Soliton equations and classical mechanics

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Abstract: The present lecture consists of two main sections. In section 1, some basic notions of hamiltonian mechanics are tersely reviewed. In particular, I recall the notions of Poisson and Symplectic Manifolds, and that of Lie-Poisson Structure; I mention some important properties of Hamiltonian Systems with symmetries, and introduce the concept of Moment Map; I end the section by saying a few words on Lax Representation and on the notion of Double Lie-Poisson Structure related with the existence of a classical r-matrix.
Section 2 is devoted to illustrate a number of techniques that allow to extract form "Soliton Equations" (infinite-dimensional integrable hamiltonian Systems) *finite-dimensional* integrable Systems. As a starting point, I consider the *stationary* or *restricted* flow technique, leading for instance from vector NLS, or from KdV, to Garnier System; then I mention other classes of finite-dimensional solution manifolds, like the rational solutions of KdV and Benjamin-Ono and their relation with Calogero-Moser System, and finally I look at the so-called *peakon-* solutions of Camassa-Holm equation, investigating in particular the (generically dynamical) r-matrix structure of the associated finite-dimensional hamiltonian System.

1 Some basic notions of hamiltonian mechanics (an overview)

In this Section, I tersely recall some notions of hamiltonian mechanics [1–3]. I tried to single out those having a deeper connection with soliton theory. The treatment might well look hand-waving, but rigor was not my main concern. For a deeper understanding, I suggest the interested reader to look at the above references, and in particular at [2], which I widely used while preparing my notes.

1.1 Lagrangian and Hamiltonian formulation

Let M be an N-dimensional manifold, modelled on the Euclidean space $\mathbb{R}^N$. We denote by $\vec{q} = (q_1, \ldots, q_N)$ a point and by $\vec{q}(t)$ a smooth curve in M; $\dot{\vec{q}} = \frac{d\vec{q}}{dt}$ represents a tangent vector to the curve, and the pair $(\vec{q}, \dot{\vec{q}})$ parameterize the tangent space to M at the point $\vec{q}$ denoted as usually as $T_{\vec{q}}M$; TM is the tangent bundle.

Let $\mathcal{L}(\vec{q}, \dot{\vec{q}})$ be a smooth function from TM to $\mathbb{R}$; we define as usual the action integral:

$$I = \int_{t_0}^{t_1} dt \mathcal{L}(\vec{q}, \dot{\vec{q}}). \tag{1.1}$$

Its stationary points (vs an arbitrary variation $\delta\vec{q}(t)$ s.t. $\delta\vec{q}(t_0) = \delta\vec{q}(t_1) = 0$) are given by the solutions of the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\vec{q}}} - \frac{\partial\mathcal{L}}{\partial\vec{q}} = 0, \tag{1.2}$$

which selects the "real" trajectory from $\vec{q}_0 = \vec{q}(t_0)$ to $\vec{q}_1 = \vec{q}(t_1)$.

The Legendre transformation

$$\mathcal{L}(\vec{q}, \dot{\vec{q}}) \to \mathcal{H}(\vec{p}, \vec{q}) = -\mathcal{L} + \vec{p}\cdot\dot{\vec{q}}, \tag{1.3}$$

where

$$\vec{p} := \frac{\partial\mathcal{L}}{\partial\dot{\vec{q}}}; \qquad (\vec{q}, \vec{p}) \in T^*M \tag{1.4}$$

defines the Hamiltonian $\mathcal{H}$ as a function from T^*M to $\mathbb{R}$.

REMARK 1. To express $\dot{\vec{q}}$ as a function of $\vec{q}, \vec{p}$ we need $D = \det|\frac{\partial^2\mathcal{L}}{\partial\dot{q}_j\partial\dot{q}_k}| \neq 0$ (*non degeneracy condition*).

Formulae (1.2), (1.4) entail the Hamilton equations

$$\dot{\vec{q}} = \frac{\partial\mathcal{H}}{\partial\vec{p}}, \qquad \dot{\vec{p}} = -\frac{\partial\mathcal{H}}{\partial\vec{q}}.$$

Now consider $T^*M \simeq \mathbb{R}^N \times \mathbb{R}^N$ and define $\vec{x} = \begin{pmatrix}\vec{q}\\ \vec{p}\end{pmatrix}$, $J_0 = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$. The Hamilton equations take the form

$$\dot{\vec{x}} = J_0\frac{\partial\mathcal{H}}{\partial\vec{x}}. \tag{1.5}$$

REMARK 2. A generalisation of the Legendre transformation to higher order Lagrangian functions has been found by Ostrogradski (see for instance again [3], p. 179). Let

$$\mathcal{L} = \mathcal{L}(\vec{q}, \vec{q}^{(1)}, \ldots, \vec{q}^{(M)}) \qquad \left(\vec{q}^{(j)} := \frac{\partial^j\vec{q}}{\partial t^j}\right).$$

The Euler–Lagrange equations read

$$\sum_{k=0}^{M}(-1)^k\frac{\partial^k}{\partial t^k}\frac{\partial\mathcal{L}}{\partial\vec{q}^{(k)}} = 0. \tag{1.6a}$$

One can define new coordinates

$$\vec{q}_k := \vec{q}^{(k-1)}, \qquad (k = 1, \ldots, M) \tag{1.6b}$$

and new momenta

$$\vec{p}_k = \sum_{j=0}^{M-k}(-1)^j\frac{\partial^j}{\partial t^j}\frac{\partial\mathcal{L}}{\partial\vec{q}^{(j+k)}}. \tag{1.6c}$$

The Legendre–Ostrogradski transformation is then given by

$$\mathcal{L}(\vec{q}, \ldots, \vec{q}^{(M)}) \to \mathcal{H}(\{\vec{q}_k, \vec{p}_k\}) = -\mathcal{L} + \sum_k \vec{p}_k\cdot\vec{q}_{(k+1)}. \tag{1.6d}$$

1.2 Poisson manifolds

Consider smooth functions $F, G, \ldots$, from T^*M into the reals. To any pair of functions, say F_1, F_2, one can associate a third function F_3, through

$$F_3 = \{F_1, F_2\},$$

where

$$\{F_1, F_2\}(\vec{x}) := \frac{\partial F_1}{\partial \vec{x}} \cdot J_0 \frac{\partial F_2}{\partial \vec{x}}. \tag{1.7}$$

Notice that

$$J_0 \frac{\partial \mathcal{H}}{\partial \vec{x}} = \{\vec{x}, \mathcal{H}\}$$

so that the Hamilton equations read

$$\dot{\vec{x}} = \{\vec{x}, \mathcal{H}\}. \tag{1.8}$$

Let us denote for a moment by Φ_M the algebra of real-valued smooth functions on T^*M: we say that a mapping $\{.,.\}$ from $\Phi_M \times \Phi_M$ to Φ_M is a Poisson bracket (P.B.), if it has the following properties:

1. skew-symmetry: $\{F, G\} = -\{G, F\}$,
2. bilinearity: $\{\alpha F_1 + \beta F_2, G\} = \alpha\{F_1, G\} + \beta\{F_2, G\}$,
3. Leibnitz property: $\{F_1 F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$,
4. Jacobi identity: $\{F_1, \{F_2, F_3\}\} + \text{cyclic perms} = 0$.

