Second-order resonant systems in optics and surfaces in $\mathbb{R}^3$

Heinz Steudel

Arbeitsgruppe "Nichtklassische Strahlung" der Max-Planck-Gesellschaft
an der Humboldt-Universität zu Berlin
Rudower Chaussee 5, 12484 Berlin, Germany
e-mail: steudel@photon.fta-berlin.de

Abstract: The Maxwell-Bloch equations of stimulated Raman scattering (SRS) or resonant two-photon propagation (TPP) are known to be integrable. With reference to a former letter [1] it is demonstrated that any solution to these equations corresponds to a surface in three-dimensional space. The common structure of the surface equations is that the mixed second-order derivative of the space vector is equal to a bilinear combination of the first-order derivatives. It is shown how particular physical systems — i.e., SRS and both non-degenerate and degenerate TPP — are reflected in the structure of the differential equations.

1 Two-photon Maxwell-Bloch equations and the corresponding linear system

We consider the propagation of two quasi–monochromatic optical waves interacting with a system of two–level atoms with the assumption of a second-order resonance. Let ω_1, ω_2 be the carrier frequencies of the waves and $\hbar\omega_0$ the energy separation of the atoms. Then we require

$$\omega_1 - \varepsilon\omega_2 = \omega_0 \tag{1}$$

and we distinguish

1. $\varepsilon = +1$, *stimulated Raman scattering* (SRS) and
2. $\varepsilon = -1$, two-photon absorption and emission or *two-photon propagation* (TPP).

The interaction dynamics is governed by the Maxwell-Bloch equations which can be written in terms of the atomic Bloch vector $\boldsymbol{R} = (R_1, R_2, R_3)$ and some second real vector $\boldsymbol{S} = (S_1, S_2, S_3)$ defined by

$$S_+ \equiv S_1 + iS_2 = \begin{cases} 2\mathcal{E}_2\mathcal{E}_1^*, & SRS, \\ 2\mathcal{E}_2^*\mathcal{E}_1^*, & TPP, \end{cases} \tag{2}$$

$$S_3 = \mathcal{E}_1\mathcal{E}_1^* - \varepsilon\mathcal{E}_2\mathcal{E}_2^*, \quad S_- = S_+^* \,. \tag{3}$$

Using properly scaled space and retarded time variables $\chi \sim x$, $\tau \sim (t - x/v)$ the equations of motion can be written in the form [2–4].

$$\partial_\chi S_+ = i(gS_+R_3 + \varepsilon S_3 R_+), \tag{4}$$

$$\partial_\tau R_+ = i(gR_+S_3 + R_3S_+), \tag{5}$$

$$\partial_\tau R_3 = -\partial_\chi S_3 = \frac{i}{2}(R_+S_- - R_-S_+)\,. \tag{6}$$

Here g is a real constant describing dynamical Stark shift and $R_\pm \equiv R_1 \pm iR_2$. We introduce entities

$$N^2(\chi) = R_1^2 + R_2^2 + R_3^2, \tag{7}$$

$$A^2(\tau) = \varepsilon(S_1^2 + S_2^2) + S_3, \tag{8}$$

and here we already made use of conservation laws following from (4–6), i.e., that N does not depend on τ, and A does not depend on χ. Physically N means the atomic density while for A we get

$$A(\tau) = \mathcal{E}_1\mathcal{E}_1^* + \varepsilon\mathcal{E}_2\mathcal{E}_2^*\,. \tag{9}$$

For $\varepsilon = -1$ we distinguish

2a) (non-degenerate) TPP with the assumption $A(\tau) > 0$ for all τ and

2b) degenerate TPP (DTPP) if $A(\tau) = 0$ for all τ. This is naturally realized when $\mathcal{E}_1 = \mathcal{E}_2$ is one and the same wave.

According to [3, 4] the differential equations (4–6) are integrability conditions of the linear system

$$\partial_\tau u = \begin{pmatrix} -i\zeta S_3 & -\varepsilon d(\zeta+\varphi)S_+ \\ d^{-1}(\zeta-\varphi)S_- & i\zeta S_3 \end{pmatrix} u, \tag{10}$$

$$\partial_\chi u = \begin{pmatrix} -i\lambda R_3 & (\lambda-\varphi)R_+ \\ -(\lambda+\varphi)R_- & i\lambda R_3 \end{pmatrix} u, \tag{11}$$

$$\varphi \equiv if = \frac{1}{2}\sqrt{g^2-\varepsilon}\,, \quad d = g + 2\varphi, \tag{12}$$

$$(2\lambda + g)(2\zeta + g) = \varepsilon, \tag{13}$$

with u denoting a two-component column. Either of the two complex parameters λ, ζ connected by (13) may be taken as the *spectral parameter*. φ is real for TPP ($\varepsilon = -1$) while for SRS ($\varepsilon = +1$) we distinguish

1a) SRS1, $g < 1$, f real, $d = g + 2if = e^{i\alpha}$, α real;

1b) SRS2, $g > 1$, φ real, d real.

