consider the following infinite hierarchy of vector fields on the loop Grassmannian $C_{\mathbf{S}^1}(M)$ generated by the Hamiltonian functionals $h_j \in \mathcal{D}(\mathcal{G}^*)$, $j \in \mathbb{Z}_+$, with respect to the Lie–Poisson bracket (1.3) $\{.,.\}_0$ reduced invariantly upon the manifold $C_{\mathbf{S}^1}(M)$:

$$X_j := [l, \nabla h_j], \quad j \in \mathbb{Z}_+. \tag{2.5}$$

The expression (2.5) is obtained having taken into account that the constrant $l^2 + il = 0$ is of the first class. The latter gives a possibility to fulfill the above reduction as that of Marsden–Weinstein [2]. Especially, on the manifold $C_{\mathbf{S}^1}(M)$ as is easy to find the following representations hold:

$$\nabla h_j(l) = [X_j, l], \tag{2.6}$$

where $j \in \mathbb{Z}_+$, $il \in C_{\mathbf{S}^1}(M)$. The representation (2.6) immediately follows from (2.4) and (2.5). As a result the following lemma holds.

LEMMA 3. *The infinite gradient-like hierarchy* (2.3) *reduced on the loop Grassmannian $C_{\mathbf{S}^1}(M)$ admits the following exact solution:*

$$\nabla h_{j+1} = [d\nabla h_j/dx, l] = [d\,[d\nabla h_{j-1}/dx, l]\,/dx, l]\,, \tag{2.7}$$

where $j \in \mathbb{Z}_+$, $l^2 + il = 0$ and $i\nabla h_0 = il \in C_{\mathbf{S}^1}(M)$.

2. As an example of using the Lemma 3, let us calculate an explicit form of the corresponding Hamiltonian flow at $j = 2$:

$$dl/dt_2 = d/dx \nabla h_1 \equiv [l, \nabla h_2] = d/dx[l_x, l] = [l_{xx}, l], \tag{2.8}$$

where $il = P \in C_{\mathbf{S}^1}(M)$, $t \in \mathbb{R}$ – an evolution parameter. Thereby one can formulate the next theorem about integrability of the flow (2.8).

THEOREM 4. *The flows* (2.5) *on the loop Grassmannian $C_{\mathbf{S}^1}(M)$ are Lax type integrable and bi-Hamiltonian.*

Proof: It is easy to show that the hierarchy of vector fields (2.5) on the manifold $C_{\mathbf{S}^1}(M)$ can be written down as the following Lax type relation:

$$\lambda^{-1} dl/dt_j := \lambda^{-1} X_j = -[l\lambda^{-1} - d/dx,\ (\lambda^{-(j-1)} \nabla h(l))_-], \tag{2.9}$$

where $t_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$, — the corresponding evolution parameters, the expression $(\ .\ )_-$ denotes taking the negative part with respect to the expansion (2.2) in the parameter $\lambda \in \mathbb{C}$ as $|\lambda| \to 0$.

In virtue of the recurrent relations (2.3) one can easily state that all the functionals $h_j \in \mathcal{D}(\mathcal{G}^*)$, $j \in \mathbb{Z}_+$, reduced on the manifold $C_{\mathbf{S}^1}(M)$ are in involution with respect to two independent and compatible Poisson brackets $\{.,.\}_0$ and $\{.,.\}_1$, that is

$$\{h_j, h_k\}_0 = 0 = \{h_j, h_k\}_1 \tag{2.10}$$

for all $j, k \in \mathbb{Z}_+$. This proves the Lax type integrability of the bi-Hamiltonian flows (2.5) on the loop Grassmannian $C_{\mathbf{S}^1}(M)$. □

3. As a simple consequence of the above proved Theorem 4 we find that the dynamical system (2.8) known in literature as the Heisenberg model [8], is also a Lax type integrable bi-Hamiltonian flow on the loop Grassmannian $C_{\mathbf{S}^1}(M)$. The problem of proving this fact within the momentum mapping aproach was, by the way, posed in [1]. It was solved completely above, if one take into account that a projector matrix $P = il \in C_{\mathbf{S}^1}(M)$ satisfies, due to (2.8), the following Heisenberg dynamical system:

$$dP/dt = i[P, P_{xx}], \tag{2.11}$$

where $P^2 = P$ and $P^+ = P$ for all $x \in \mathbb{R}$, $t \in \mathbb{R}$.