Φ_M acquires then a *Lie-algebra structure*, the Lie commutator being given by the Poisson bracket. A manifold (isomorphic to an open set of) $\mathbb{R}^{2N}$, endowed with an $N \times N$ matrix J, which is skew-symmetric and fulfills the Jacobi identity, is a *Poisson manifold*; J_0 is just a *special case* of such matrix.

Canonical Transformations

A meaningful question to ask is: how does a given J transforms by a change of variables in $\mathbb{R}^{2N}$? The answer is very simple.

Let $y_j = f_j(x_1 \ldots x_N)$; for any $F \in \Phi_M$ we have $\frac{\partial F}{\partial x_k} = \sum_j \frac{\partial y_j}{\partial x_k} \frac{\partial F}{\partial y_j}$, or

$$\frac{\partial F}{\partial \vec{x}} = A \frac{\partial \mathcal{F}}{\partial \vec{y}} \qquad \left(A_{kj} := \frac{\partial y_j}{\partial x_k}\right)$$

hence

$$\{F, G\}(\vec{x}) = \frac{\partial F}{\partial \vec{y}} \cdot \tilde{J} \frac{\partial G}{\partial \vec{y}}$$

with $\tilde{J} = A^t J A$. The transformation $\vec{x} \to \vec{y}$ which leaves J invariant (i.e. $\tilde{J} = J$, so that the Poisson bracket stays unchanged), is called *canonical.* Such transformations form a group, the isotropy group of J; for $J = J_0$, it is usually called the *symplectic group.*

REMARK 3. Taking now $\mathbb{R}^{2N}$ as our new configuration space, the pair $(\dot{\vec{x}}, \vec{x})$, singles out a point in the *tangent space* to $\mathbb{R}^{2N}$ in $\vec{x}$, while $(\frac{\partial F}{\partial \vec{x}}, \vec{x})$ denotes a point in the *cotangent space*. In this sense, $J(\vec{x})$ is a *tensor* field mapping the *cotangent bundle* into the *tangent bundle*.

REMARK 4. J needs not be nonsingular, though of course this property is coordinate free. If J is singular, the linear equation $J\vec{v} = 0$ has a *nontrivial* solution. Assuming $\vec{v} = \frac{\partial C}{\partial \vec{x}}$ for some function C, it follows

$$\{F, C\} = 0 \qquad \forall F.$$

Such a function C is called "*Casimir*" function .

Hamiltonian vector fields

Suppose we are given a hamiltonian system

$$\dot{\vec{x}} = \{\vec{x}, \mathcal{H}\} := X_{\mathcal{H}}.$$

The vector field

$$X_{\mathcal{H}} := J\frac{\partial \mathcal{H}}{\partial \vec{x}} \tag{1.9}$$

is called a hamiltonian vector field. A function $F(\vec{x})$, such that

$$\dot{F}(\vec{x}) = \frac{\partial F}{\partial \vec{x}} \cdot \dot{\vec{x}} = \{F, \mathcal{H}\}(\vec{x}) = 0 \tag{1.10}$$

is called an "*integral of motion*". Clearly, due to Jacobi identity, integrals of motions form an abelian subalgebra of the whole Poisson algebra.

A Lie algebra is also built up by the hamiltonian vector fields, associated to smooth functions on our Poisson manifold, once we define their commutator as follows:

$$[X_F, X_G] := X'_F[X_G] - X'_G[X_F], \tag{1.11a}$$

where

$$(X'[Y])_j = \frac{\partial X_j}{\partial x_k} Y_k.$$

Indeed, it holds

$$[X_F, X_G] = X_{\{F,G\}} \tag{1.11b}$$

or, more explicitly,

$$\left[J\frac{\partial F}{\partial \vec{x}}, J\frac{\partial G}{\partial \vec{x}}\right] = J\frac{\partial}{\partial \vec{x}}\{F, G\}.$$

We have the properties

$$\{F, G\} = 0 \Rightarrow [X_F, X_G] = 0$$

$$[X_F, X_G] = 0 \Rightarrow \{F, G\} = \text{constant } \textit{“co-boundary”}$$

So, we have a Lie-algebra homomorphism.

Lie–Poisson structure

So far, we have just given an example (the standard one) of Poisson tensor. An interesting class of Poisson tensors is given by linear (in $\vec{x}$) Poisson tensors of a special type [2].

Suppose $\mathcal{G}$ is a Lie algebra (finite-dimensional: think for instance of a *matrix* Lie algebra), and let $\{e_j\}$ be a basis in $\mathcal{G}$. One has

$$[e_i, e_k] = C_{ik}^l e_l.$$

Consider its dual $\mathcal{G}^*$ (i.e., the space of linear functionals on $\mathcal{G}$). Denote by ξ an element of $\mathcal{G}$, by x an element of $\mathcal{G}^*, \langle x, \xi\rangle = x(\xi)$ being the duality form, by η^j the basis in $\mathcal{G}^*$ dual to the basis e_i:

$$\xi = \sum \xi^j e_i, \qquad x = \sum x_i \eta^i, \qquad \langle \eta^i, e_j\rangle = \delta^i_j,$$

$\mathcal{G}^*$ becomes a Poisson manifold if we define, in a natural way

$$\{x_i, x_j\} = C_{ij}^k x_k. \tag{1.12}$$

Smooth functions on $\mathcal{G}^*$ are then equipped with the Poisson bracket

$$\{F, G\}(x) = C_{ij}^k x_k \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j},$$

which can be written in the simple invariant form

$$\{F, G\} = \langle x, [\nabla F, \nabla G]\rangle \qquad (1.13),$$

once we realize that if x belongs to $\mathcal{G}^*, \nabla F$ belongs to its dual, i.e. to $\mathcal{G}$.

The above Poisson structure, of great importance in the theory of integrable systems, can be generalized to infinite-dimensional spaces and is known as a Lie–Poisson structure.