Bäcklund transformations were established for the SRS/TPP system [5, 6]. Some peculiarities arise for DTPP, see case 2b) above, and were discussed recently [7]. In order to simplify the equations as much as possible it is of interest to note that N always can be transformed to 1 by a scaling of χ while A can be normalized by a scaling of τ if $A(\tau) > 0$ but not, of course, if $A = 0$, i.e., not for DTPP.

2 Surfaces in $\mathbb{R}^3$

Here we essentially follow [1]. Because of the conservation law (6) there is some function $F_3(\chi,\tau)$ such that

$$F_{3\chi} = R_3\,, \quad F_{3\tau} = -S_3\,. \tag{14}$$

Another conservation law,

$$\partial_\tau(R_+ e^{2fF_3}) = \varepsilon\, d\, \partial_\chi(S_+ e^{2fF_3})\,, \tag{15}$$

is found and leads to

$$R_+ = \phi_\chi\, e^{-2fF_3}, \tag{16}$$

$$S_+ = \frac{\varepsilon}{d}\phi_\tau\, e^{-2fF_3}, \tag{17}$$

with $\phi(\chi,\tau)$ being some complex-valued function. Now the equations of motion (4–6) may be written in terms of F_3 and ϕ, i.e., for three real functions depending on two variables. Thus we arrive at

$$\phi_{\chi\tau} = i\left(\frac{\varepsilon}{d}F_{3\chi}\phi_\tau - d\phi_\chi F_{3\tau}\right), \tag{18}$$

$$e^{i(d^*-d)F_3}F_{3\chi\tau} = \frac{\varepsilon i}{2}\left(d\phi_\chi\phi_\tau^* - d^*\phi_\chi^*\phi_\tau\right) \tag{19}$$

where d is either real or of modulus 1. Now let us take

$$\phi = F_1 + iF_2 \tag{20}$$

and write (18, 19) in real form. As well we will write (7, 8) in terms of F_1, F_2, F_3. For SRS1 with $\varepsilon = +1$, $d = e^{i\alpha}$, f real we get

$$F_{1\chi\tau} = \cos\alpha(F_{2\chi}F_{3\tau} - F_{3\chi}F_{2\tau}) + \sin\alpha(F_{3\chi}F_{1\tau} + F_{1\chi}F_{3\tau}), \tag{21}$$

$$F_{2\chi\tau} = \cos\alpha(F_{3\chi}F_{1\tau} - F_{1\chi}F_{3\tau}) + \sin\alpha(F_{3\chi}F_{2\tau} + F_{2\chi}F_{3\tau}), \tag{22}$$

$$e^{2F_3\sin\alpha}F_{3\chi\tau} = \cos\alpha(F_{1\chi}F_{2\tau} - F_{2\chi}F_{1\tau}) - \sin\alpha(F_{1\chi}F_{1\tau} + F_{2\chi}F_{2\tau}), \tag{23}$$

$$(F_{1\chi}^2 + F_{2\chi}^2)e^{-4fF_3} + F_{3\chi}^2 = N^2(\chi), \tag{24}$$

$$(F_{1\tau}^2 + F_{2\tau}^2)e^{-4fF_3} + F_{3\tau}^2 = A^2(\tau)\,. \tag{25}$$

System (21–23) is derivable as Euler–Lagrange equations from the Lagrangian

$$L = \frac{1}{8f}\left(d\phi_\chi\phi_\tau^* + d^*\phi_\chi^*\phi_\tau\right) + F_{3\chi}F_{3\tau}. \tag{26}$$

For SRS2 ($\varepsilon = +1$) and TPP ($\varepsilon = -1$), d real we get

$$F_{1\chi\tau} = dF_{2\chi}F_{3\tau} - \frac{\varepsilon}{d}F_{3\chi}F_{2\tau}, \tag{27}$$

$$F_{2\chi\tau} = \frac{\varepsilon}{d} F_{3\chi} F_{1\tau} - d F_{1\chi} F_{3\tau}, \tag{28}$$

$$F_{3\chi\tau} = \frac{\varepsilon}{d} (F_{1\chi} F_{2\tau} - F_{2\chi} F_{1\tau}), \tag{29}$$

$$F_{1\chi}^2 + F_{2\chi}^2 + F_{3\chi}^2 = N^2(\chi), \tag{30}$$

$$\frac{\varepsilon}{d^2}(F_{1\tau}^2 + F_{2\tau}^2) + F_{3\tau}^2 = A^2(\tau)\,. \tag{31}$$

Here DTPP is included. It is distinguished by $A \equiv 0$.

For the system (27–29) no Lagrangian is known. We remind that according to the remark at the end of section 1 without loss in generality we may put $N = 1$ and $A = 1$ or 0.

Summarizing we state that any solution to the two-photon Maxwell–Bloch equations (4–6) can be represented by a surface in three-dimensional space, and any surface (F_1, F_2, F_3) in $\mathbb{R}^3$ fulfilling (18, 20) defines a solution to (4–6).

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