4. In the framework of the present analysis of dynamical systems on the loop Grassmann manifold $C_{\mathbf{S}^1}(M)$ the following interesting problem arose: how to generalize the above momentum mapping technique for another nonlinear model:

$$d\vec{S}/dt = \vec{S} \times \vec{S}_{xx} + \vec{S} \times J\vec{S}, \tag{2.12}$$

where the vector $\vec{S} \in \mathbb{R}^3$, $\langle \vec{S}, \vec{S} \rangle = 1$, and the matrix $J := \operatorname{diag}(J_1, J_2, J_3)$ is arbitrary monotonic and constant. If one defines the following Hermitian matrix

$$S := \sum_{a=1}^{3} S_a \sigma^a \in C_{\mathbf{S}^1}(u^*(H^2)),$$

where

$$\sigma^1 = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|, \quad \sigma^2 = \left\| \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right\|, \quad \sigma^3 = \frac{1}{2} \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|,$$

then this matrix evidently satisfies the condition $S^2 = 1$. Thus the matrix $P = (1 - S)/2$ belongs to the loop Grassmannian manifold $C_{\mathbf{S}^1}(M)$ and satisfies some dynamical system generalizing (2.11) and being equivalent to (2.12).

To find the corresponding Lax type representation for the dynamical system (2.12), we shall make use of the Lie–Poisson bracket (1.4) in the following slightly different form:

$$\{\gamma, \mu\}_z := (l\frac{1}{z}, [\nabla\gamma, \nabla\mu]) + (\nabla\gamma, d\nabla/dx), \tag{2.13}$$

where $\gamma, \mu \in \mathcal{D}(\hat{\mathcal{G}}^*)$, $z \in \mathbb{C}$ — a complex parameter. Let θ_1 and θ_2 be some automorphisms of the loop Lie algebra $C_{\mathbf{S}^1}(u(H^2))$ having no common fixed points. Making then the usual averaging [8] of the bracket (2.13) over a grating of periods

$$\operatorname{gr}(\mathbb{C}) := \{m_1\omega_1 + m_2\omega_2 \in \mathbb{C} : \ Im\, \omega_1/\omega_2 > 0, \ m_1, m_2 \in \mathbb{Z}\},$$

one can obtain a Poisson bracket on $\mathcal{D}(\hat{\mathcal{G}}^*)$ compatible for all $z \in \mathbb{C}$:

$$\{\gamma, \mu\}_\theta := (l_\theta(z), [\nabla\gamma, \nabla\mu]) + (\nabla\gamma, d\nabla\mu/dx), \tag{2.14}$$

where by definition, $\theta_1\xi := \sigma_1\xi\sigma_1$, $\theta_2\xi := \sigma_3\xi\sigma_3$, $\xi \in \mathcal{G}$,

$$l_\theta(z) := \sum_{m_1,m_2\in\mathbb{Z}} \varrho\sigma_3^{m_2}\sigma_1^{m_1} l \sigma_1^{m_1}\sigma_3^{m_2}(z - m_1\omega_1 - m_2\omega_2)^{-1}. \tag{2.15}$$

As a result of computing the matrix (2.15), we get the following expression [3,6,8]

$$l_\theta(z) = -i\varrho/\mathrm{sn}(z;k)\left[S_1\sigma_1 + \mathrm{dn}(z;k)S_2\sigma_2 + \mathrm{cn}(z;k)S_3\sigma_3\right], \tag{2.16}$$

with the elliptic functions sn, dn and cn, built in the case $\omega_1 = 2K$, $\omega_2 = i2K'$, where K and K' — full elliptic integrals of moduli k and $k' := \sqrt{1-k^2}$ correspondingly, and $0 < k < 1$, $k^2 := (J_2 - J_1)/(J_3 - J_1)$, $\rho := \sqrt{J_3 - J_1}$.

Let us now suppose that a functional $h(z) \in I_\theta(\hat{\mathcal{G}}^*)$, where

$$d\nabla h(z)/dx = [l_\theta(z), \nabla h(z)] \tag{2.17}$$

for all $z \in \mathbb{C}$, possesses the following asymptotic expansion as $|z| \to \infty$:

$$h(z) \simeq \sum_{j\in\mathbb{Z}_+}\left(h_j^{(1)}\frac{\xi^j}{\nu} + h_j^{(2)}\xi^j\right), \tag{2.18}$$

in the elliptic curve parametrization: $\nu^2 := 1/4\xi(\xi+\alpha)(\xi+\beta)$, $\alpha := -1/4(J_1 - J_3)$, $\beta = -1/4(J_2 - J_3)$, and $\xi = \varrho^2\mathrm{cn}^2(z;k)/\mathrm{sn}^2(z;k)$, $z \in \mathbb{C}$.

Substituting expansion (2.18) into (2.17) one can obtain two infinite hierarchies on components $\nabla h_j^{(1)}$ and $\nabla h_j^{(2)} \in \hat{\mathcal{G}}$, $j \in \mathbb{Z}_+$, which are recurrently solved upon the loop submanifold $C_{\mathbf{S}^1}(M)$ imbedded into the adjoint loop space $\hat{\mathcal{G}}^* = C_{\mathbf{S}^1}(u^*(H))$. As a result of such computations one obtains an infinite hierarchy of Lax type integrable flows on the loop Grassmannian $C_{\mathbf{S}^1}(M)$, known in literature as Landau–Lifshitz hierarchy [6, 10].

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