Symplectic manifolds

If a Poisson tensor is nonsingular, its inverse, usually denoted as ω, is called a symplectic tensor. It maps the tangent bundle into the cotangent bundle and makes our "configuration" space a symplectic manifold. Obviously, it is skew-symmetric. The condition that ensures that it gives rise to a Poisson bracket is now linear and reads

$$\partial_k\omega_{ij} + \partial_i\omega_{jk} + \partial_j\omega_{ki} = 0 \qquad \forall ijk. \tag{1.14}$$

Usually, one defines the symplectic form (again denoted as ω)

$$\omega = \sum_{jk} \omega_{jk} dx^j \wedge dx^k$$

so that

$$\omega(X_F, X_G) = \sum_{jk} \omega_{ik} X_F^j X_G^k = -X_F \cdot \omega X_G$$

$$= -J\nabla F \cdot \omega J\nabla G = \nabla F \cdot J\nabla G = \{F, G\}. \tag{1.15}$$

A theorem by Darboux ensures that — at least locally — on a symplectic manifold there exists a system of coordinates $(\vec{q}, \vec{p})$ such that

$$\omega = \sum_i dp_i \wedge dq_i, \qquad \text{i.e.} \qquad (\omega_{jk}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and the Poisson bracket takes again the canonical form.

Complete integrability (Arnold–Liouville)

If (M, ω) is a symplectic manifold, hence a Poisson manifold of (necessarily) even dimension $2n$, and there exist smooth globally defined independent functions $F_1, \ldots, F_n$ in involution, then

1. M_a, i.e. the manifold of constant values of $F_i : F_i = a_i$ is invariant under the flows induced by any $\mathcal{H} = \mathcal{H}(F_1, .., F_n)$.
2. (Every connected component of) M_a is basically a Cartesian product of a k-torus and an $n-k$ Euclidean space. If M_a compact, it is an n-dim. torus.
3. There exist coordinates φ_j such that $\{\varphi_j, \mathcal{H}\} = \nu_j(\mathcal{H})$. Usually $\{\varphi_j\}$ are not canonically conjugated to $\{F_k\}$: $\{\varphi_j, F_k\} \neq \delta_{jk}$. However, there exists a set of action variables $\{I_k\}$ $(I_k = I_k(F_1, \ldots, F_n))$ such that

$$\{\varphi_j, I_k\} = \delta_{jk},$$

i.e. ω takes the Darboux form:

$$\omega = \sum_j dI_j \wedge d\varphi_j$$

which implies:

$$\dot{\varphi}_j = \{\varphi_j, \mathcal{H}\} = \nu_j(\mathcal{H}), \qquad \left(\nu_j = \frac{\partial \mathcal{H}}{\partial I_J}\right), \tag{1.16a}$$

$$\dot{I}_j = \{I_j, \mathcal{H}\} = 0. \tag{1.16b}$$

To have a deeper insight into these kind of systems, let us consider a hamiltonian system in $\mathbb{R}^{2n}$ with the standard P.B. (or sympl. structure). Assume we have found in some way an involutive family of independent integrals F_j. We can choose F_j as the new "momenta" P_j and construct their conjugate variables Q_j by looking for a generating function S of the canonical transformation $(p, q) \to (P, Q)$

$$\sum_j p_i dq_j + \sum_j Q_j dP_j = dS(q, P), \tag{1.17a}$$

whence we get

$$p_j = \frac{\partial S}{\partial q_j}, \qquad Q_j = \frac{\partial S}{\partial P_j}. \tag{1.17b}$$

Assuming one is able to express p_j as functions of (q, P) (i.e. to invert the expressions: $P_j = P_j(q, p)$), from (1.17a) one founds S up to arbitrary function of P:

$$S = S_0(P) + \sum_j \int^{q_j} p_j(q, P) dq_j. \tag{1.17c}$$

Thus

$$Q_j = \frac{\partial S_0}{\partial P_j} + \sum_k \int^{q_k} \frac{\partial p_k}{\partial P_j} dq_k \tag{1.17d}$$

and, in the compact case

$$\nu_j = \sum_k \oint dq_k \frac{\partial p_k}{\partial P_j}. \tag{1.17e}$$

A simple instance is the one where we have *separation of variables*, i.e.:

$$S = \sum_j S_j(q_j; F_1, \ldots, F_n).$$

Then the integration of the equations of motion is reduced to quadratures already in the variables (q, p).

In fact

$$\mathcal{H} = \mathcal{H}(p, q) = \mathcal{H}\left(\frac{\partial S_1}{\partial q_1}, \ldots, \frac{\partial S_n}{\partial q_n}, q_1, \ldots, q_n\right), \tag{1.18a}$$

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j} = f_j(q_1, \ldots, q_n, F_1, \ldots, F_n). \tag{1.18b}$$

On each manifold M_a of constant values of F_j we have a system of n differential equations. In particular, if we are dealing with a "natural hamiltonian system"

$$\mathcal{H} = \sum_j p_j^2 + V(q_1, \ldots, q_n), \tag{1.19a}$$

then

$$\dot{q}_j = p_j = \frac{\partial S_j}{\partial q_j}(q_j, F_1, \ldots, F_n) = f_j(q_j, F_1, \ldots, F_n) \tag{1.19b}$$

and the problem is then "trivially" solved through

$$\int_{\bar{q}_j}^{q_j} \frac{dq_j}{f_j} = t - \bar{t}. \tag{1.19c}$$

1.3 A few words on group theoretical aspects

Hamiltonian group action and moment map

Let (M, ω) be a symplectic manifold and G a Lie group acting on M:

$$x \in M \to gx \in M, \qquad g \in G.$$

We may think of g as $g(t) = e^{t\xi}$ for some ξ belonging to $\mathcal{G}$, the Lie algebra of G. To each $\xi \in \mathcal{G}$ one can associate a vector field on M (intuitively: $x(t) = g(t)x$ describes a curve in M, starting from x). Differentiating with respect to t we get a tangent vector at the point $x(t)$. At $t = 0$ we get the tangent vector $\xi x = X_\xi$.

If there exists a function $\mathcal{H}_\xi$ such that

1. $\omega X_\xi = \frac{\partial \mathcal{H}_\xi}{\partial x}$,
2. $\mathcal{H}_{\xi+\eta} = \mathcal{H}_\xi + \mathcal{H}_\eta$ (linearity),
3. $\mathcal{H}_{[\xi,\eta]} = \{\mathcal{H}_\xi, \mathcal{H}_\eta\}$,

we speak of a Hamiltonian action of the group G. For instance, if ω is constant, we have

$$\mathcal{H}_\xi = \frac{1}{2} x \cdot \omega \xi x,$$

and in general

$$\mathcal{H}_\xi = x \cdot \int_0^1 d\lambda\ \lambda\ \omega(\lambda x)\ \xi x.$$

In any case $\mathcal{H}_\xi$ is linear in ξ, so that we can certainly consider it as a linear functional on $\mathcal{G}$:

$$\mathcal{H}_\xi = \langle \mu(x), \xi \rangle \tag{1.20}$$

$\mu(x)$ being element of $\mathcal{G}^*$; thus, we have a map: $\mu : M \to \mathcal{G}^*$ called the *moment map.*

Example 1: *the moment map for the rotation group.* Consider the ordinary Euclidean space $\mathbb{R}^3$, the Poisson manifold $T^*\mathbb{R}^3 \simeq \mathbb{R}^6$ equipped with the canonical bracket, and the action of the $SO(3)$ group on it, through

$$\phi_g(p, q) = (gp, gq)$$

with $g \in SO(3)$ (i.e. g is an orthogonal unimodular matrix). The corresponding vector field is

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} Ap \\ Aq \end{pmatrix}.$$

A being some $so(3)$ traceless skew-symmetric matrix. The corresponding Hamilton equations read

$$Ap = -\frac{\partial \mathcal{H}}{\partial q}, Aq = \frac{\partial \mathcal{H}}{\partial p} \Rightarrow \mathcal{H}_A = (Ap) \cdot q = \langle a, q \wedge p \rangle.$$

Hence for the moment map we have $\mu(q,p) = q \wedge p$ *(angular momentum).*

Example 2: *hamiltonian action of* $U(n)$. The configuration space is now the set X of hermitian $n \times n$ matrices. The Poisson (symplectic) manifold is T^*X, whose elements are the pairs (x,y) of hermitian matrices. G is the group $U(n)$ of unitary $n \times n$ matrices, which acts through

$$(x,y) \to (gxg^+, gyg^+), \qquad gg^+ = 1.$$

The corresponding Poisson bracket is given by:

$$\{F_1, F_2\} = \mathrm{tr}\left(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_2}{\partial y}\frac{\partial F_1}{\partial x}\right).$$

The linearization of the group action ($g = e^{t\xi}, \xi = -\xi^+$) yields the vector field

$$X_\xi = ([\xi, x], [\xi, y])$$

which is hamiltonian for

$$\mathcal{H}_\xi = \mathrm{tr}[x,y]\xi$$

thus yielding the moment map: $\mu(x,y) = [x,y]$. One can prove the following theorem:

THEOREM: *If a Hamiltonian* $\mathcal{H}$ *is invariant under the action of a group* G, *then the corresponding moment map is an integral of motion. In fact, the invariance property*

$$\mathcal{H}(x) = \mathcal{H}(gx) \quad \forall g \in G$$

entails, by taking $g = \exp(t\xi)$ *and differentiating with respect to* t

$$\frac{\partial \mathcal{H}}{\partial x} \cdot J \frac{\partial \mathcal{H}_\xi}{\partial x} = 0 \Leftrightarrow \{\langle \mu(x), \xi \rangle, \mathcal{H}\} = 0, \quad \forall \xi \in \mathcal{G}$$

hence $\{\mu(x), \mathcal{H}\} = 0$.

1.4 Lax representation

Historically (starting from late 60's) many hamiltonian systems have been discovered to be integrable because one has succeeded in casting them in Lax form. Suppose one is given a dynamical system:

$$\dot{x}_j = f_j(x_1, \ldots, x_N). \tag{1.21a}$$

Assume one can find two matrices L, M of dimension m, depending on the dynamical variables x, such that (1.21a) can be rewritten as

$$\dot{L} = [L, M]. \tag{1.21b}$$

Then (1.21a) has certainly m integrals of motions. Indeed, $L(t) = U(t)L(0)U^{-1}(t)$ with $U(0) = I$, $M = U^{-1}\dot{U}$, which implies for instance $\mathrm{tr}L^k(t) = \mathrm{tr}L^k(0)$.

Note that the above mentioned *Lie–Poisson structure* leads naturally to a Lax representation. Indeed recall (1.13):

$$\{F, G\}(x) = \langle x, [\nabla F, \nabla G]\rangle,$$

where $\langle , \rangle$ is the pairing between $\mathcal{G}^*$ and $\mathcal{G}$. Identifying $\mathcal{G}$ and $\mathcal{G}^*$ via the above pairing, assumed to be invariant under the adjoint action $x \to gxg^{-1} = \tilde{x}$, we have

$$\langle x, y\rangle = \langle \tilde{x}, \tilde{y}\rangle = \langle x, y\rangle + t(\langle [\xi, x], y\rangle + \langle x, [\xi, y]\rangle) + 0(t^2)$$
$$\Rightarrow \langle [\xi, x], y\rangle + \langle x, [\xi, y]\rangle = 0.$$

We can write

$$\{F, G\} = \langle \nabla G, [x, \nabla F]\rangle,$$

and then

$$\dot{x} = \{\mathcal{H}, x\} \Rightarrow \dot{x} = -\langle \nabla x, [x, \nabla \mathcal{H}]\rangle = -[x, \nabla \mathcal{H}].$$

Double Lie–Poisson structure and classical r-matrix

Suppose our Poisson manifold is $\mathcal{G} \times \mathcal{G}$, equipped with the Lie-Poisson structure (1.13). Suppose we can define a linear map on $\mathcal{G}$, say r, such that the expression:

$$[x, r(y)] + [r(x), y] := [x, y]_r$$

is again a Lie product, i.e. satisfies the Jacobi identity. Then r is called a *classical r-matrix*. In the first formulation [4,5] r was assumed to be "constant", i.e. independent of the point x. However, this is not necessary, as it has been shown later [6]. We have the following propositions.

PROPOSITION 1. *Ad-invariant function on G are in involution with respect to the r bracket defined as:*

$$\{F_1, F_2\} = \langle x, [\nabla F_1, \nabla F_2]_r\rangle.$$

PROPOSITION 2. *The equations of motion can be written in Lax form.*

Indeed, suppose $\mathcal{H}$ is an Ad-invariant function $\mathcal{H}(x) = \mathcal{H}(gxg^{-1})$. Then

$$\forall \xi : \ \langle \nabla \mathcal{H}, [\xi, x]\rangle = \langle x, [\nabla \mathcal{H}, \xi]\rangle = 0.$$

Hence

$$\langle x, [\nabla F_1, \nabla F_2]_r\rangle = \langle x, [\nabla F_1, r(\nabla F_2)] + \langle x, [r(\nabla F_1), \nabla F_2]\rangle$$

and both quantities on the r.h.s. are zero if F_1, F_2 are invariant functions. Moreover

$$\dot{x} = \{x, \mathcal{H}\} = [x, r(\nabla(\mathcal{H})].$$

The most important classes of r-matrices arise from a Lie algebra splitting. Namely, let: $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$, $\quad \mathcal{G}_\pm := \pi_\pm(\mathcal{G})$, $\pi_\pm$ being the corresponding projections; then $r = \pi_+ - \pi_-$ is an r-matrix and $[x, y]_r = [x_+, y_+] - [x_-, y_-]$.

2 Soliton equations and finite-dimensional reductions

Question: what has Part 1 to do with soliton equations?
Answer: Soliton equations are *integrable* infinite-dimensional hamiltonian systems, and suitable finite-dimensional reductions lead to integrable finite-dimensional hamiltonian systems. We are now giving some examples of such finite-dimensional reduction.

2.1 Stationary flows

A soliton hierarchy is a hierarchy of evolution equations of the form

$$\frac{\partial u}{\partial t_N} = K^{(N)}(u, u_x, \ldots), \qquad N = 1, 2, \ldots, \tag{2.1}$$

where $u = u(x, \vec{t})$ is a (possibly vector or matrix-valued) function, x is the "space" variable, $\vec{t}$ is a set of evolution parameters, and $K^{(N)}$ are hamiltonian vector fields

$$K^{(N)} = J \frac{\delta \mathcal{H}^{(N)}}{\delta u},$$

the Hamiltonians $\mathcal{H}^{(N)}$ being mutually in involution. The vector fields $K^{(N)}$ yield an infinite-dimensional abelian Lie algebra

$$[K^{(N)}, K^{(M)}] = K_{\{\mathcal{H}^{(N)}, \mathcal{H}^{(M)}\}} = 0. \tag{2.2a}$$

Any manifold (finite-dimensional, because it is spanned by u and a finite number of its x-derivates) of the form

$$\sum_j c_j K^{(j)}[u] = 0, \qquad (c_j = \text{constant}) \tag{2.2b}$$

is an *invariant submanifold* for the whole soliton hierarchy: namely, if at some time $\vec{t_0}$ the field u belongs to (2.2b) (i.e. it fulfills a certain finite-order *ordinary* differential equation) it will keep on doing that at any time, provided it will evolve according to any equation of the soliton hierarchy. In fact

$$\frac{\partial}{\partial t_N} \sum_j c_j K^{(j)} = \sum_j c_j \frac{\partial K^{(j)}}{\partial t_N} = \sum_j c_j K^{(j)'}[K^{(N)}] = (\text{commutativity})$$

$$\sum_j c_j K^{(N)'}[K^{(j)}] = K^{(N)'} \left[\sum_j c_j K^{(j)} \right] = 0$$

whence the previous statement follows. Such invariant submanifolds define in fact finite-dimensional integrable hamiltonian systems [7]. First of all, they are Lagrangian systems,

$$\sum_j c_j K^{(j)} = 0 \Rightarrow J \frac{\delta}{\delta u} \sum_j c_j \mathcal{H}^{(j)} = 0. \tag{2.3}$$

If J is non degenerated (i.e. $\ker J = 0$) then the Lagrangian is just the hamiltonian density $\mathcal{L} = \sum_j c_j h_j$ ($\mathcal{H} = \int dx h$), otherwise the situation is a bit more complicated. Assuming to be in the simplest case, we have something like

$$\mathcal{L} = \mathcal{L}(u, u^{(1)}, \ldots, u^{(m)}).$$

We can define canonically conjugated (hamiltonian) variables through the Ostrogradski prescription (1.6).

We have thus a hamiltonian system, possessing (infinitely many!) commuting integrals of motion.

Indeed it holds

$$\left\{\mathcal{H}^{(r)}, \sum_j c_j \mathcal{H}^{(j)}\right\} = 0 \Rightarrow$$

$$\left\langle \frac{\delta \mathcal{H}^{(r)}}{\delta u}, J\left(\sum_j c_j \frac{\delta \mathcal{H}^{(j)}}{\delta u}\right)\right\rangle = 0 \Rightarrow$$

$$\int_{-\infty}^{+\infty} dx \left(\gamma^{(r)}(u) \cdot \sum_j c_j K^{(j)}\right)(x) = 0, \qquad \left(\gamma^{(r)} = \frac{\delta \mathcal{H}^{(r)}}{\delta u}\right)$$

hence $\sum_j c_j(\gamma^{(r)} \cdot K^{(j)})$ is an exact derivative

$$\sum_j c_j(\gamma^{(r)} \cdot K^{(j)}) = \sum_j c_j \frac{d}{dx} I^{(r,j)}.$$

But $\sum_j c_j K^{(j)} = 0$, thus $\sum_j c_j I^{(r,j)}$ is an integral of motion.

It can be shown that

1. these integrals are in involution;

2. a sufficient number of them are functionally independent.

Hence one constructs in this way finite-dimensional integrable hamiltonian system. A simple example arising from the above strategy is the *Garnier* system [8]

$$\frac{\partial^2 q_j}{\partial x^2} + [\langle q, q\rangle - \Lambda] q_j = 0, \tag{2.4}$$

$$\langle q, q\rangle = \sum_j q_j^2, \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

Its Hamiltonian reads

$$\mathcal{H} = \frac{1}{2}\sum_j p_j^2 + V(q)$$

with

$$V(q) = \frac{1}{4}\langle q, q\rangle^2 - \frac{1}{2}\langle q, \Lambda q\rangle.$$

It comes out (for instance) as a stationary flows of the vector nonlinear Schrödinger equations. Several Lax representation are known, involving $(n+1)\times(n+1)$ matrices or just 2×2 matrices [9]. A family of involutive integrals is given by

$$F_j = \sum_{k\neq j} \frac{(p_j q_k - p_k q_j)^2}{\lambda_j - \lambda_k} + p_j^2 - \lambda_j q_j^2 + q_j^2 \langle q, q\rangle. \tag{2.5}$$

Its 2×2 Lax pair reads

$$L = \begin{pmatrix} \lambda + \langle p, R_\lambda q\rangle & \langle p, R_\lambda p\rangle \\ \langle q, R_\lambda q\rangle + 1 & -(\lambda + \langle p, R_\lambda q\rangle) \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & \lambda + \langle q, q\rangle \\ 1 & 0 \end{pmatrix}, \tag{2.6}$$

where $R_\lambda = (\lambda I - \Lambda)^{-1}$. The system is separable, the separation coordinates μ_j are defined as:

$$L_{21}(\mu_j) = 0,$$

while the corresponding conjugated moment π_j are defined as

$$\pi_j = L_{11}(\mu_j).$$

The separation equations read

$$-\pi_j^2 = \Delta(\mu_j, F_1, \ldots, F_N),$$

where $\Delta(\lambda, F_1, \ldots, F_N) = \det L(\lambda)$. L belongs to the loop algebra

$$\mathcal{G} = s\ell(2)[\lambda, \lambda^{-1}],$$

given by the set of (truncated) Laurent series with coefficients in $s\ell(2)$. A convenient inner product in such loop algebra is $\langle x, y\rangle := \mathrm{res}\,\mathrm{tr} xy$. One can define the splitting

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-,$$

$\mathcal{G}_+$: only nonnegative powers of $\lambda = \pi_+(\mathcal{G})$,

$\mathcal{G}_-$: only negative powers of $\lambda = \pi_-(\mathcal{G})$.

With the above pairing $\mathcal{G}_- = \mathcal{G}_+^*$, and we have the r-matrix $r = \pi_+ - \pi_-$. However, one can show that the r-matrix for the Garnier system is not just the above difference of projections, but contains a constant extra term. Namely, we have

$$r(x) = x_+ - x_- + x_{21}^{(-1)}\sigma^{(+)}, \qquad \sigma^{(+)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and the superscript (-1) denotes the coefficient of λ^{-1} in the Laurent series.

2.2 Rational solutions of soliton equations

There are other finite-dimensional manifolds preserved by the flows of a soliton equation. Here, we want to quote the manifold of rational solutions of KdV and Benjamin–Ono equations.

The KdV equation:

$$u_t = 6uu_x - u_{xxx}$$

admits rational solutions of the form

$$u(x,t) = 2\sum_{j=1}^{n} \frac{1}{[x - q_j(t)]^2}$$

(clearly a finite-dimensional manifold) if and only if [10]

1. $n = \frac{d(d+1)}{2}$, d — (positive) integer,
2. the set $\{(q_1, \ldots, q_N)(t_0), (p_1, \ldots, p_N)(t_0)\}$ with $p_j = \dot{q}_j$, fulfills the conditions
$$p_j(t_0) = 0, \qquad \sum_{k\neq j} [q_j(t_0) - q_k(t_0)]^{-3} = 0,$$
3. $q_j(t), p_j(t)$ are solutions of the Hamilton equation for $\mathcal{H}_3 = \frac{1}{3}\mathrm{tr}\,L^3$, where
$$L_{jk} = p_j\delta_{jk} + i(1 - \delta_{jk})\frac{1}{q_j - q_k}.$$

The Hamiltonian $\mathcal{H} = \frac{1}{2}\mathrm{tr}\,L^2$ yields the celebrated Calogero–Moser system [11]:

$$\dot{q_j} = p_j, \qquad \dot{p}_j = 2\sum_{k\neq j} \frac{1}{(q_j - q_k)^3}.$$

A further similar example is provided by the Benjamin–Ono equation:

$$u_t + uu_x + \frac{1}{2}Hu_{xx} = 0,$$

where $(Hf)(x) := \frac{1}{\pi}\int_{-\infty}^{+\infty} dy \frac{f(y)}{y-x}$. It has rational solutions of the form

$$u = i\sum_i \left(\frac{1}{x - q_j(t)} - \frac{1}{q - \bar{q}_j(t)} \right),$$

$$q_j \neq q_k\ , \qquad (j \neq k), \qquad \mathrm{Im}\, q_j > 0.$$

One gets, after some manipulation,

$$\ddot{q}_j = 2\sum_{k\neq j} \frac{1}{(q_j - q_k)^3},$$

again the flow corresponding to CM system.

To end this section, we think it is worthwhile to sketch briefly the technique that has been used [2, 12] to solve the CM system. It is the so-called *projection* method which amounts to construct a (trivially) integrable system in a large phase space, and then project it down (in a nontrivial way) in a smaller phase space, getting a (possibly) interesting and still integrable system. A typical scheme is to start from a geodesic flow and to get a motion in a potential field.

For the CM system, we consider a free motion in the space of hermitian matrices with trace zero (or, equivalently, in a space of skew-hermitian matrices)

$$\ddot{x} = 0. \tag{2.7a}$$

It can be integrated to

$$\dot{x} = y, \qquad x = x_0 + ty. \tag{2.7b}$$

The Hamilton equations are obviously

$$\begin{matrix} \dot{x} = y \\ \dot{y} = 0 \end{matrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J \begin{pmatrix} \partial\mathcal{H}/\partial x \\ \partial\mathcal{H}/\partial y \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \mathcal{H} = \frac{1}{2}\mathrm{tr}\ y^2. \tag{2.7c}$$

Clearly, the unitary group G has a hamiltonian action on the Poisson (or symplectic) manifold of above; its moment map is $[x, y]$, and the moment map is preserved by the $\mathcal{H}$ flow, as $\mathcal{H}$ is G-invariant. In fact $\{[x, y], \mathcal{H}\} = \mathrm{tr}\frac{\partial}{\partial x}[x, y]\frac{\partial\mathcal{H}}{\partial y} = 0$. Thanks to this conservation laws, we can look at the flow on a surface of constant level of the angular momentum $[x, y]$. Let us choose $[x, y] = i(I - e \otimes e), e = (1, \ldots, 1)$ (degenerate matrix with $n - 1$ coinciding eigenvalues). We now reduce the hermitian matrix x to diagonal form $Q = \mathrm{diag}(q_1, .., q_N)$ through a unitary transformation

$$x(t) = U(t)Q(t)U^+(t).$$

$U(t)$ can be chosen such that it *preserves* the moment map: defining $L(t) := U(t)YU^+(t)$ we can thus require

$$\begin{aligned} [Q, L] = U[x, y]U^+ &= iU(I - e \otimes e)U^+ \\ &= i(I - E \otimes E) = i(I - e \otimes e) \end{aligned}$$

so that $L_{jk} = \frac{i}{q_j - q_k}, (j \neq k)$. On the other hand, the hamiltonian action of the unitary group ensures

$$\begin{aligned} \omega_{x,y} = \mathrm{tr}\, dx \wedge dy &= \omega_{QL} = \mathrm{tr}\, dQ \wedge dL \\ &= \sum_i dq_j \wedge dL_{jj} \Rightarrow L_{jj} = p_j. \end{aligned}$$

Hence $L_{jk} = p_j\delta_{jk} + i(1 - \delta_{jk})\frac{1}{q_j - q_k}$ and

$$\dot{L} = [\dot{U}U^+, L] = [M, L], \tag{2.8a}$$

$$\dot{Q} = [M, Q] + L \Rightarrow M_{jk} = m_j\delta_{jk} + (1 - \delta_{jk})\frac{L_{jk}}{q_j - q_k}. \tag{2.8b}$$

The solution is very simple: Q_j are the eigenvalues of the matrix $L(0) + Q(0)t$.

2.3 Rapidly decreasing solutions of soliton equations; the peakons

The title of this subsection is not a mistake. We shall try to clarify its meaning right now. It is well known that for each soliton equation, the N-soliton sector is an invariant $2N$-dimensional submanifold, parameterized by their positions x_j and velocities $v_j (j = 1, \dots, N)$. In most cases, the behaviour of such dynamical variables is trivial: the velocities are constant and consequently the positions undergo a uniform motion. However, there are examples in which the time behaviour of the soliton parameters is nontrivial: historically, the first instance was provided by the "boomerons" and "trappons", which are solutions of the *matrix* KdV hierarchy, and have been discovered and exhaustively described by Calogero and Degasperis [13]. Here, we want focus our attention on a more recent example, introduced by Camassa and Holm [14]. They have found that an integrable dispersive equation for water waves has N-peakon solutions whose dynamics is described, in canonical coordinates, by the following Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} p_j p_k (\lambda + \mu \exp[-\nu|q_j - q_k|]). \tag{2.9}$$

The corresponding Hamilton equations read

$$\dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j} = p_j \sum_k p_k \{\mu \ \mathrm{sign}(q_j - q_k)\exp[-\nu|q_j - q_k|]\}, \tag{2.10a}$$

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j} = \sum_k p_k \{\lambda + \mu \exp[-\nu|q_j - q_k|]\}. \tag{2.10b}$$

With no restriction, we can moreover rescale μ to 1.

The system (2.10) can be written in Lax form [14, 15]

$$\dot{L} = [L, M], \tag{2.11a}$$

where L, M are $N \times N$ matrices, whose entries read:

$$L_{jk} = \sqrt{p_j p_k}\,\alpha(q_j - q_k), \tag{2.11b}$$

$$M_{jk} = -2\sqrt{p_j p_k}\,\alpha'(q_j - q_k). \tag{2.11c}$$

In (2.11c) a prime denoted differentiation with respect to the argument, and the function $\alpha(x)$ is given by

$$\alpha(x) = \exp\left[-\frac{\nu|x|}{2}\right]. \tag{2.12}$$

The hamiltonian (2.9) is then the following simple function of the invariants of the matrix L

$$\mathcal{H} = -\mathrm{tr} L^2 + (\lambda + 2)(\mathrm{tr} L)^2. \tag{2.13}$$

If we rewrite it for a moment in the form

$$\mathcal{H} = \sum_{j,k} p_j p_k f(q_j - q_k),$$

with

$$f(x) = \lambda + 2 - \alpha^2(x)$$

we readily discover that, due to the even nature of the function $f(x)$:

1. *The N coordinates q_j keep their own initial ordering, thus we can assume "natural ordering": $q_1 \leq q_2 \leq \cdots q_N$. Physically, we would say that the particles do not cross each other.*

2. *The conjugated momenta p_j keep their own initial sign so that, in particular, there are no "dynamical ambiguities" in the expression $\sqrt{p_j p_k}$ occurring in the Lax matrices.*

By introducing the Lagrangian through a Legendre transformation

$$\mathcal{L}(q, \dot{q}) = \ -\mathcal{H}(p, q) + \sum_k p_k \dot{q}_k = \mathcal{H}(p, q) = \frac{1}{2} \sum_{jk} \dot{q}_j \dot{q}_k G_{jk}(q), \tag{2.14}$$

$G(q)$ being the inverse of the matrix $(F(q))_{jk} \equiv f(q_j - q_k)$, the dynamics under scrutiny is naturally interpreted as a geodesic motion on a manifold characterized by the metric tensor $G(q)$.

The complete integrability of the peakons system can be for instance proved by using the r-matrix approach.

We recall that whenever we have a Lie algebra $\mathcal{G}$, the trace form on it allows one to identify $\mathcal{G}$ with its dual (the space of linear functions on $\mathcal{G}$), and to consider the r-matrix as a linear endomorphism on $\mathcal{G}$, that induces the Lie–Poisson bracket (1.13)

$$\{f, g\}_L \ = \ \langle L, [\nabla f, \nabla g]_r \rangle$$

with

$$[x, y]_r = [x, r(y)] + [r(x), y], \qquad (x, y \ \in \mathcal{G})$$

and

$$r(x) \ = \ \sum_{j,k} r_{jk} e^{(i)} \langle e^{(k)}, x \rangle, \tag{2.15}$$

$e^{(i)}$ being a basis in $\mathcal{G}$. In our case, everything is very simple. The algebra $\mathcal{G}$ is just $gl(N, \mathbb{R})$, so that we work with the standard (Weyl) basis

$$(E_{jk})_{lm} = \delta_{jl} \delta_{km}$$

and we have

$$r \ = \ -\frac{\nu}{8} \sum_{jk} \mathrm{sign}(q_j - q_k) E_{jk} \otimes (E_{jk} + E_{kj})$$

so that

$$r(x) = \frac{\nu}{4}(\pi_-(X^{(s)}) - \pi_+(X^{(s)})). \tag{2.16}$$

In (2.16) $\pi_-(\pi_+)$ denotes the strictly lower-triangular (upper-triangular) projection and $X^{(s)}$ is the symmetric part of the matrix X. The Lax representation for the commuting hamiltonian flows generated by the invariant functions $F^{(k)} \equiv \frac{1}{k+1}\mathrm{tr}L^{k+1}$ has the form

$$\frac{\partial L}{\partial t_k} \equiv \{F^{(k)}, L\} = [L, r(L^k)]. \tag{2.17}$$

On the other hand, the r-matrix (2.16) is just one of the r-matrices that can be associated with the finite nonperiodic Toda lattice [2,16]. Such striking coincidence strongly suggests to give a closer look to the peakons lattice, to uncover possible further connections with the Toda system.

To this aim, we first notice that the inverse of the L matrix (2.11b) is in this case easily computable, yielding

$$L^{-1} = \begin{pmatrix} B_1 & A_1 & 0 & \dots \\ A_1 & B_2 & A_2 & \dots \\ 0 & \ddots & \ddots & \ddots \\ \vdots & \dots & A_{N-1} & B_N \end{pmatrix} \tag{2.18}$$

where

$$A_j = -\frac{1}{p_j p_{j+1}} \frac{e_j}{1-e_j^2}, \qquad B_j = \frac{1}{p_j} \frac{1-e_{j-1}^2 e_j^2}{(1-e_{j-1}^2)(1-e_j^2)}, \tag{2.19}$$

with $e_j = \exp[-\nu(q_j - q_{j+1})]$ $(j = 1, \cdots, N-1)$; $e_0 = e_N = 0$. The isospectral flows of L^{-1} are of course characterized by the dynamical equations

$$\frac{\partial L^{-1}}{\partial t_k} = [L^{-1}, r(L^k)]$$

and in particular we have

$$\{\mathcal{H}, L^{-1}\} = [L^{-1}, r(L)].$$

The above equation can be written in terms of the new dynamical variable $\Lambda \equiv L^{-1}$, taking the form

$$\dot{\Lambda} = [\Lambda, r(\Lambda^{-1})] = [\Lambda, r(\nabla_\Lambda \ln\det\Lambda)]$$

Thus we have the following

PROPOSITION: *When the variables (q,p) evolve according to the Hamiltonian (2.1), the "Flaschka variables" (A,B) (see for instance [17]), defined through (2.19) evolve according to the Hamiltonian* $\ln\det\Lambda$.

In this sense, *the peakons lattice is just one of the commuting flows of the (finite, nonperiodic) Toda hierarchy.*

The connection with the Toda system can be further exploited to derive an integrable discretisation of the peakons lattice, i.e. a symplectic map preserving the conserved quantities of the continuous-time model. Like in the Toda system, the basic tool is the factorisation of the Lax matrix [18]. Taking care of the natural ordering, it can be written as

$$L = \sum_j p_j E_{jj} + \sum_{k>j} \sqrt{p_j p_k} \exp\left[-\frac{\nu}{2}(q_k - q_j)\right] E_{jk}$$
$$+ \sum_{k<j} \sqrt{p_j p_k} \exp\left[-\frac{\nu}{2}(q_j - q_k)\right] E_{jk}. \tag{2.20}$$

It is then possible to find out explicitly (and uniquely, up to signs) an upper triangular matrix $\mathcal{U}(L)$

$$\mathcal{U}(L) \equiv \sum_j \sum_{k \geq j} u_{jk} E_{jk}$$

such that

$$L \ = \ \mathcal{U}(L) \times [\mathcal{U}(L)]^t \tag{2.21}$$

where the superscript t denotes transposition. In fact one finds

$$u_{jk}^2 \ = \ p_j[\exp[-\nu(q_k - q_j)] - \exp[-\nu(q_{k+1} - q_j)]. \tag{2.22}$$

The above expression holds for $1 \leq k \leq N$, provided we set $q_{N+1} = -\infty$. The new (or "Miura transformed") L, say $\tilde{L}$, is then given by

$$\tilde{L} \ = \ [\mathcal{U}(L)]^t \times \mathcal{U}(L), \tag{2.23}$$

which yields for the entries u_{jk} the alternative expression

$$u_{jk}^2 \ = \ \tilde{p}_k[\exp[-\nu(\tilde{q}_k - \tilde{q}_j)] - \exp[-\nu(\tilde{q}_k - \tilde{q}_{j-1})]. \tag{2.24}$$

Identifying (2.22) and (2.24) we get, after some trivial manipulation,

$$\frac{\tilde{p}_k}{\exp[-\nu(q_k - \tilde{q}_k)] - \exp[-\nu(q_{k+1} - \tilde{q}_k)]}$$
$$= \frac{p_j}{\exp[-\nu(q_j - \tilde{q}_j)] - \exp[-\nu(q_j - \tilde{q}_{j-1})]}. \tag{2.25}$$

The r.h.s. and the l.h.s. of (2.25) must then be independent of the label, i.e. they have to be equal to the same constant, say β. Therefore, (2.25) is in fact equivalent to the canonical transformation:

$$\tilde{p}_k \ = \ \beta(\exp[-\nu(q_k - \tilde{q}_k)] - \exp[-\nu(q_{k+1} - \tilde{q}_k)]),$$
$$p_k = \beta(\exp[-\nu(q_k - \tilde{q}_k)] - \exp[-\nu(q_k - \tilde{q}_{k-1})]),$$

whose generating function is

$$S(q,\tilde{q}) \;=\; \frac{\beta}{2}\sum_k(\exp[-\nu(q_k-\tilde{q}_k)]-\exp[-\nu(q_{k+1}-\tilde{q}_k)]).$$

The above canonical transformation can be interpreted as a discrete-time flow, by setting

$$(p_j,q_j)=(p_j(n),q_j(n));\;\;(\tilde{p}_j,\tilde{q}_j)=(p_j(n+1),q_j(n+1))$$

and written in the following discrete Newton form in terms of $q=q(n),\ \tilde{q}=q(n+1),\ \hat{q}=q(n-1)$

$$\begin{aligned}&\exp[-\nu(\hat{q}_j-q_j)]-\exp[-\nu(\hat{q}_{j+1}-q_j)]\\&\quad=\exp[-\nu(q_j-\tilde{q}_j)]-\exp[-\nu(q_j-\tilde{q}_{j-1})].\end{aligned}\tag{2.26}$$

Moreover, as in the Toda case, such discrete flow can be explicitly integrated. Namely, if we denote $L=L(p,q)=L_n,\ \tilde{L}=L(\tilde{p},\tilde{q})=L_{n+1}$, we get [18]

$$L_n \;=\; \mathcal{U}(L_0^n)L_0[\mathcal{U}(L_0^n)]^{-1}.\tag{2.27}$$

From (2.27), looking at n as at a discrete time, we can read off the interpolating Hamiltonian $F(L)$ defined through

$$L^n=\exp[n\nabla_L F],$$

whence it follows

$$F(L) \;=\; trL(\ln L-I).$$

It is remarkable that the same system, up to trivial notational changes and discrete-time reversal, has been obtained by factorisation of the Toda–Lax matrix in [19]. This coincidence is easily explained in terms of the interpolating hamiltonian flow, which is the same up to the interchange $L\to L^{-1}$, giving rise to the time-reversal. In connection with the above result, it might be worthwhile to recall that interest in integrable time-discretisation has been recently revived by a number of stimulating papers [20].

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